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THE SCHWARZ LEMMA AT THE BOUNDARY FOR THE INTERSECTION OF TWO BALLS

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Abstract. Schwarz lemma at the non-smooth boundary point for holomorphic self-map on the intersection of two balls in $\mathbb{C}^2$ is obtained. At the complex tangent point in the corner of the boundary of the domain, the tangential eigenvalue of the complex Jacobian of the holomorphic map is estimated if the map is transversal.

1. Introduction

Classic Schwarz lemma says that if any holomorphic self-map f on the unit disc Δ fixes the origin, then $|f'(0)| \leq 1$. Furthermore f is a simply rotation around the origin if and only if $|f'(0)| = 1$. What if we focus on the boundary points instead of the interior points? What can we say about the derivative of f at its fixed boundary point of the unit disc if f is also holomorphic at the boundary point? It was G. Julia who first considered and answered this question.

Theorem 1.1 (Julia-Carathéodory, [3], [5]). *Let $f : \Delta \to \Delta$ be a holomorphic map. If f is holomorphic at $z = 1$ with $f(1) = 1$ and $f(0) = \alpha$, then*

$$f'(1) \geq \frac{|1-\alpha|^2}{1-|\alpha|^2}.$$

Moreover, the equality holds if and only if $f(z) \equiv \varphi_\alpha(e^{i\theta}z)$ where φ_α is a Möbius transform sending α to 0 and $e^{i\theta} = \varphi_\alpha(1)$.

Recently, several extensions of this type of boundary Schwarz lemma have been accomplished. If we list some of them Schwarz lemma at the boundary was proved to hold for the unit balls [8], the complex ellipsoids [10], the strongly pseudoconvex domains [2], [6], the convex domain of finite type [13], and the bounded symmetric domains [12]. All known results assume that boundary points are smooth. Few results are known for non-smooth boundary points (see [7]). As explained in [11] if the domain D is the universal covering of

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a compact complex manifold or is homogeneous with respect to $Aut(D)$ then D must have some non-smooth boundary points. Thus it is natural to ask if the boundary Schwarz lemma holds at non-smooth boundary points of such domains.

This article presents Schwarz lemma at the boundary for the intersection of two complex balls in $\mathbb{C}^2$. Such a domain has been particular interest in various complex analysis problems ([1], [4]). It is also a simplest explicit model for piecewise smooth convex domains. In order to see features of main theorems of this paper, it can be helpful to see Schwarz lemma at the boundary for the unit ball B^n $(n \geq 2)$ case which reveals typical issues for extension to multi-dimensional domains. Denote the complex Jacobian of a mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ at the point p by $J_f(p)$.

Theorem 1.2 ([8]). *Let $f : B^n \to B^n$ be a holomorphic map satisfying $f(z_0) = w_0$ for $z_0, w_0 \in \partial B^n$. Assume that f is also holomorphic at z_0. Then the followings hold*

1. *There exists a nonzero real number λ such that $\overline{J_f(z_0)}^T {w_0}^T = \lambda {z_0}^T$ and $\lambda = \overline{w_0} J_f(z_0) {z_0}^T \geq \frac{|1-\langle w_0, \overline{f(0)}\rangle|^2}{1-\|f(0)\|^2} > 0$.*
2. *Suppose that $z_0 = w_0$. Let $\mu_2, \ldots, \mu_n$ be eigenvalues of $J_f(z_0)$ associated to holomorphic tangent direction. Then $|\mu_j| \leq \sqrt{\lambda}$ for $j = 2, \ldots, n$.*

Boundary Schwarz lemma is about holomorphic directional derivative of f at the boundary point. Normal direction and tangential directions are well defined at smooth boundary point. On the other hand, the intersection of two balls has non-smooth boundary points where the tangent space can not be defined. Instead of considering normal vector to the tangent space at the boundary point, we consider the vectors belonging to a certain normal cone at non-smooth boundary points. Genericallay, non-smooth boundary points of the intersection of two balls do not allow holomorphic tangent vectors, but there are two exceptional boundary points which allow them. There we can obtain a result similar to part (ii), Theorem 1.2. We present the main theorems and their proofs on the next section.

