Honam Mathematical J. **45** (2023), No. 4, pp. 629–647
https://doi.org/10.5831/HMJ.2023.45.4.629

SCREEN GENERIC LIGHTLIKE SUBMERSIONS

Gaurav Sharma, Sangeet Kumar*, and Dinesh Kumar Sharma

Abstract. We introduce the study of a new class of a lightlike submersion $\phi : K_1 \to K^{'}$ from a screen generic lightlike submanifold of an indefinite Kaehler manifold K_2 onto an indefinite almost Hermitian manifold $K^{'}$, and show that for this case $K^{'}$ must be an indefinite Kaehler manifold. Then, we derive a relationship between the holomorphic sectional curvatures of K_2 and $K^{'}$. Finally, we present a classification theorem for a screen generic lightlike submersion, giving the relationship between the sectional curvatures of the total space K_2 and the fibers.

1. Introduction

The theory of Riemannian and semi-Riemannian submersions has emerged as one of the most fruitful areas of research in differential geometry, and its contribution to the advancement of the subject has been significant. The geometry of submersions is observed to have a wide range of applications in differential geometry and theoretical physics, including the Kaluza-Klein theory, Yang-Mills theory, supergravity, and superstring theory (for details, see [2], [10] and [13]).

The concept of Riemannian submersions was introduced and developed by O'Neill [15] and Grey [8]. A Riemannian submersion $\phi : K_1 \to K'$ naturally generates two distributions on K_1, referred as the horizontal and vertical distributions, respectively. For a Riemannian submersion, the integrability of vertical distribution is necessary, giving rise to the fibres of the submersion, which are closed submanifolds of K_1. Then Kobayashi [11] observed that for a CR-submanifold of a Kaehler manifold, the totally real distribution is always integrable. Kobayashi noted this similarity between the total space of a Riemannian submersion and a CR-submanifold, and defined the notion of a CR-submersion.

On the other hand, Sahin [17] introduced a new kind of submersion, specifically, a lightlike submersion defined from a semi-Riemannian manifold onto an

Received March 2, 2023. Revised April 17, 2023. Accepted May 10, 2023.
2020 Mathematics Subject Classification. 53A05.
Key words and phrases. lightlike submersion; indefinite Kaehler manifold, holomorphic bisectional curvature.
*Corresponding author

r-lightlike manifold. To address the comparable situation for a screen generic lightlike submanifold of an indefinite Kaehler manifold, we have used the same approach and introduced a new class of a lightlike submersion, which is called a screen generic lightlike submersion. As we know in case of a screen generic lightlike submanifold, the radical distribution $Rad(TK_1) = S(TK_1) \cap S(TK_1^{\perp})$ is invariant and there exists a sub-bundle D_0 of $S(TK_1)$ such that $D_0 = S(TK_1) \cap \bar{J}S(TK_1)$. In this way, we find a distribution $D = D_0 \perp Rad(TK_1)$, which is invariant in $S(TK_1)$. Consequently, there exists a complementary distribution D' of $S(TK_1)$ such that $S(TK_1) = D \oplus D'$.

One challenge to define a screen generic lightlike submersion $\phi : K_1 \to K^{'}$, where K_1 is a screen generic lightlike submanifold of an indefinite Kaehler manifold K_2 and K' is an indefinite almost Hermitian manifold, is that in this case the distribution D' may not be integrable to satisfy the condition of a submersion. To overcome this challenge, we presume that the distribution D' is integrable. Literature suggests the study of lightlike submersions has many applications across a variety of fields, and a very limited number of reports are available on this subject. This motivated us to introduce and investigate the concept of screen generic lightlike submersions.

This paper is organised as follows: In Section 2, we recall the basic theory of a lightlike submanifold given by Duggal et. al. [4]. In Section 3, after defining a screen generic lightlike submanifold, we review some basic theorems on integrability of distributions D and D'. In Section 4, a screen generic lightlike submersion ϕ is defined from a screen generic lightlike submanifold K_1 of an indefinite Kaehler manifold K_2 onto an indefinite almost Hermitian manifold $K^{'}$. Furthermore, we prove that if an indefinite almost Hermitian manifold $K^{'}$ admits a lightlike submersion $\phi : K_1 \to K^{'}$ of a screen generic lightlike submanifold K_1 of an indefinite Kaehler manifold K_2 then $K^{'}$ must be an indefinite Kaehler manifold. Also the relation between the holomorphic sectional curvature of K_2 and that of $K^{'}$ is established.

2. Preliminaries

2.1. Lightlike Submanifolds

Let (K_1^n, g_1) be an isometrically immersed submanifold of a semi-Riemannian manifold (K_2^{m+n}, g_2) of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$. The metric g_1 is the induced metric of g_2 on K_1. K_1 is called a lightlike submanifold of K_2 if the metric g_1 becomes degenerate on the tangent bundle TK_1 of K_1. Locally, a lightlike vector field $\zeta \in \Gamma(TK_1), \zeta \neq \{0\}$ exists so that $g_1(\zeta, Y_2) = 0$ for every $Y_2 \in \Gamma(TK_1)$. Then, for each tangent space T_yK_1, we have

$$T_yK_1^{\perp} = \cup\{u \in T_yK_2 : g_2(u, v) = 0, \ \forall\ v \in T_yK_1, y \in K_1\},$$

where T_yK_1 is an n-dimensional degenerate subspace of T_yK_2. As a result, even though the subspaces TyK_1 and $TyK_1^{\perp}$ are no longer complimentary, i.e. $T_yK_1 \cap T_yK_1^{\perp} \neq 0$, they are both degenerate and orthogonal. In this case, there exists a subspace $Rad(T_yK_1) = T_yK_1 \cap T_yK_1^{\perp}$, named as the radical subspace defined as:

$$Rad(T_yK_1) = \{\zeta_y \in T_yK_1 : g_1(\zeta_y, Y_2) = 0, \quad \forall\, Y_2 \in T_yK_1\}.$$

$Rad(TK_1)$ is known as the radical distribution on K_1 and K_1 is referred as an r-lightlike submanifold of K_2, if the mapping

$$Rad(TK_1) : y \in K_1 \longrightarrow Rad(T_yK_1),$$

defines a smooth distribution on K_1 of rank $r > 0$. For an r-lightlike submanifold K_1, we find $S(TK_1)$ is a complementary orthogonal vector subbundle to $Rad(TK_1)$ in TK_1, which is a non-degenerate screen distribution. Thus, we can write

$$TK_1 = Rad(TK_1) \perp S(TK_1). \tag{1}$$

Since $S(TK_1)$ is canonically isomorphic to the vector bundle $TK_1/Rad(TK_1)$, however, it is not unique. Let us use the notation

$$(K_1, g, S(TK_1), S(TK_1^{\perp}))$$

to represent a r-lightlike submanifold, where $S(TK_1^{\perp})$ is a complementary vector subbundle to $Rad(TK_1)$ in $TK_1^{\perp}$.

Theorem 2.1. [4] *For an r-lightlike submanifold*

$$(K_1, g, S(TK_1), S(TK_1^{\perp}))$$

of a semi-Riemannian manifold (K_2, g_2), there exists a complementary vector bundle $ltr(TK_1)$ of $Rad(TK_1)$ in $S(TK_1^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TK_1) \mid_u)$ consisting of smooth sections $\{N_i\}$ of $S(TK_1^{\perp})^{\perp} \mid_u$, where u is a coordinate neighborhood of K_1 such that

$$g_2(N_i, N_j) = 0, \quad g_2(N_i, \zeta_j) = \delta_{ij}, \text{ for } \quad i, j \in \{1, 2, .., r\},$$

where $\{\zeta_1, \ldots, \zeta_r\}$ is lightlike basis of $\Gamma(Rad(TK_1))$.

