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A WORK ON INEXTENSIBLE FLOWS OF SPACE CURVES WITH RESPECT TO A NEW ORTHOGONAL FRAME IN E^3

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Abstract. In this study, we bring forth a new general formula for inextensible flows of Euclidean curves as regards modified orthogonal frame (MOF) in E^3. For an inextensible curve flow, we provide the necessary and sufficient conditions, which are denoted by a partial differential equality containing the curvatures and torsion.

1. Introduction

The theory of curves has praxis in mathematics, engineering and physics [5, 10]. Since space curves are one-dimensional manifolds, they are the simplest and most basic structures of differential geometry. Studies on curves have started about 400 years ago. The opinion of characterizing the curve with its tangent belongs to Fermat [4]. Descartes first gave the definition of an algebraic curve [6]. After Descartes, Euler gave the parametric definition of curves in 1748 [7]. Frenet and Serret were the first to consider the idea of characterizing curves with tangent, principal normal and binormal vectors [8, 14].

Although a Frenet-Serret framework is central to curve characterization, it has some shortcomings. Bishop described an alternative framework called a new relative parallel framework (RPAF) adapted to the Frenet framework [2]. Frenet and Bishop frames give curves their unique properties such as curvature κ and torsion τ, and these characteristics are invariant. Frenet-Serret formulas are used to analyze the kinematic characteristics of a particle travelling along a space curve in E_1^3. In summary, Frenet formulas characterize the curve by defining derivatives of unit vectors derived as tangential, normal and binormal.

In summary, Frenet formulas make use of tangent, principal normal, and binormal vectors of space curves, which are interconnected. In [13], Sasai finally worked on an orthogonal frame and came up with a formulation corresponding to the Frenet-Serret equation. The resulting frame was called the MOF. In

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most studies, external factors were ignored while performing curve characterization. However, later research has revealed that these external factors play an important role. One of these external factors is curve flow. If the arc-length of a curve is conserved, the flow of the curve is named to be inelastic or inextensible. First, Kwon and Park investigated non-extensible curve flows in Euclidean 3-space E^3 [11, 12]. After them, inextensible flows of curves are worked [1].

In our work, we introduce a standard formula for inextensible flows of space curves relative to the MOF in E^3. Our aim is to this study, expressing the requisite and adequate conditions for an inextensible curve flow in the form of a partial differential equation involving curvature and torsion.

2. Preliminaries

Initially, we will review the traditional fundamental definition and theorems of space curves, which applies to space curves in E^3. Assume that s is the arc-length parameter and F is a curve of class C^3. Furthermore, we consider that curvature $\kappa(s)$ not disappear anywhere. Thus an orthonormal frame $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ exists that fulfills the Frenet-Serret equation

$$(1) \qquad \begin{cases} \overrightarrow{t}' &= \kappa \overrightarrow{n} \\ \overrightarrow{n}' &= -\kappa \overrightarrow{t} + \tau \overrightarrow{b} \\ \overrightarrow{b}' &= -\tau \overrightarrow{n} \end{cases}$$

where the vectors $\overrightarrow{t}$, $\overrightarrow{n}$ and $\overrightarrow{b}$ denote the tangent, primary normal and binormal unit vectors, respectively. And also κ and τ are curvature and torsion, respectively. When we take two function κ of class C^1 and continuous function τ, there is a curve of class that admits an orthonormal frame $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ satisfying (1) with provided and as its curvature κ and torsion τ, respectively. A space curve can only be determined in one way in E^3 [9].
Let the coordinates in Euclidean 3-space be F_i $(i = 1, 2, 3)$. Assume that $F(s)$ is an analytical curve and that s spans a certain interval. We consider F to be non-singular, i.e.