2. Main theorems and proofs

Let $B_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2 - a|^2 < 1\}$ and $B_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2 + a|^2 < 1\}$ where $a \in \mathbb{R}$ and $0 < a < 1$. Let $\Omega = B_1 \cap B_2$. Then $S = \partial B_1 \cap \partial B_2$ is the set of non-smooth boundary points of Ω. It is a real 2-dimensional sphere and is called the corner of the boundary of Ω. By solving the equations $|z_1|^2 + |z_2 - a|^2 = 1$, $|z_1|^2 + |z_2 + a|^2 = 1$,

$$S = \{(z, it) : z \in \mathbb{C}, t \in \mathbb{R}, |z|^2 + t^2 = 1 - a^2\}.$$

Define a directional cone $C_p(\Omega)$ at a point $p \in S$ by

$$C_p(\Omega) = \{w \in \mathbb{C}^n : \|w\| = 1, \exists r > 0 \text{ s.t. } p - rw \in \Omega\}.$$

For $w \in C_p$, the range of angle between $-w$ and the inward normal vector $(-z, -it)$ is determined by real inner product of $(0, a) - (z, it)$ with $(-z, -it)$ and $(0, -a) - (z, it)$ with $(-z, -it)$. If $w \in C_p$, then the angle between $-w$ and $(-z, -it)$ is less than $\arccos \sqrt{1 - a^2}$.

Define complex inner product by $\langle z, w \rangle = \sum_{j=1}^n z_j w_j$ for $z, w \in \mathbb{C}^n$. Any point z in C^2 is understood as row vector $z = (z_1, z_2)$. Whenever we need to express it as column vector we take $z^T = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Theorem 2.1. *Let $f : \Omega \to \Omega$ be a holomorphic mapping. Assume that f is holomorphic at $p \in S$ and $f(p) = p$. Then the followings hold.*

1. *For each $w \in C_p(\Omega)$, there exists a nonzero vector $\hat{w} \in \mathbb{C}^2$ satisfying $\hat{w}/\|\hat{w}\| \in C_p(\Omega)$ such that*

$$\overline{(p - (0, a))} J_f(p) \hat{w}^T \geq \frac{|1 - \alpha_+|^2}{1 - |\alpha_+|^2} \tag{1}$$

$$\overline{(p + (0, a))} J_f(p) \hat{w}^T \geq \frac{|1 - \alpha_-|^2}{1 - |\alpha_-|^2} \tag{2}$$

where $\alpha_+ = \langle \overline{p - (0, a)}, f(p - \hat{w}) - (0, a) \rangle$ and $\alpha_- = \langle \overline{p + (0, a)}, f(p - \hat{w}) + (0, a) \rangle$.

2. *For $p = (z, 0) \in S$, it holds that*

$$\overline{(p - (0, a))} J_f(p) p^T \geq \frac{|1 - \beta_+|^2}{1 - |\beta_+|^2} \tag{3}$$

$$\overline{(p + (0, a))} J_f(p) p^T \geq \frac{|1 - \beta_-|^2}{1 - |\beta_-|^2} \tag{4}$$

where $\beta_+ = \langle \overline{p - (0, a)}, f(0) - (0, a) \rangle$, $\beta_- = \langle \overline{p + (0, a)}, f(0) + (0, a) \rangle$.

Proof. For $w \in C_p(\Omega)$, there exists a number $r > 0$ such the map $\varphi(\zeta) = p - rw + r\zeta w$ satisfies $\varphi(\Delta) \subset \Omega$ and $\varphi(1) = p$. It can be done by choosing affine line realizing an analytic disc inside the ball with vertex at p and with its direction satisfying the angle condition defining the cone $C_p(\Omega)$. Take $\hat{w} = rw$. Define holomorphic maps on Δ by

$$g_+(\zeta) = \overline{(f(p) - (0, a))} (f(\varphi(\zeta)) - (0, a))^T,$$

$$g_-(\zeta) = \overline{(f(p) + (0, a))} (f(\varphi(\zeta)) + (0, a))^T.$$

It holds that $|g_\pm(\zeta)| < 1$ for $\zeta \in \Delta$ because

$$|g_\pm(\zeta)| \leq \|f(p) - (0, \pm a)\| \|f(\varphi(\zeta)) - (0, \pm a)\| < 1$$

for $\zeta \in \Delta$. It also holds that $g_\pm(1) = 1$ because

$$g_\pm(1) = \|f(p) - (0, \pm a))\|^2 = \|p - (0, \pm a)\|^2 = 1.$$

By Theorem 1.1,

$$g'_+(1) = \overline{(p-(0,a))}J_f(p)\hat{w}^T \geq \frac{|1-g_+(0)|^2}{1-|g_+(0)|^2}$$

where

$$\begin{aligned} g_+(0) &= (\overline{f(p)}-(0,a))(f(p-\hat{w})-(0,a))^T \\ &= \langle \overline{p-(0,a)}, f(p-\hat{w})-(0,a)\rangle. \end{aligned}$$

In a similar way, we obtain the inequality (2).