It follows that there exists a lightlike transversal vector bundle $ltr(TK_1)$ locally spanned by $\{N_i\}$. Let $tr(TK_1)$ and $ltr(TK_1)$ be the vector bundles complementary (but not orthogonal) to TK_1 in $TK_2|_{K_1}$ and to $Rad(TK_1)$ in $S(TK_1^{\perp})$, respectively. Then we have

$$tr(TK_1) = ltr(TK_1) \perp S(TK_1^{\perp}), \tag{2}$$

$$\begin{aligned} TK_2 \mid_{K_1} &= TK_1 \oplus tr(TK_1) \\ &= (Rad(TK_1) \oplus ltr(TK_1)) \perp S(TK_1) \perp S(TK_1^{\perp}). \end{aligned} \tag{3}$$

Let

$$(K_1, g, S(TK_1), S(TK_1^{\perp}))$$

be an r-lightlike submanifold of a semi-Riemannian manifold (K_2, g_2). Consider the Levi-Civita connection defined on K_2 as $\bar{\nabla}$. The Gauss and Weingarten formulae are then derived by using the decomposition Eq. (2) as

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h(Y_1, Y_2),$$

$$\bar{\nabla}_{Y_1} V = -A_V Y_1 + \nabla^t_{Y_1} V,$$

where $\{h(Y_1, Y_2), \nabla^t_{Y_1} V\}$ and $\{\nabla_{Y_1} Y_2, A_V Y_1\}$ belong to $\Gamma(tr(TK_1))$ and $\Gamma(TK_1)$, respectively. Here, h is a symmetric bilinear second fundamental form on $\Gamma(TK_1)$ and A_V is linear shape operator on K_1. In view of Eq. (3), the Gauss and Weingarten formulae become

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{4}$$

$$\bar{\nabla}_{Y_1} N = -A_N Y_1 + \nabla^l_{Y_1} N + D^s(Y_1, N), \tag{5}$$

$$\bar{\nabla}_{Y_1} V = -A_V Y_1 + D^l(Y_1, V) + \nabla^s_{Y_1} V, \tag{6}$$

where $Y_1, Y_2 \in \Gamma(TK_1), N \in \Gamma(ltr(TK_1))$ and $V \in \Gamma(S(TK_1^{\perp}))$. Further by employing Eqs. (4)-(6), we derive

$$g_1(A_V Y_1, Y_2) = g_2(h^s(Y_1, Y_2), V) + g_2(Y_2, D^l(Y_1, V)), \tag{7}$$

$$g_2(D^s(Y_1, N), V) = g_2(A_V Y_1, N).$$

If P is considered to be the projection morphism of TK_1 on $S(TK_1)$, then on the screen distribution $S(TK_1)$ of K_1, we can introduce few new geometric objects. Therefore as a result of employing Eq. (2), we have

$$\nabla_{Y_1} PY_2 = \nabla^*_{Y_1} PY_2 + h^*(Y_1, Y_2), \quad \nabla_{Y_1} \zeta = -A^*_{\zeta} Y_1 + \nabla^{*t}_{Y_1} \zeta,$$

for any $Y_1, Y_2 \in \Gamma(TK_1)$ and $\zeta \in \Gamma(Rad(TK_1))$, where $\{\nabla^*_{Y_1} PY_2, A^*_{\zeta} Y_1\}$ and $\{h^*(Y_1, Y_2),\ \nabla^{*t}_{Y_1} \zeta\}$ belong to $\Gamma(S(TK_1))$ and $\Gamma(Rad(TK_1))$, respectively. Using Eqs. (4), (5) and (6), we obtain

$$g_2(h^l(Y_1, PY_2), \zeta) = g_1(A^*_{\zeta} Y_1, PY_2), g_2(h^*(Y_1, PY_2), N) = g_1(A_N Y_1, PY_2),$$

for any $Y_1, Y_2 \in \Gamma(TK_1), \zeta \in \Gamma(Rad(TK_1))$ and $N \in \Gamma(ltr(TK_1))$.
Next, consider $\bar{\nabla}$ is a metric connection and using Eqs. (4) – (6), for $Y_1, Y_2, X \in \Gamma(TK_1)$ and $V_1, V_2 \in \Gamma(tr(TK_1))$, we obtain

$$(\nabla_{Y_1} g_2)(Y_2, X) = g_2(h^l(Y_1, Y_2), X) + g_2(h^l(Y_1, X), Y_2)$$

and

$$(\nabla^t_{Y_1} g_2)(V_1, V_2) = -g_2(A_{V_1} Y_1, V_2) - g_2(A_{V_2} Y_1, V_1),$$

which implies that the transversal linear connection ∇^t on $tr(TK_1)$ and the induced linear connection ∇ on K_1 are generally not the metric connections.

Let $\bar{R}$ and R denote the curvature tensors of $\bar{\nabla}$ and ∇, respectively. Then we have

$$\begin{aligned}\bar{R}(Y_1,Y_2)X &= R(Y_1,Y_2)X + A_{h^l(Y_1,X)}Y_2 - A_{h^l(Y_2,X)}Y_1 + A_{h^s(Y_1,X)}Y_2\\ &\quad + A_{h^s(Y_2,X)}Y_1 + (\nabla_{Y_1}h^l)(Y_2,X) - (\nabla_{Y_2}h^l)(Y_1,X)\\ &\quad + D^l(Y_1,h^s(Y_2,X)) - D^l(Y_2,h^s(Y_1,X))\\ &\quad + (\nabla_{Y_1}h^s)(Y_2,X) - (\nabla_{Y_2}h^s)(Y_1,X) + D^s(Y_1,h^l(Y_2,X))\\ &\quad - D^s(Y_2,h^l(Y_1,X)),\end{aligned} \tag{8}$$

where

$$(\nabla_{Y_1}h^s)(Y_2,X) = \nabla^s_{Y_1}h^s(Y_2,X) - h^s(\nabla_{Y_1}Y_2,X) - h^s(Y_2,\nabla_{Y_1}X),$$

$$(\nabla_{Y_1}h^l)(Y_2,X) = \nabla^l_{Y_1}h^l(Y_2,X) - h^l(\nabla_{Y_1}Y_2,X) - h^l(Y_2,\nabla_{Y_1}X),$$

for $Y_1, Y_2, X \in \Gamma(TK_1)$.

2.2. Indefinite Kaehler Manifold

Definition 2.2. [1] *Let K_2 be an indefinite almost Hermitian manifold, $\bar{J}$ be an almost complex structure of the type (1, 1) with Hermitian metric g_2 such that for $Y_1, Y_2 \in \Gamma(TK_2)$, we have*

$$\bar{J}^2 = -I, \quad g_2(\bar{J}Y_1, \bar{J}Y_2) = g_2(Y_1, Y_2). \tag{9}$$

If $\bar{\nabla}$ is considered to be a Levi-Civita connection of K_2 with respect to g_2, then the covariant derivative of $\bar{J}$ is defined by

$$(\bar{\nabla}_{Y_1}\bar{J})Y_2 = \bar{\nabla}_{Y_1}\bar{J}Y_2 - \bar{J}\bar{\nabla}_{Y_1}Y_2, \tag{10}$$

for any $Y_1, Y_2 \in \Gamma(TK_2)$. Then K_2 is called an indefinite Kaehler manifold, if

$$(\bar{\nabla}_{Y_1}\bar{J})Y_2 = 0, \tag{11}$$

for each $Y_1, Y_2 \in \Gamma(TK_2)$.

3. Screen Generic Lightlike Submanifold

Definition 3.1. [3] *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a real r-lightlike submanifold of K_2. Then we say that K_1 is a screen lightlike submanifold of K_2 if the following conditions are satisfied:*

(A) *$Rad(TK_1)$ is invariant with respect to $\bar{J}$, that is,*

$$\bar{J}(Rad(TK_1)) = Rad(TK_1).$$

(B) *There exists a subbundle D_0 of $S(TK_1)$ such that*

$$D_0 = \bar{J}(S(TK_1)) \cap S(TK_1),$$

where D_0 is a non-degenerate distribution on K_1.