$$\sum_{i=1}^{3} \left(\frac{dF_i}{ds}\right)^2$$

is not at all zero. As a result, F may be parametrized by its arc-length s. Henceforth, we can just write F in the following form:

$$F = F(s) = (F_1, F_2, F_3), \ s \in I,$$

where I is a open interval and $F(s)$ is analytical in s . Let the curvature $\kappa(s)$ of F is differently zero.Thus we can give an MOF $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ as noted below:

$$\overrightarrow{T} = \frac{dF}{ds}, \ \overrightarrow{N} = \frac{d\overrightarrow{T}}{ds}, \ \overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N},$$

where the vector $\overrightarrow{T} \times \overrightarrow{N}$ are orthogonal both the vector $\overrightarrow{T}$ and the vector $\overrightarrow{N}$. Now we define the relationships between $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$and given classic Frenet frame $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ as follows;

$$\begin{cases} \overrightarrow{T} & = \overrightarrow{t} \\ \overrightarrow{N} & = \kappa \overrightarrow{n} \\ \overrightarrow{B} & = \kappa \overrightarrow{b} \end{cases}$$

where $\kappa \neq 0$. Hence, when $\kappa(s_0) = 0$ and the length squares of $\overrightarrow{N}$ and $\overrightarrow{B}$ vary analytically in s, $\overrightarrow{N}(s_0) = \overrightarrow{B}(s_0) = 0$. According to MOF $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$, a elementary computation shows that the Frenet derivative formulas take the next formula

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}, \tag{2}$$

and

$$\tau = \tau(s) = \frac{\det\left(F', \ F'' \ F'''\right)}{\kappa^2},$$

where τ is torsion of F. As we know from the classic Frenet-Serret equation, an extractable singularity of τ is any zero point of κ^2. In the usual situation, equation (2) relates to the Frenet-Serret equation [3]. Also, $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ satisfies:

$$\begin{gathered} \left\langle \overrightarrow{T}, \overrightarrow{N} \right\rangle = \left\langle \overrightarrow{T}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{N}, \overrightarrow{B} \right\rangle = 0. \\ \left\langle \overrightarrow{T}, \overrightarrow{T} \right\rangle = 1, \left\langle \overrightarrow{N}, \overrightarrow{N} \right\rangle = \left\langle \overrightarrow{B}, \overrightarrow{B} \right\rangle = \kappa^2, \end{gathered} \tag{3}$$

where $\langle , \rangle$ stands for E^3 is inner product. Let us emphasize here that the values of κ^2 and τ in (2) and (3) are analytical in s.

3. Inextensible Flows of Curve in Modified Orthogonal Frame

Throughout of this work, unless indicated otherwise, we take the transformation F as follows;

$$F : [0,l] \times [0,\omega) \to E^3$$
$$(u,t) \to F(u,t),$$

which is a family of one-parameter differentiable and regular curves in the three-dimensional Euclidean space and where the beginning curve's arc length is l. Let $u \in [0,l]$. If the speed of curve F is expressed $v = \left\| \frac{\partial F}{\partial u} \right\|$. In this instance, the arc-length of curve F denote as follows;

$$S(u) = \int_0^u \left\| \frac{\partial F}{\partial u} \right\| du = \int_0^u v du.$$

We can re-express the $\frac{\partial}{\partial s}$ operator according to the variable u as follows;

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}.$$

So, $ds = vdu$ is the arclength parameter. The following form can be defined as any flow of the curve F .

Definition 3.1. *Flow of the curve F in E^3 is given by*

$$\frac{\partial F}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B},$$

where $\left\{ \overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B} \right\}$ is MOF and $f_1,\ f_2,\ f_3$ are scalar velocity functions of the curve$F$. The arc-length variation is given by

$$S(u) = \int_0^u v du.$$

The case of the curve not undergoing any compression or extension can be represented in E^3 by the situation;

$$\int_0^u S(u,t) = \int_0^u \frac{\partial v}{\partial s} du = 0, \quad \forall u \in [0,l] .$$

Definition 3.2. *$F(u,t)$ is curve evolution and its flow $\frac{\partial F}{\partial t}$ on the MOF in three-dimensional Euclidean space are considered inextensible if*

$$\frac{\partial}{\partial t} \left\| \frac{\partial F}{\partial u} \right\| = 0.$$

We are currently looking into what conditions must exist for a flow to be inextensible. We need the next theorem in this situation.