Proof of the second statement is as follows. Define

$$\begin{aligned} h_+(\zeta) &= \overline{(f(p)-(0,a))}(f(\zeta p)-(0,a))^T, \\ h_-(\zeta) &= \overline{(f(p)+(0,a))}(f(\zeta p)+(0,a))^T. \end{aligned}$$

Since $|z|^2 = 1-a^2$ and $|\zeta|^2|z|^2+a^2 < 1$ for $|\zeta|<1$, $f(\zeta p)\in\Omega$ for $|\zeta|<1$. Thus from the similar argument of the proof of the first statement, h_+ and h_- are clearly self-maps on unit disc and send 1 to itself. Again by Theorem 1.1, we have

$$h'_+(1) = \overline{(p-(0,a))}J_f(p)p^T \geq \frac{|1-h_+(0)|^2}{1-|h_+(0)|^2}$$

where $h_+(0) = \overline{(p-(0,a))}(f(0)-(0,a))^T$. In a similar way, we obtain the inequality (4). □

For $q\in\Omega$ and X, the Kobayashi pseudo-metric is defined by

$$K_\Omega(q;X) = \inf\{t>0 : \exists\varphi\in \mathrm{Hol}(\Delta,\Omega), \varphi(0)=q, t\varphi'(0)=X\}$$

Theorem 2.2 ([9]). *Let Ω be a $\mathbb{C}$-convex domain containing no complex line through $q\in\Omega$ in the direction of X. Then*

$$\frac{1}{4d_\Omega(q,X)} \leq K_\Omega(q;X) \leq \frac{1}{d_\Omega(q,X)}$$

where $d_\Omega(q,X) = \sup\{r>0 : q+\zeta X\in\Omega \text{ if } |\zeta|<r\}$.

There are two different type of points in S. If $J(T_pS) = T_pS$ then the point $p\in S$ is called complex tangent point (or exceptional point). Otherwise, the point p is called generic boundary point. We can find all the complex tangent points as follows. Let $r_1 = |z_1|^2+|z_2-a|^2-1$ and $r_2 = |z_1|^2+|z_2+a|^2-1$. If $(z_1,z_2)\in S$ is a complex tangent point, then there exists nonzero holomorphic vector $V = V_1\frac{\partial}{\partial z_1}+V_2\frac{\partial}{\partial z_2}$ such that $Vr_1=0$ and $Vr_2=0$. The equations are

$$\begin{aligned} V_1\bar{z}_1 + V_2(\bar{z}_2-a) &= 0, \\ V_1\bar{z}_1 + V_2(\bar{z}_2+a) &= 0. \end{aligned}$$

It is solvable only if $V_2a=0$ and it implies that $V_2=0$. The equation reduces to $V_1\bar{z}_1=0$. It implies that $z_1=0$ because $V_1\neq 0$. On the other hand, any boundary point in S has coordinates (z,it) with $|z|^2+t^2 = 1-a^2$. Thus the complex tangent points in S are only $(0,\pm i\sqrt{1-a^2})$.

Definition 2.3. *Let $f : \Omega \to \Omega$ be a holomorphic map. Suppose that it is also holomorphic at $p \in \partial\Omega$ and $f(p) = p$. The map f is called transversal at p if there exists a nonzero real number λ and non-negative real numbers s_1, s_2, at least one of which is nonzero such that*

$$\overline{J_f(p)}^T(s_1\frac{\partial r_1}{\partial \bar z} + s_2\frac{\partial r_2}{\partial \bar z}) = \lambda(s_1\frac{\partial r_1}{\partial \bar z} + s_2\frac{\partial r_2}{\partial \bar z})$$

Remark 2.4. *For the unit ball case, the normal vector at each boundary point is the boundary point itself. For the general pseudoconvex domains with smooth boundary, the holomorphic gradient of the defining function is taken as normal vector at the boundary point. For our case, the holomorphic map $f : \Omega \to \Omega$ preserves the real tangent cone to $p \in S$. Thus it preserves the real normal cone to p. To estimate horizontal eigenvalue at complex tangent points in S, an extra condition to f needs to be imposed regarding the mapping property inside real normal cone. At smooth boundary point in ∂B_j near the corner S, $\frac{\partial r_j}{\partial \bar z}$ is a transversal eigenvector of $\overline{J_f(p)}^T$. Thus we demand the map f infinitesimally interpolate $\frac{\partial r_j}{\partial \bar z}, j = 1, 2$ on S*

For the notation, $A \lesssim B$ means that there exists a positive constant C which depends on only Ω such that $A \le CB$. We denote holomorphic tangent vector $\partial/\partial z_1$ with Z_1 whenever it is defined on $\overline{\Omega}$.