We observe that there exists a complementary non-degenerate distribution D' in $S(TK_1)$ such that

$$S(TK_1) = D_0 \oplus D',$$

where

$$\bar{J}(D') \nsubseteq S(TK_1) \quad and \quad \bar{J}(D') \nsubseteq S(TK_1^{\perp}).$$

Let P_0, P_1, and Q be the projection morphism on D_0, $Rad(TK_1)$ and D', respectively. Then for each $Y_1 \in \Gamma(TK_1)$, we have

$$\begin{aligned} Y_1 &= P_0 Y_1 + P_1 Y_1 + Q Y_1 \\ &= P Y_1 + Q Y_1, \end{aligned} \tag{12}$$

where $D = D_0 \perp Rad(TK_1)$, D is invariant and $PY_1 \in \Gamma(D)$, $QY_1 \in \Gamma(D')$. From Eq. (12), we have

$$\bar{J}Y_1 = fY_1 + \omega Y_1,$$

where fY_1 and ωY_1 are the tangential and transversal component of $\bar{J}Y_1$ respectively. Moreover it is obvious that $\bar{J}(D') \neq D'$. Whereas, for vector field $Y_2 \in \Gamma(D')$, we have

$$\bar{J}Y_2 = fY_2 + \omega Y_2,$$

such that $fY_2 \in \Gamma(D')$ and $\omega Y_2 \in \Gamma(S(TK_1^{\perp}))$. In the same way for $V_1 \in \Gamma(tr(TK_1))$, we have

$$\bar{J}V_1 = EV_1 + FV_1, \tag{13}$$

where EV_1 is the tangential part and FV_1 is the transversal part of $\bar{J}V_1$, respectively. Next, we recall the conditions for the integrability of the distributions D_0, D and D' associated with a screen generic lightlike submanifold.

Theorem 3.2. [3] *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a real r-lightlike submanifold of K_2. Then the distribution D_0 is integrable if and only if the following conditions hold:*

$$g_1(\nabla^*_{Y_1}\bar{J}Y_2 - \nabla^*_{Y_2}\bar{J}Y_1, fZ) = g_1(E(h^s(Y_1, \bar{J}Y_2) - h^s(Y_2, \bar{J}Y_1)), Z) \tag{14}$$

and

$$h^*(Y_1, \bar{J}Y_2) = h^*(Y_2, \bar{J}Y_1),$$

for each $Y_1, Y_2 \in \Gamma(D_0)$ and $Z \in \Gamma(D')$. Further, the distribution D is integrable if and only if Eq. (14) holds for each $Y_1, Y_2 \in \Gamma(D), Z \in \Gamma(D')$.

Theorem 3.3. [3] *Let K_2 be an indefinite Kaehler manifold and let K_1 be a real r-lightlike submanifold of K_2. Then the distribution D' is integrable if and only if*

$$\nabla_Z fV - \nabla_V fZ - A_{\omega V}Z + A_{\omega Z}V \in \Gamma(D'), \tag{15}$$

for each $Z, V \in \Gamma(D')$.

Definition 3.4. [5] *A lightlike submanifold* (K_1, g) *of a semi-Riemannian manifold* (K_2, g_2) *is totally umbilical in* K_2 *if there is a smooth transversal vector field* $H \in \Gamma(ltr(TK_1))$ *on* K_1*, called the transversal curvature vector field of* K_1*, such that, for each* $Y_1, Y_2 \in \Gamma(TK_1)$*,*

$$h(Y_1, Y_2) = Hg_1(Y_1, Y_2). \tag{16}$$

It is clear from the Gauss and Weingarten formulae (4) of K_2 that K_1 is totally umbilical if and only if there are smooth vector fields $H^l \in \Gamma(ltr(TK_1))$ and $H^s \in \Gamma(S(TK_1^{\perp}))$ in each coordinate neighbourhood U, such that

$$h^l(Y_1, Y_2) = H^l g_1(Y_1, Y_2), D^l(Y_1, W) = 0, h^s(Y_1, Y_2) = H^s g_1(Y_1, Y_2),$$

for each $Y_1, Y_2 \in \Gamma(TK_1)$ and $W \in \Gamma(S(TK_1^{\perp}))$.

Theorem 3.5. *Suppose that* K_2 *is an indefinite Kaehler manifold and* K_1 *is a totally umbilical screen generic lightlike submanifold of* K_2*. Then* $H^l = 0$*.*

Proof. For $Y_1, Y_2 \in \Gamma(D_0)$, using Eq. (11) along with the hypothesis and then considering the lightlike transversal components, we get

$$H^l g_1(Y_1, JY_2) = CH^l g_1(Y_1, Y_2).$$

Taking $Y_1 = JY_2$ and in view of the non-degeneracy of D_0, above equation yields that $H^l = 0$. □

4. Screen Generic Lightlike Submersion

Definition 4.1. *Assume that* K_2 *is an indefinite Kaehler manifold and* (K_1, g_1, D) *is a screen generic lightlike submanifold of* K_2 *such that* D' *is integrable and* $(K^{'}, g_2)$ *is an indefinite almost Hermitian manifold. Then a smooth mapping* $\phi : (K_1, g_1, D) \to (K^{'}, g_2)$ *is called a lightlike submersion if*

(a) *at each* $p \in K_1, \mathcal{V}_p = ker(\phi_*)_p = D'$*,*
(b) *at every* $p \in K_1$*, the differential operator* ϕ_* *restricts to an isometry of the horizontal space* $\mathcal{H}_p = D_p$ *onto* $T_{\phi(p)}K^{'}$*, that is*

$$g_1(Y_1, Y_2) = g_2(\phi_*(Y_1), \phi_*(Y_2))$$

for any given vector fields $Y_1, Y_2 \in \Gamma(D)$*.*

The restriction of ϕ_{*p} to $\mathcal{H}_p = S(TK_1)_p$ maps the space isomorphically onto $T_{\phi(p)}K^{'}$, as the definition implies. Then for any vector $\tilde{Y_1} \in T_{\phi(p)}K^{'}$, we note that the vector $Y_1 \in S(TK_1)_p$ is a horizontal lift of $\tilde{Y_1}$. On the other hand, if $\tilde{Y_1}$ is a vector field on an open set U of $K^{'}$ then the horizontal lift $\tilde{Y_1}$ is the vector field $Y_1 \in \Gamma(S(TK_1))$ on $\phi^{-1}(U)$ such that $\phi_*(Y_1) = \tilde{Y}_1 o\phi$ and is called the basic vector field.

Lemma 4.2. *Consider a screen generic lightlike submersion $\phi : K_1 \to K^{'}$ defined from a screen generic lightlike submanifold of an indefinite Kaehler manifold K_2 onto an indefinite almost hermitian manifold $K^{'}$. If Y_1 and Y_2 are basic vector field ϕ-related to $\tilde{Y_1}, \tilde{Y_2}$, respectively, then*

(i) $g_1(Y_1, Y_2) = g_2(\tilde{Y_1}, \tilde{Y_2}) o\phi$.

(ii) $[Y_1, Y_2]^{\mathcal{H}}$ *is the basic vector field and ϕ-related to* $[\tilde{Y_1}, \tilde{Y_2}]$.

(iii) $(\nabla^{K_1}_{Y_1} Y_2)^{\mathcal{H}}$ *is a basic vector field and ϕ-related to* $(\tilde{\nabla}^{K'}_{\tilde{Y_1}} \tilde{Y_2})$.