Theorem 3.3. *Let* $\left\{\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}\right\}$ *be a MOF and* $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ *be smooth flow of the curve* F *in 3-dimensional Euclidean space. The flow is inextensible necessary and sufficient condition*

$$\frac{\partial f_1}{\partial s}=f_2\kappa^2.$$

Proof. We have $\left\langle \frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\right\rangle=v^2$. The operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute because vary coordinates u and t are linear independent coordinates.Thus we get

$$\begin{aligned}
2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\right\rangle \\
&= 2\left\langle \frac{\partial F}{\partial u},\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial t}\right)\right\rangle \\
&= 2\left\langle \frac{\partial F}{\partial u},\frac{\partial}{\partial u}\left(f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}\right)\right\rangle \\
&= 2v\left\langle \overrightarrow{T},\frac{\partial}{\partial u}(f_1)\overrightarrow{T}+vf_1\overrightarrow{N}\right\rangle \\
&\quad + 2v\left\langle \overrightarrow{T},\frac{\partial}{\partial u}(f_2)\overrightarrow{N}+vf_2(-\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})\right\rangle \\
&\quad + 2v\left\langle \overrightarrow{T},\frac{\partial}{\partial u}(f_3)\overrightarrow{B}+vf_3\left(-\tau\overrightarrow{N}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}\right)\right\rangle \\
&= 2v\left(\frac{\partial}{\partial u}(f_1)-\kappa^2 f_2 v\right).
\end{aligned}$$

Thus we get

$$\frac{\partial v}{\partial t}=2v\left(\frac{\partial}{\partial u}(f_1)-\kappa^2 f_2 v\right). \tag{4}$$

Now let $\frac{\partial F}{\partial t}$ be extensible. From eq.(4) we find

$$\frac{\partial}{\partial t}S(u,t)=\int_0^u \frac{\partial v}{\partial t}du=\int_0^u\left(\frac{\partial}{\partial u}(f_1)-\kappa^2 f_2 v\right)du=0$$

for all $u\in[0,l]$. This come to mean that $\frac{\partial}{\partial u}(f_1)=\kappa^2 f_2 v$. In other words $\frac{\partial}{\partial s}(f_1)=\kappa^2 f_2$. To complete the proof, the argument can be inverted to demonstrate sufficiency. □

We constrain the parameterized arc length curve. Namely, If we take $v=1$ and the local coordinate u subtend to the curve arc-length s.

Theorem 3.4. *Let* $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ *be a differentiable and regular flow of the curve* F *with MOF* $\left\{\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}\right\}$*in 3-dimensional Euclidean space.*

In this case

$$\frac{\partial \overrightarrow{T}}{\partial t} = \left(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 - \tau f_3 \right) \overrightarrow{N} + \left(\left(\tau f_2 + \frac{\partial}{\partial s}(f_3) \right) + \frac{\kappa^{'}}{\kappa} f_3 \right) \overrightarrow{B},$$

$$\frac{\partial \overrightarrow{N}}{\partial t} = -\left(\kappa^2 f_1 + \kappa^2 \frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'} f_2 - \kappa^2 \tau f_3 \right) \overrightarrow{T} + \frac{1}{\kappa^2} \lambda \overrightarrow{B},$$

$$\frac{\partial \overrightarrow{B}}{\partial t} = -\left(\kappa^2 \tau f_2 + \kappa^2 \frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'} f_3 \right) \overrightarrow{T} - \frac{1}{\kappa^2} \lambda \overrightarrow{N}.$$

Proof. Using the MOF and Theorem 3.3, we calculate

$$\begin{aligned} \frac{\partial \overrightarrow{T}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial F}{\partial s} \\ &= \frac{\partial}{\partial s}\left(f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B} \right) \\ &= \frac{\partial}{\partial s}(f_1)\overrightarrow{T} + f_1 \overrightarrow{N} + \frac{\partial}{\partial s}(f_2)\overrightarrow{N} + f_2(-\kappa^2 \overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau \overrightarrow{B}) \\ &\quad + \frac{\partial}{\partial s}(f_3)\overrightarrow{B} + f_3 \left(-\tau \overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B} \right), \end{aligned}$$

that is

$$\frac{\partial \overrightarrow{T}}{\partial t} = \left(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 - \tau f_3 \right) \overrightarrow{N} + \left(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3 \right) \overrightarrow{B}. \tag{5}$$