Theorem 2.5. *Assume that $0 < a < 1/\sqrt{2}$. Let $p \in S$ be a complex tangent point. Suppose that $f : \Omega \to \Omega$ is transversal at p. Then there exists $\mu \in \mathbb{C}$ such that $J_f(p)Z_1 = \mu Z_1$ for $Z_1 \in T_p^{(1,0)}S$ and $|\mu| \lesssim \sqrt{\lambda}$ where λ is an transversal eigenvalue of $\overline{J_f(p)}^T$.*

Proof. It suffices to prove for the complex tangent point $p = (0, ib)$. Since $df_p(T_p^{(1,0)}S) \subset T_p^{(1,0}S$ for complex tangent point p and $\dim_{\mathbb{C}} T_p^{\mathbb{C}}S = 1$, it holds that $df_p(Z_1) = \mu Z_1$ for some number μ. Let $v_p = s_1\frac{\partial r_1}{\partial \bar z} + s_2\frac{\partial r_2}{\partial \bar z}$ be a transversal eigenvector of $\overline{J_f(p)}^T$ associated with λ. Then $v_p = (0, s_1(ib-a)+s_2(ib+a)) \in \mathbb{C}^2$. Let $\beta = s_1(ib - a) + s_2(ib + a)$. Take an inward line $p_t = p - tv_p$, ($0 < t < \epsilon$, ϵ is sufficiently small) inside Ω. Express $Z_1 = (1, 0)$ as a vector in $\mathbb{C}^2$. Take a complex line $\zeta \to p_t + \zeta Z_1 = (\zeta, ib - t\beta)$. We can determine the directional distance $d_\Omega(p_t, Z_1)$ along the inward line by estimating $|\zeta|$ satisfying $r_j(p_t + \zeta Z_1) = 0$ for $j = 1$ or $j = 2$. It is

$$|\zeta|^2 + |ib \pm a - t\beta|^2 = 1.$$

It deduces that

$$|\zeta|^2 = 2t\Re(ib \pm a)\bar\beta - t^2,$$

which implies that

$$d_\Omega(p_t, Z_1) = |\zeta| = \sqrt{2kt}\,(1 + O(t)), \quad t \to 0+ \tag{5}$$

where $k = \Re(ib \pm a)\bar\beta = s_1 + s_2(1 - 2a^2) > 0$.

For $z \in \Omega$ and $X \in \mathbb{C}^2$, if z is near p, $d_{B_j}(z, X) \lesssim (-r_j(z))^{1/2}$, j=1,2 ([9, Proposition 4]). It implies that

$$d_\Omega(z, X)^2 \lesssim s_1(-r_1(z)) + s_2(-r_2(z))$$

for z near p. Since

$$\begin{aligned} r_j(f(p_t)) &= r_j(p - t df_p(v_p) + O(t^2)) \\ &= -2t\Re\langle df_p(v_p), \frac{\partial r_j}{\partial z}\rangle + O(t^2), \qquad t \to 0+ \end{aligned}$$

for $j = 1, 2$, we have

$$\begin{aligned} d_\Omega(f(p_t), Z_1)^2 &\lesssim s_1(-r_1(f(p_t)) + s_2(-r_2(f(p_t)) \\ &= (2t)\Re\langle \overline{df_p(v_p)}, v_p\rangle + O(t^2) \\ &= (2t)\lambda\|v_p\|^2 + O(t^2), \qquad t \to 0+. \end{aligned}$$

Using similar argument in [13, proof pt 2 of Theorem 1.4], we have

$$\begin{aligned} &K_\Omega(f(p_t); df_{p_t}(Z_1)) \\ &= K_\Omega(f(p_t); df_p(Z_1)) - K_\Omega(f(p_t); (df_{p_t} - df_p)(Z_1)) \\ &\gtrsim \frac{|\mu|}{d_\Omega(f(p_t), Z_1)} - \frac{O(t)}{d_\Omega(f(p_t), Z_1)} \\ &\gtrsim \frac{(|\mu| - O(t))}{\sqrt{2t\lambda}\,(\|v_p\|^2 + O(t))^{1/2}}, \qquad t \to 0+ \end{aligned}$$

By decreasing property of Kobayashi metric under holomorphic maps, Theorem 2.2, and the equation (5) we have

$$\begin{aligned} 1 &\geq \frac{K_\Omega(f(p_t); df_{p_t}(v_p))}{K_\Omega(p_t; v_p)} \\ &\gtrsim \frac{|\mu| - O(t)}{\sqrt{\lambda}} \frac{\sqrt{k} + O(t)}{(\|v_p\|^2 + O(t))^{1/2}}, \qquad t \to 0+ \end{aligned}$$

which implies that $|\mu| \lesssim \sqrt{\lambda}$.

□

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