(iv) *For any vertical vector field V, $[Y_1, V]$ is vertical.*

Proof. If Y_1 and Y_2 are the basic vector fields of K_1, then (i) obviously follows from part (b) of the definition (4.1). Given that P and Q are projections from TK_1 on the distribution of a screen generic lightlike submanifold of an indefinite Kaehler manifold, D and D', respectively, then $[Y_1, Y_2] = P[Y_1, Y_2] + Q[Y_1, Y_2]$. Since, the horizontal component of $[Y_1, Y_2]$ which is $P[Y_1, Y_2]$ is a basic vector field and correspond to $[\tilde{Y_1}, \tilde{Y_2}]$, that is

$$\phi_*(P[Y_1, Y_2]) = [\phi_*(Y_1), \phi_*(Y_2)],$$

Then from Koszul's formula, we have

$$\begin{aligned} 2g_1(\nabla_{Y_1} Y_2, Z) &= Y_1(g_1(Y_2, Z)) + Y_2(g_1(Z, Y_1)) - Z(g_1(Y_1, Y_2)) \\ &\quad -g_1(Y_1, [Y_2, Z]) + g_1(Y_2, [Z, Y_1]) + g_1(Z, [Y_1, Y_2]). \end{aligned} \tag{17}$$

for any $Y_1, Y_2, Z \in \Gamma(D)$.

Let Y_1, Y_2, and Z be the horizontal lifts of the vector fields $\tilde{Y_1}$, $\tilde{Y_2}$, and $\tilde{Z}$, respectively. Then $Y_1(g_1(Y_2, Z) = \tilde{Y_1}(g_2(\tilde{Y_2}, \tilde{Z}) o\phi$ and

$$g_1(Z, [Y_1, Y_2]) = g_2(\tilde{Z}, [\tilde{Y_1}, \tilde{Y_2}]) o\phi.$$

Thus from Eq. (17) we obtain

$$\begin{aligned} 2g_1(\nabla^{K_1}_{Y_1} Y_2, Z) &= \tilde{Y_1}(g_2(\tilde{Y_2}, \tilde{Z}) o\phi + \tilde{Y_2}(g_2(\tilde{Z}, \tilde{Y_1}) o\phi - \tilde{Z}(g_2(\tilde{Y_1}, \tilde{Y_2}) \\ &\quad o\phi - g_2(\tilde{Y_1}, [\tilde{Y_2}, \tilde{Z}]) o\phi + g_2(\tilde{Y_2}, [\tilde{Z}, \tilde{Y_1}]) o\phi \\ &\quad + g_2(\tilde{Z}, [\tilde{Y_1}, \tilde{Y_2}]) o\phi \\ &= 2g_2(\nabla^{K^{'}}_{\tilde{Y_1}} \tilde{Y_2}, \tilde{Z}). \end{aligned} \tag{18}$$

Given that $\tilde{Z}$ is an arbitrary vector field and ϕ is surjective, therefore condition (iii) follows from Eq.(18). Next, let V $\in \Gamma(D')$ then $[Y_1, V]$ is ϕ related to $[\tilde{Y_1}, 0]$, which proves (iv) and this concludes the proof. □

Let $\nabla^{K^{'}}$ be the covariant differentiation on $K^{'}$. Then we define corresponding operator $\tilde{\nabla}^{K^{'}}$ by assuming

$$\tilde{\nabla}^{K^{'}}_{Y_1} Y_2 = (\tilde{\nabla}^{K_1}_{Y_1} Y_2)^{\mathcal{H}}$$

for any basic vector field Y_1 and Y_2. Thus from (iii) of Lemma (4.2), $\tilde{\nabla}^{K^{'}}_{Y_1} Y_2$ is a basic vector field and $\phi_*(\nabla^{K^{'}}_{Y_1} Y_2) = \phi_*(\tilde{\nabla}^{K^{'}}_{Y_1} Y_2) = \tilde{\nabla}^{K^{'}}_{\tilde{Y_1}} \tilde{Y_2}$. Thus we have a tensor field C, using Eq. (12) as

$$\nabla^{K_1}_{Y_1} Y_2 = \tilde{\nabla}^{K'}_{Y_1} Y_2 + C(Y_1, Y_2) \tag{19}$$

for any $Y_1, Y_2 \in \Gamma(D)$, where $C(Y_1, Y_2)$ denote the vertical part of $\nabla^{K_1}_{Y_1} Y_2$. It is easy to check that C is a billinear map from $D \times D \to D'$.

Lemma 4.3. *The tensor field C is skew-symmetric and satisfies*

$$C(Y_1, Y_2) = \frac{1}{2}\mathcal{V}[Y_1, Y_2].$$

Proof. Let $Z \in \Gamma(D')$ be any vertical vector field. Then, for any $Y_1 \in \Gamma(D)$ consider $(\bar{\nabla}_Z g_2)(Y_1, Y_1) = 0$, which further implies

$$\begin{aligned} 0 &= Zg_1(Y_1, Y_1) = 2g_2(\bar{\nabla}_Z X, Y_1) \\ &= 2g_1(\nabla^{K_1}_Z Y_1, Y_1) = 2g_1(\nabla^{K_1}_{Y_1} Z - [Y_1, Z], Y_1) \\ &= 2g_1(\nabla^{K_1}_{Y_1} Z, Y_1) = -2g_1(Z, \nabla^{K_1}_{Y_1} Y_1) = -2g_1(Z, \tilde{\nabla}^{K'}_{Y_1} Y_1 + C(Y_1, Y_1)) \\ &= -2g_1(Z, C(Y_1, Y_1)), \end{aligned}$$

Since D' is non-degenerate distribution, $g_1(Z, C(Y_1, Y_1)) = 0$ if and only if $C(Y_1, Y_1) = 0$, that is, if and only if, C is skew-symmetric. Also for $Y_1, Y_2 \in \Gamma(D)$, we have

$$\begin{aligned} [Y_1, Y_2] &= \nabla^{K_1}_{Y_1} Y_2 - \nabla^{K_1}_{Y_2} Y_1 = (\tilde{\nabla}^{K'}_{Y_1} Y_2 - \tilde{\nabla}^{K'}_{Y} Y_1) + C(Y_1, Y_2) \\ &\quad -C(Y_2, Y_1) \\ &= (\tilde{\nabla}^{K'}_{Y_1} Y_2 - \tilde{\nabla}^{K'}_{Y} Y_1) + 2C(Y_1, Y_2). \end{aligned}$$

On comparing the vertical components on both sides, we get

$$C(Y_1, Y_2) = \frac{1}{2}\mathcal{V}[Y_1, Y_2]. \tag{20}$$

□

Next we define a new tensor field T as

$$\nabla^{K_1}_{Y_1} Z = T_{Y_1} Z + (\nabla^{K_1}_{Y_1} Z)^{\mathcal{V}}, \tag{21}$$

for any $Y_1 \in \Gamma(D)$ and $Z \in \Gamma(D')$. Clearly, T is a bilinear map defined from $D \times D' \to D$. Since $[Y_1, Z] = \nabla^{K_1}_{Y_1} Z - \nabla^{K_1}_Z Y_1$ is vertical, therefore we have

$$\mathcal{H}(\nabla^{K_1}_{Y_1} Z) = \mathcal{H}(\nabla^{K_1}_Z Y_1) = T_{Y_1} Z. \tag{22}$$

Lemma 4.4. *For each $Y_1, Y_2 \in \Gamma(D)$ and $Z \in \Gamma(D')$, we have*

$$g_1(T_{Y_1} Z, Y_2) = -g_1(Z, C(Y_1, Y_2)). \tag{23}$$

Proof. For each $Y_1, Y_2 \in \Gamma(D), Z \in \Gamma(D')$ and using Eqs.(21) and (22), we have

$$\begin{aligned} g_1(T_{Y_1}Z, Y_2) &= g_1(\nabla^{K_1}_{Y_1}Z, Y_2) = g_1(\bar{\nabla}_{Y_1}Z, Y_2) = -g_1(Z, \bar{\nabla}_{Y_1}Y_2) \\ &= -g_1(Z, \nabla_{Y_1}Y_2) = -g_1(Z, \tilde{\nabla}^{K'}_{Y_1}Y_2 + C(Y_1, Y_2)) \\ &= -g_1(Z, C(Y_1, Y_2)). \end{aligned}$$

Thus, the result follows. □

Theorem 4.5. *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a screen generic lightlike submanifold of K_2. If a screen generic lightlike submersion $\phi : K_1 \to K^{'}$ is defined from K_1 onto an indefinite almost Hermitian manifold $K^{'}$ such that D' is integrable, then $K^{'}$ is necessarily an indefinite Kaehler manifold.*