Now differentiate the MOF by t and by using the (5)

$$\begin{aligned} \frac{\partial}{\partial t}\left\langle \overrightarrow{T}, \overrightarrow{N} \right\rangle &= 0 = \left(\kappa^2 f_1 + \kappa^2 \frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'} f_2 - \kappa^2 \tau f_3 \right) + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{N}}{\partial t} \right\rangle, \\ \frac{\partial}{\partial t}\left\langle \overrightarrow{T}, \overrightarrow{B} \right\rangle &= 0 = \left(\kappa^2 \tau f_2 + \kappa^2 \frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'} f_3 \right) + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{B}}{\partial t} \right\rangle, \\ \frac{\partial}{\partial t}\left\langle \overrightarrow{N}, \overrightarrow{B} \right\rangle &= 0 = \lambda + \left\langle \overrightarrow{N}, \frac{\partial \overrightarrow{B}}{\partial t} \right\rangle. \end{aligned} \tag{6}$$

From (5) and (6), we obtain

$$\frac{\partial \overrightarrow{T}}{\partial t} = \left(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 - \tau f_3 \right) \overrightarrow{N} + \left(\left(\tau f_2 + \frac{\partial}{\partial s}(f_3) \right) + \frac{\kappa^{'}}{\kappa} f_3 \right) \overrightarrow{B},$$

$$\frac{\partial \overrightarrow{N}}{\partial t} = -\left(\kappa^2 f_1 + \kappa^2 \frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'} f_2 - \kappa^2 \tau f_3 \right) \overrightarrow{T} + \frac{1}{\kappa^2} \lambda \overrightarrow{B},$$

$$\frac{\partial \overrightarrow{B}}{\partial t} = -\left(\kappa^2 \tau f_2 + \kappa^2 \frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'} f_3 \right) \overrightarrow{T} - \frac{1}{\kappa^2} \lambda \overrightarrow{N},$$

respectively, where $\left\langle \frac{\partial \overrightarrow{N}}{\partial t}, \overrightarrow{B} \right\rangle = \lambda$. □

The conditions on partial differential equality involving the curvatures and torsion are given in the following theorem for the curve flow $F(s,t)$.

Theorem 3.5. *Assume that* $\left\{\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}\right\}$ *is modified orthogonal frame in 3-dimensional Euclidean space and curve flow* $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ *is inextensible. Then the partial differential equations system is given by;*

$$\begin{aligned}\lambda =&\kappa^2\tau f_1+\kappa^2\tau\frac{\partial}{\partial s}f_2+\kappa\kappa^{'}\tau f_2-\kappa^2\tau^2 f_3+\kappa^2\frac{\partial}{\partial s}(\tau f_2)+\kappa^2\frac{\partial^2}{\partial s^2}f_3\\&+\kappa^2\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3)+\kappa\kappa^{'}\tau f_2+\kappa\kappa^{'}\frac{\partial}{\partial s}f_3+\kappa^{'^2}f_3,\end{aligned}$$

$$\begin{aligned}\frac{\partial\kappa^2}{\partial t}=&\frac{\partial}{\partial s}(\kappa^2 f_1)+\frac{\partial}{\partial}(\kappa^2\frac{\partial}{\partial}(f_2))+\frac{\partial}{\partial s}(\kappa\kappa^{'}f_2)-\frac{\partial}{\partial s}(\kappa^2\tau f_3)-\kappa\kappa^{'}f_1\\&-\kappa\kappa^{'}\frac{\partial}{\partial s}(f_2)-\kappa^{'^2}f_2+\kappa\kappa^{'}\tau f_3-\kappa^2\tau^2 f_2-\kappa^2\tau\frac{\partial}{\partial s}(f_3)-\kappa\kappa^{'}\tau f_3,\end{aligned}$$

$$\frac{\partial\tau}{\partial t}=\kappa^2\tau f_2+\kappa^2\frac{\partial}{\partial s}(f_3)+\kappa\kappa^{'}f_3+\lambda\frac{\partial}{\partial s}\left(\frac{1}{\kappa^2}\right)+\frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)$$