Proof. Let $Y_1, Y_2 \in \Gamma(D)$ be basic vector fields. Then from Eqs. (4) and (19), we have

$$\bar{\nabla}_{Y_1}Y_2 = \tilde{\nabla}^{K'}_{Y_1}Y_2 + C(Y_1, Y_2) + h^l(Y_1, Y_2) + h^s(Y_1, Y_2). \tag{24}$$

Applying $\bar{J}$ in Eq. (24), we get

$$\begin{aligned} \bar{J}\bar{\nabla}_{Y_1}Y_2 &= \bar{J}\tilde{\nabla}^{K'}_{Y_1}Y_2 + \bar{J}C(Y_1, Y_2) + \bar{J}h^l(Y_1, Y_2) + \bar{J}h^s(Y_1, Y_2), \\ &= \bar{J}\tilde{\nabla}^{K'}_{Y_1}Y_2 + fC(Y_1, Y_2) + \omega C(Y_1, Y_2) + \bar{J}h^l(Y_1, Y_2) \\ &\quad + fh^s(Y_1, Y_2). \end{aligned} \tag{25}$$

On replacing Y_2 by $\bar{J}Y_2$ in Eq. (24), we get

$$\bar{\nabla}_{Y_1}\bar{J}Y_2 = \tilde{\nabla}^{K'}_{Y_1}\bar{J}Y_2 + C(Y_1, \bar{J}Y_2) + h^l(Y_1, \bar{J}Y_2) + h^s(Y_1, \bar{J}Y_2). \tag{26}$$

Since K_2 is an indefinite Kaehler manifold, therefore we have

$$\bar{\nabla}_{Y_1}\bar{J}Y_2 = \bar{J}\bar{\nabla}_{Y_1}Y_2.$$

Then from Eqs. (25) and (26), we acquire

$$\begin{aligned} &\tilde{\nabla}^{K'}_{Y_1}\bar{J}Y_2 + C(Y_1, \bar{J}Y_2) + h^l(Y_1, \bar{J}Y_2) + h^s(Y_1, \bar{J}Y_2) \\ &= \bar{J}\tilde{\nabla}^{K'}_{Y_1}Y_2 + fC(Y_1, Y_2) + \omega C(Y_1, Y_2) + \bar{J}h^l(Y_1, Y_2) + fh^s(Y_1, Y_2). \end{aligned}$$

On comparing the components of horizontal, vertical and normal vector fields, we get

$$\tilde{\nabla}^{K'}_{Y_1}\bar{J}Y_2 = \bar{J}\tilde{\nabla}^{K'}_{Y_1}Y_2, \tag{27}$$

$$C(Y_1, \bar{J}Y_2) = fC(Y_1, Y_2) + fh^s(Y_1, Y_2), \tag{28}$$

$$h^l(Y_1, \bar{J}Y_2) = \bar{J}h^l(Y_1, Y_2), \tag{29}$$

$$h^s(Y_1, \bar{J}Y_2) = \omega C(Y_1, Y_2). \tag{30}$$

From Eq. (27), we have $\tilde{\nabla}^{K'}_{Y_1}\bar{J}Y_2 = \bar{J}\tilde{\nabla}^{K'}_{Y_1}Y_2$ that is $(\tilde{\nabla}^{K'}_{Y_1}\bar{J})Y_2 = 0$, which proves that K' is also an indefinite Kaehler manifold. □

Corollary 4.6. *If $\phi : K_1 \to K'$ is a submersion of screen generic lightlike submanifold of an indefinite Kaehler manifold onto an indefinite almost Hermitian manifold such that D' is integrable, then*

$$C(Y_1, \bar{J}Y_2) + h(Y_1, \bar{J}Y_2) = \bar{J}C(Y_1, Y_2) + \bar{J}h(Y_1, Y_2).$$

Proposition 4.7. *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a totally umbilical screen generic lightlike submanifold of K_2. If K' is an indefinite Kaehler manifold such that $\phi : K_1 \to K'$ is a lightlike submersion from K_1 onto K', then*

$$T_{\bar{J}Y_1}V = \bar{J}T_{Y_1}V,$$

for each $Y_1 \in \Gamma(D), V \in \Gamma(D')$.

Proof. Let Y_1 be a basic vector field, $Y_2 \in \Gamma(D), V \in \Gamma(D')$. Then we have

$$\begin{aligned} g_1(T_{\bar{J}Y_1}V, Y_2) &= g_1(\mathcal{H}(\nabla_{\bar{J}Y_1}V), Y_2) = g_1(\nabla_{\bar{J}Y_1}V, Y_2) \\ &= g_1([\bar{J}Y_1, V] + \nabla_V\bar{J}Y_1, Y_2) \\ &= g_1(\nabla_V\bar{J}Y_1, Y_2) = g_1(\bar{\nabla}_V\bar{J}Y_1, Y_2) \\ &= g_1(\bar{J}\bar{\nabla}_VY_1, Y_2) = -g_1(\bar{\nabla}_VY_1, \bar{J}Y_2) \\ &= -g_1(\nabla_VY_1, \bar{J}Y_2) = -g_1(T_VY_1, \bar{J}Y_2) \\ &= g_1(\bar{J}T_VY_1, Y_2). \end{aligned}$$

Then using non-degeneracy of D_0 in $S(TK_1)$, we have $T_{\bar{J}Y_1}V = \bar{J}T_{Y_1}V$. □

Proposition 4.8. *Assume that K_2 is an indefinite Kaehler manifold and K_1 is a totally umbilical screen generic lightlike submanifold of K_2. If K' is an indefinite Kaehler manifold such that $\phi : K_1 \to K'$ is a lightlike submersion from K_1 onto K', then we have $C(\bar{J}Y_1, \bar{J}Y_2) = C(Y_1, Y_2)$.*

Proof. For $Y_1, Y_2 \in \Gamma(D), V \in \Gamma(D')$ and using Lemma (4.4) and Eq. (22), we have

$$\begin{aligned} g_1(V, C((\bar{J}Y_1, \bar{J}Y_2)) &= -g_1(T_{\bar{J}Y_1}V, \bar{J}Y_2) = -g_1(\bar{J}T_VY_1, \bar{J}Y_2) \\ &= -g_1(T_VY_1, Y_2) = -g_1(T_{Y_1}V, Y_2) \\ &= g_1(V, C(Y_1, Y_2)). \end{aligned}$$

Then using the non-degeneracy of D', we have $C(\bar{J}Y_1, \bar{J}Y_2) = C(Y_1, Y_2)$. □

Corollary 4.9. *For horizontal vector field Y_1 and Y_2, we have*

$$C(Y_1, \bar{J}Y_2) = -C(\bar{J}Y_1, Y_2).$$

Now for $U, V \in \Gamma(D')$, we define L by

(31) $$\nabla_UV = L(U.V) + \hat{\nabla}_UV.$$

where $L(U,V) = \mathcal{H}(\nabla_U V)$, $\hat{\nabla}_U V = \mathcal{V}(\nabla_U V)$. For $V \in \Gamma(D'), Y_1 \in \Gamma(D)$, define $\mathcal{A}$ as

$$\nabla_V Y_1 = \mathcal{H}(\nabla_U Y_1) + \mathcal{A}_V Y_1. \tag{32}$$

Now for basic vector field Y_1 and $V \in \Gamma(D')$,

$$\mathcal{H}(\nabla_V Y_1) = \mathcal{H}(\nabla_{Y_1} V) = T_{Y_1} V.$$

Thus from Eq. (32), we have

$$\nabla_V Y_1 = T_{Y_1} V + \mathcal{A}_V Y_1. \tag{33}$$

The operators L and $\mathcal{A}$ are related by

$$g_1(\mathcal{A}_V Y_1, W) = -g_1(L(V,W), Y_1). \tag{34}$$

Theorem 4.10. *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a totally umbilical screen generic lightlike submanifold of K_2. Let $K^{'}$ be an indefinite Kaehler manifold and $\phi : K_1 \to K^{'}$ be a lightlike submersion from K_1 onto $K^{'}$ such that D' is integrable. If $\bar{H}$ and $H^{K^{'}}$ represent the holomorphic sectional curvature of K_2 and $K^{'}$ respectively, then for any unit basic vector $Y_1 \in \Gamma(\mathcal{H})$ of K_1, we have*

$$\bar{H} = H^{K^{'}} + 4||H^s||^2.$$

Proof. For $Y_1, Y_2, X \in \Gamma(D)$, using Eqs. (19) and (21), we have

$$\nabla_{Y_1}\nabla_{Y_2} X = \tilde{\nabla}^{K^{'}}_{Y_1}\tilde{\nabla}^{K^{'}}_{Y_2} X + T_{Y_1} C(Y_2, X) + (\nabla_{Y_1}\nabla_{Y_2} X)^{\mathcal{V}}, \tag{35}$$

Replacing Y_1 with Y_2 in Eq. (35), we have

$$\nabla_{Y_2}\nabla_{Y_1} X = \tilde{\nabla}^{K^{'}}_{Y}\tilde{\nabla}^{K^{'}}_{Y_1} X + T_{Y} C(Y_1, X) + (\nabla_{Y_2}\nabla_{Y_1} X)^{\mathcal{V}}, \tag{36}$$

Also

$$\nabla_{[Y_1,Y_2]} X = \tilde{\nabla}^{K^{'}}_{\mathcal{H}[Y_1,Y_2]} X + 2T_Z(Y_1, Y_2) + (\nabla_{[Y_1,Y_2]} X)^{\mathcal{V}}, \tag{37}$$

Further using Eqs. (35)-(37), we have

$$\begin{aligned} R^{K_1}(Y_1,Y_2)X &= (R^{K^{'}}(\tilde{Y_1},\tilde{Y_2})\tilde{X}))^* + T_X C(Y_2,X) - T_Y C(Y_1,X) \\ &\quad -2T_Z(Y_1,Y_2) + (R^{K_1}(Y_1,Y_2)X)^{\mathcal{V}}, \end{aligned} \tag{38}$$

where $(R^{K^{'}}(\tilde{Y_1},\tilde{Y_2})\tilde{X}))^*$ denotes the basic vector field of K_1 corresponding to $R^{K^{'}}(\tilde{Y_1},\tilde{Y_2})\tilde{X})$, therefore using Eq. (38) in Eq. (8), we get

$$\begin{aligned} \bar{R}(Y_1,Y_2)X &= (R^{K^{'}}(\tilde{Y_1},\tilde{Y_2})\tilde{X}))^* + T_X C(Y_2,X) - T_Y C(Y_1,X) \\ &\quad -2T_Z C(Y_1,Y_2) + A_{h^l(Y_1,X)}Y_2 - A_{h^l(Y_2,X)}Y_1 + \\ &\quad A_{h^s(Y_1,X)}Y_2 - A_{h^s(Y_2,X)}Y_1 + (\nabla_{Y_1}h^l)(Y_2,X) - \\ &\quad (\nabla_{Y_2}h^l)(Y_1,X) + D^l(Y_1,h^s(Y_2,X)) - D^l(Y_2,h^s(Y_1,X)) \\ &\quad +(\nabla_{Y_1}h^s)(Y_2,X) - (\nabla_{Y_2}h^s)(Y_1,X) + D^s(Y_1,h^l(Y_2,X)) \\ &\quad +D^s(Y_2,h^l(Y_1,X)) + (\bar{R}(Y_1,Y_2)X)^{\mathcal{V}}. \end{aligned} \tag{39}$$

Now for basic vector field $W \in \Gamma(D)$, we have

$$\bar{R}(Y_1, Y_2, X, W) = g_1(\bar{R}(Y_1, Y_2)X, W),$$

therefore using Eq. (39), we get

$$\begin{aligned}\bar{R}(Y_1, Y_2, X, W) &= g_1((R^{K'}(\tilde{Y_1}, \tilde{Y_2})\tilde{X})*, W) + g_1(T_X C(Y_2, X), W) \\ &\quad - g_1(T_Y C(Y_1, X), W) - 2g_1(T_Z C(Y_1, Y_2), W) \\ &\quad + g_1(A_{h^l(Y_1,X)}Y_2, W) - g_1(A_{h^l(Y_2,X)}Y_1, W) \\ &\quad + g_1(A_{h^s(Y_1,X)}Y_2, W) - g_1(A_{h^s(Y_2,X)}Y_1, W), \end{aligned} \tag{40}$$

Now using Lemma (4.4), we have

$$g_1(T_X C(Y_2, X), W) = -g_1(C(Y_2, X), C(Y_1, W)), \tag{41}$$

$$g_1(T_Y C(Y_1, X), W) = -g_1(C(Y_1, X), C(Y_2, W)), \tag{42}$$

and

$$g_1(T_Z C(Y_1, Y_2), W) = -g_1(C(Y_1, Y_2), C(X, W)). \tag{43}$$

Since K_1 is totally umbilical, thus using Eq. (5), we have

$$g_1(A_{h^l(Y_1,X)}Y_2, W) = -g_1(\bar{\nabla}_Y h^l(Y_1, X), W) = g_1(h^l(Y_1, X), \bar{\nabla}_Y W) = 0. \tag{44}$$

Similarly, we have

$$g_1(A_{h^l(Y_2,X)}Y_1, W) = 0. \tag{45}$$

Also we have

$$g_1(A_{h^s(Y_1,X)}Y_2, W) = g_2(h^s(Y_2, W), h^s(Y_1, X)) \tag{46}$$

and

$$g_1(A_{h^s(Y_2,X)}Y_1, W) = g_2(h^s(Y_1, W), h^s(Y_2, X)). \tag{47}$$

Now using Eqs. (41) - (47) in Eq. (40), we get

$$\begin{aligned}\bar{R}(Y_1, Y_2, X, W) &= \bar{R}^{K'}(\tilde{Y_1}, \tilde{Y_2}, \tilde{X}, \tilde{W}) - g_1(C(Y_2, X), C(Y_1, W)) \\ &\quad + g_1(C(Y_1, X), C(Y_2, W)) \\ &\quad + 2g_1(C(Y_1, Y_2), C(X, W)) \\ &\quad + g_2(h^s(Y_2, W), h^s(Y_1, X)) \\ &\quad - g_2(h^s(Y_1, W), h^s(Y_2, X). \end{aligned} \tag{48}$$

Now putting $Y_2 = \bar{J}Y_1, X = Y_1, W = \bar{J}Y_1$ in Eq. (48) and using skew symmetric property of C along with Eqs. (28) and (30), we get

$$\begin{aligned}\bar{R}(Y_1, \bar{J}Y_1, Y_1, \bar{J}Y_1) &= \bar{R}^{K'}(\tilde{Y_1}, \tilde{\bar{J}Y_1}, \tilde{Y_1}, \tilde{\bar{J}Y_1}) - g_1(C(\bar{J}Y_1, Y_1), \\ &\quad C(Y_1, \bar{J}Y_1)) + 2g_1(C(Y_1, \bar{J}Y_1), C(Y_1, \bar{J}Y_1)) \\ &\quad + g_2(h^s(\bar{J}Y_1, \bar{J}Y_1), h^s(Y_1, Y_1)), \end{aligned} \tag{49}$$

Since

$$C(\bar{J}Y_1, Y_1) = -C(Y_1, \bar{J}Y_1), g_1(C(Y_1, \bar{J}Y_1), C(Y_1, \bar{J}Y_1)) \\ = g_1(\bar{J}h^s(Y_1, Y_1), \bar{J}h^s(Y_1, Y_1)) \\ = g_1(h^s(Y_1, Y_1), h^s(Y_1, Y_1)),$$

using this together with hypothesis and Proposition (4.8), we have

$$\bar{R}(Y_1, \bar{J}Y_1, Y_1, \bar{J}Y_1) = \bar{R}^{K'}(\tilde{Y_1}, \bar{J}\tilde{Y_1}, \tilde{Y_1}, \bar{J}\tilde{Y_1}) + 4||H^s||^2,$$

that is

$$\bar{H} = H^{K'} + 4||H^s||^2.$$

□

Theorem 4.11. *Suppose that K_2 is an indefinite Kaehler manifold and K_1 is a totally umbilical screen generic lightlike submanifold of K_2. If K' is an indefinite Kaehler manifold such that $\phi : K_1 \to K'$ is a lightlike submersion from K_1 onto K' such that D' is integrable, then the sectional curvature of K_2 and of the fibre are related by*

$$\begin{aligned} \bar{K}(Z \wedge W) &= \hat{K}(Z \wedge W) - g_1(L(Z, W), L(W, Z)) \\ &\quad + g_1(L(W, W), L(Z, Z)), \end{aligned}$$

where $Z, W \in \Gamma(D')$.