Proof. We know that $\frac{\partial}{\partial s}\frac{\partial\overrightarrow{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\overrightarrow{T}}{\partial s}$. In this case we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial\overrightarrow{T}}{\partial t}=&\frac{\partial}{\partial s}\left[\left(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3\right)\overrightarrow{N}+\left(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3\right)\overrightarrow{B}\right]\\=&\left(\frac{\partial}{\partial s}(f_1)+\frac{\partial^2}{\partial s^2}(f_2)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_2)-\frac{\partial}{\partial s}(\tau f_3)\right)\overrightarrow{N}\\&+\left(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3\right)(-\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})\\&+\left(\frac{\partial}{\partial s}(\tau f_2)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3)\right)\overrightarrow{B}\\&+\left(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3\right)\left(-\tau\overrightarrow{N}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}\right),\end{aligned}$$

while

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\overrightarrow{T}}{\partial s}=&\frac{\partial}{\partial t}\overrightarrow{N}\\=&-\left(\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2-\kappa^2\tau f_3\right)\overrightarrow{T}+\frac{1}{\kappa^2}\lambda\overrightarrow{B},\end{aligned}$$

thereby

$$\lambda = \kappa^2\tau f_1 + \kappa^2\tau\frac{\partial}{\partial s}f_2 + \kappa\kappa^{'}\tau f_2 - \kappa^2\tau^2 f_3 + \kappa^2\frac{\partial}{\partial s}(\tau f_2) + \kappa^2\frac{\partial^2}{\partial s^2}f_3 + \kappa^2\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3) + \kappa\kappa^{'}\tau f_2 + \kappa\kappa^{'}\frac{\partial}{\partial s}f_3 + \kappa^{'^2}f_3.$$

Since $\frac{\partial}{\partial s}\frac{\partial\overrightarrow{N}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial\overrightarrow{N}}{\partial s}$, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial\overrightarrow{N}}{\partial t} =& \frac{\partial}{\partial s}\left[-\left(\kappa^2 f_1 + \frac{\partial}{\partial s}(\kappa^2 f_2) + \kappa\kappa^{'}f_2 - \kappa^2\tau f_3\right)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B}\right]\\
=& -\left(\frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'}f_2) - \frac{\partial}{\partial s}(\kappa^2\tau f_3)\right)\overrightarrow{T}\\
&- \left(\kappa^2 f_1 + \kappa^2\frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'}f_2 - \kappa^2\tau f_3\right)\overrightarrow{N}\\
&+ \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})\overrightarrow{B} + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)\overrightarrow{B} + \frac{1}{\kappa^2}\lambda\left(-\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B}\right),\end{aligned}$$

while

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\overrightarrow{N}}{\partial s} =& \frac{\partial}{\partial t}\left[(-\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B})\right]\\
=& -\frac{\partial}{\partial t}(\kappa^2)\overrightarrow{T} - \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{N}\\
&- \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B} + \frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})\overrightarrow{N}\\
&- \frac{\kappa^{'}}{\kappa}\left(\kappa^2 f_1 + \kappa^2\frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'}f_2 - \kappa^2\tau f_3\right)\overrightarrow{T} + \frac{\kappa^{'}}{\kappa^3}\lambda\overrightarrow{B}\\
&+ \frac{\partial}{\partial t}(\tau)\overrightarrow{B} - \tau\left(\kappa^2\tau f_2 + \kappa^2\frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'}f_3\right)\overrightarrow{T} - \frac{\tau}{\kappa^2}\lambda\overrightarrow{N}.\end{aligned}$$

Thereby we see that

$$\begin{aligned}\frac{\partial\kappa^2}{\partial t} =& \frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial}(\kappa^2\frac{\partial}{\partial}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'}f_2) - \frac{\partial}{\partial s}(\kappa^2\tau f_3) - \kappa\kappa^{'}f_1\\
&- \kappa\kappa^{'}\frac{\partial}{\partial s}(f_2) - \kappa^{'^2}f_2 + \kappa\kappa^{'}\tau f_3 - \kappa^2\tau^2 f_2 - \kappa^2\tau\frac{\partial}{\partial s}(f_3) - \kappa\kappa^{'}\tau f_3\end{aligned}$$

and

$$\frac{\partial\tau}{\partial t} = \kappa^2\tau f_2 + \kappa^2\frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'}f_3 + \lambda\frac{\partial}{\partial s}\left(\frac{1}{\kappa^2}\right) + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda).$$

From the relation $\frac{\partial}{\partial s}\frac{\partial\overrightarrow{B}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial\overrightarrow{B}}{\partial s}$ no additional new formulas are obtained. □

4. Conclusions

Our results are a generalization of the inflexible curve flow formulas obtained using the known Frenet framework. If we take $\kappa = 1$ in modified orthogonal frame, the properties of the curves given relative to known Frenet frame are obtained.

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