Proof. For $Z, W, V \in \Gamma(D')$, using Eqs. (31) and (32) we have

$$\begin{aligned} R(Z, W)V &= \nabla_Z \nabla_W V - \nabla_W \nabla_Z V - \nabla_{[Z,W]} V \\ &= \mathcal{A}_Z L(W, V) + \hat{\nabla}_Z \hat{\nabla}_W V \\ &\quad - \mathcal{A}_W L(Z, V) - \hat{\nabla}_W \hat{\nabla}_Z V - \hat{\nabla}_{[Z,W]} V \\ &\quad + horizontal \quad part \\ &= \hat{R}(Z, W)V + \mathcal{A}_Z L(W, V) - \mathcal{A}_W L(Z, V) \\ &\quad + horizontal \quad part. \end{aligned} \tag{50}$$

Let $Z, W, V, S \in \Gamma(D')$ and using Eqs. (31)-(34) in Eq. (50), one has

$$\begin{aligned} R(Z, W, V, S) &= g_1(R(Z, W)V, S) = g_1(\hat{R}(Z, W)V + \mathcal{A}_Z L(W, V) \\ &\quad - \mathcal{A}_Z L(W, V), S) \\ &= g_1(\hat{R}(Z, W)V, S) + g_1(\mathcal{A}_Z L(W, V), S) \\ &\quad - g_1(\mathcal{A}_W L(Z, V), S) \\ &= \hat{R}(Z, W, V, S) - g_1(L(Z, S), L(W, V)) \\ &\quad + g_1(L(W, S), L(Z, V). \end{aligned}$$

Taking $V = Z$ and $S = W$ in above equation, we acquire

$$\begin{aligned} R(Z, W, Z, W) &= \hat{R}(Z, W, Z, W) - g_1(L(Z, W), L(W, Z)) \\ &\quad + g_1(L(W, W), L(Z, Z). \end{aligned} \tag{51}$$

From Eq. (8), setting $Y_1 = Z, Y_2 = W$, we obtain

$$\begin{aligned} \bar{R}(Z,W)Z &= R(Z,W)Z + A_{h^l(Z,Z)}W - A_{h^l(W,Z)}Z + A_{h^s(Z,Z)}W \\ &\quad -A_{h^s(W,Z)}Z + (\nabla_Z h^l)(W,Z) - (\nabla_W h^l)(Z,Z) \\ &\quad +D^l(Z,h^s(W,Z)) - D^l(W,h^s(Z,Z)) + (\nabla_Z h^s)(W,Z) \\ &\quad -(\nabla_W h^s)(Z,Z) + D^s(Z,h^l(W,Z)) + D^s(W,h^l(Z,Z)), \end{aligned}$$

Then considering the inner product of the above equation with $W \in \Gamma(D')$, we have

$$\begin{aligned} \bar{R}(Z,W,Z,W) &= g_1(\bar{R}(Z,W)Z,W) = g_1(R(Z,W)Z,W) \\ &\quad +g_1(A_{h^l(Z,Z)}W,W) - g_1(A_{h^l(W,Z)}Z,W) \\ &\quad +g_1(A_{h^s(Z,Z)}W,W) - g_1(A_{h^s(W,Z)}Z,W) \\ &\quad +g_1((\nabla_Z h^l)(W,Z),W) - g_1((\nabla_W h^l)(Z,Z),W) \\ &\quad +g_1(D^l(Z,h^s(W,Z)),W) - g_1(D^l(W,h^s(Z,Z)),W) \\ &\quad +g_1((\nabla_Z h^s)(W,Z),W) - g_1(D^l(W,h^s(Z,Z)),W) \\ &\quad +g_1((\nabla_Z h^s)(W,Z),W) - g_1((\nabla_W h^s)(Z,Z),W) \\ &\quad +g_1(D^s(Z,h^l(W,Z)),W) + g_1(D^s(W,h^l(Z,Z)),W), \end{aligned}$$

which further becomes

$$\begin{aligned} \bar{R}(Z,W,Z,W) &= R(Z,W,Z,W) + g_1(A_{h^l(Z,Z)}W,W) \\ &\quad -g_1(A_{h^l(W,Z)}Z,W) + g_1(A_{h^s(Z,Z)}W,W) \\ &\quad -g_1(A_{h^s(W,Z)}Z,W). \end{aligned} \tag{52}$$

Using Eqs. (7) and (51) in Eq. (52), we get

$$\begin{aligned} \bar{R}(Z,W,Z,W) &= \hat{R}(Z,W,Z,W) - g_1(L(Z,W),L(W,Z)) \\ &\quad +g_1(L(W,W),L(Z,Z) + g_1(A_{h^l(Z,Z)}W,W) \\ &\quad -g_1(A_{h^l(W,Z)}Z,W) + g_1(h^s(W,W),h^s(Z,Z)) \\ &\quad -g_1(h^s(Z,W),h^s(Z,W)). \end{aligned} \tag{53}$$

As K_1 is a totally umbilical lightlike manifold, therefore from theorem (3.5) and $h^s(Z,W) = H^s g_1(Z,W)$, thus Eq. (53) reduces to

$$\begin{aligned} \bar{R}(Z,W,Z,W) &= \hat{R}(Z,W,Z,W) - g_1(L(Z,W),L(W,Z)) \\ &\quad +g_1(L(W,W),L(Z,Z)). \end{aligned}$$

Thus the proof follows. □

Theorem 4.12. *Let K_2 be an indefinite Kaehler manifold and K_1 be a totally umbilical screen generic lightlike submanifold of K_2. If $K^{'}$ is an indefinite Kaehler manifold such that $\phi : K_1 \to K^{'}$ is a lightlike submersion from K_1*

onto $K^{'}$, *then for* $Y_1, Y_2 \in \Gamma(D)$ *and* $V_1, V_2 \in \Gamma(D')$

$$\begin{aligned}\bar{R}(Y_1, V_1, Y_2, V_2) &= g_1(T_{Y_1}V_2, P(\nabla_{V_1}Y_2)) - g_1(A_{V_1}Y_2, Q\nabla_{Y_1}V_2) \\ &\quad -g_1(L(V_1, V_2), \tilde{\nabla}^{K'}_{Y_1}Y_2) + g_1(L([Y_1, Y_2], V_2), Y_2) \\ &\quad +g_1(\hat{\nabla}_{V_1}C(Y_1, Y_2), V_2) + g_2(h^s(V_1, W), h^s(Y_1, Y_2)).\end{aligned}$$

Proof. For $Y_1, Y_2 \in \Gamma(D)$ and $V_1, V_2 \in \Gamma(D')$, we have

$$\nabla_{Y_1}\nabla_{V_1}Y_2 = C(Y_1, \mathcal{H}(\nabla_{V_1}Y_2) + \nabla_{Y_1}(\mathcal{A}_{V_1}Y_2) + \text{horizontal part.} \tag{54}$$

Similarly,

$$\nabla_{V_1}\nabla_{Y_1}Y_2 = \mathcal{A}_{V_1}\tilde{\nabla}^{K'}_{Y_1}Y_2 + \hat{\nabla}_{V_1}C(Y_1, Y_2) + \text{horizontal part,} \tag{55}$$

and

$$\nabla_{[Y_1,V_1]}Y_2 = \mathcal{A}_{[Y_1,V_1]}Y_2 + \text{horizontal part.} \tag{56}$$

We know that

$$R(Y_1, V_1)Y_2 = \nabla_{Y_1}\nabla_{V_1}Y_2 + \nabla_{V_1}\nabla_{Y_1}Y_2 - \nabla_{[Y_1,V_1]}Y_2,$$

further using Eqs. (54) - (56) in above equation, we acquire

$$\begin{aligned}R(Y_1, V_1)Y_2 &= C(Y_1, \mathcal{H}(\nabla_{V_1}Y_2) + \nabla_{Y_1}(\mathcal{A}_{V_1}Y_2) + \mathcal{A}_{V_1}\tilde{\nabla}^{K'}_{Y_1}Y_2 \\ &\quad +\hat{\nabla}_{V_1}C(Y_1, Y_2) - \mathcal{A}_{[Y_1,V_1]}Y_2 + \text{horizontal part.}\end{aligned}$$

Now taking the inner product of the above equation with $V_2 \in \Gamma(D')$ and using Eqs.(23) and (34), we obtain

$$\begin{aligned}R(Y_1, V_1, Y_2, V_2) &= g_1(R(Y_1, V_1)Y_2, V_2) \\ &= g_1(C(Y_1, \mathcal{H}(\nabla_{V_1}Y_2)), V_2) + g_1(\nabla_{Y_1}(\mathcal{A}_{V_1}Y_2), V_2) \\ &\quad +g_1(\mathcal{A}_{V_1}\tilde{\nabla}^{K'}_{Y_1}Y_2, V_2) + g_1(\hat{\nabla}_{V_1}C(Y_1, Y_2), V_2) \\ &\quad -g_1(\mathcal{A}_{[Y_1,V_1]}Y_2, V_2) \\ &= g_1(T_{Y_1}V_2, \mathcal{H}(\nabla_{V_1}Y_2)) + g_1(\nabla_{Y_1}\mathcal{A}_{V_1}Y_2, V_2) \\ &\quad -g_1(L(V_1, V_2), \tilde{\nabla}_{Y_1}Y_2) + g_1(L([Y_1, V_1], V_2), Y_2) \\ &\quad +g_1(\hat{\nabla}_{V_1}C(Y_1, Y_2), V_2).\end{aligned} \tag{57}$$

In Eq. (8), setting $Y_2 = V_1$ and $X = Y_2$, one has

$$\begin{aligned}\bar{R}(Y_1, V_1)Y_2 &= R(Y_1, V_1)Y_2 + A_{h^l(Y_1,Y_2)}V_1 - A_{h^l(V_1,Y_2)}Y_1 \\ &\quad +A_{h^s(Y_1,Y_2)}V_1 - A_{h^s(V_1,Y_2)}Y_1 \\ &\quad +(\nabla_{Y_1}h^l)(V_1, Y_2) - (\nabla_{V_1}h^l)(Y_1, Y_2) \\ &\quad +D^l(Y_1, h^s(V_1, Y_2)) - D^l(V_1, h^s(Y_1, Y_2)) \\ &\quad +(\nabla_{Y_1}h^s)(V_1, Y_2) - (\nabla_{V_1}h^s)(Y_1, Y_2) \\ &\quad +D^s(Y_1, h^l(V_1, Y_2)) - D^s(V_1, h^l(Y_1, Y_2)).\end{aligned} \tag{58}$$

Now using Eq. (58), we have

$$\begin{aligned}\bar{R}(Y_1,V_1,Y_2,V_2) &= g_1(\bar{R}(Y_1,V_1)Y_2,V_2)\\ &= R(Y_1,V_1,Y_2,V_2)+g_1(A_{h^l(Y_1,Y_2)}V_1,V_2)\\ &\quad -g_1(A_{h^l(V_1,Y_2)}Y_1,V_2)+g_1(A_{h^s(Y_1,Y_2)}V_1,V_2)\\ &\quad -g_1(A_{h^s(V_1,Y_2)}Y_1,V_2). \end{aligned}\tag{59}$$

Using Eq. (57) in Eq. (59), we derive

$$\begin{aligned}\bar{R}(Y_1,V_1,Y_2,V_2) &= g_1(T_{Y_1}V_2,\mathcal{H}(\nabla_{V_1}Y_2))+g_1(\nabla_{Y_1}\mathcal{A}_{V_1}Y_2,V_2)\\ &\quad -g_1(L(V_1,V_2),\tilde{\nabla}^{K'}_{Y_1}Y_2)+g_1(L([Y_1,Y_2],V_2),Y_2)\\ &\quad +g_1(\hat{\nabla}_{V_1}C(Y_1,Y_2),V_2)+g_1(A_{h^l(Y_1,Y_2)}V_1,V_2)\\ &\quad -g_1(A_{h^l(V_1,Y_2)}Y_1,V_2)+g_1(A_{h^s(Y_1,Y_2)}V_1,V_2)\\ &\quad -g_1(A_{h^s(V_1,Y_2)}Y_1,V_2).\end{aligned}$$

Further using Eq. (7), we acquire

$$\begin{aligned}\bar{R}(Y_1,V_1,Y_2,V_2) &= g_1(T_{Y_1}V_2,P(\nabla_{V_1}Y_2))+g_1(\nabla_{Y_1}\mathcal{A}_{V_1}Y_2,V_2)\\ &\quad -g_1(L(V_1,V_2),\tilde{\nabla}^{K'}_{Y_1}Y_2)+g_1(L([Y_1,Y_2],V_2),Y_2)\\ &\quad +g_1(\hat{\nabla}_{V_1}C(Y_1,Y_2),V_2)+g_1(A_{h^l(Y_1,Y_2)}V_1,V_2)\\ &\quad -g_1(A_{h^l(V_1,Y_2)}Y_1,V_2)+g_2(h^s(V_1,V_2),h^s(Y_1,Y_2))\\ &\quad -g_2(h^s(Y_1,V_2),h^s(V_1,Y_2)),\end{aligned}\tag{60}$$

Now for $Y_1,Y_2\in\Gamma(D)$ and $V_1,V_2\in\Gamma(D')$, we have

$$\begin{aligned}g_1(\nabla_{Y_1}\mathcal{A}_{V_1}Y_2,V_2) &= g_2(\bar{\nabla}_{Y_1}\mathcal{A}_{V_1}Y_2,V_2)=-g_1(\mathcal{A}_{V_1}Y_2,\bar{\nabla}_{Y_1}V_2)\\ &= -g_1(\mathcal{A}_{V_1}Y_2,\nabla_{Y_1}V_2)\\ &= -g_1(\mathcal{A}_{V_1}Y_2,Q\nabla_{Y_1}V_2).\end{aligned}$$

Using above result and totally umbilical property of K_1, Eq. (60) becomes

$$\begin{aligned}\bar{R}(Y_1,V_1,Y_2,V_2) &= g_1(T_{Y_1}V_2,P(\nabla_{V_1}Y_2))-g_1(\mathcal{A}_{V_1}Y_2,Q\nabla_{Y_1}V_2)\\ &\quad -g_1(L(V_1,V_2),\tilde{\nabla}^{K'}_{Y_1}Y_2)+g_1(L([Y_1,Y_2],V_2),Y_2)\\ &\quad +g_1(\hat{\nabla}_{V_1}C(Y_1,Y_2),V_2)+g_2(h^s(V_1,V_2),h^s(Y_1,Y_2)).\end{aligned}\tag{61}$$

□

Theorem 4.13. *Assume that K_2 is an indefinite Kaehler manifold and K_1 is a totally umbilical screen generic lightlike submanifold of K_2. If K' is an indefinite Kaehler manifold such that $\phi: K_1\to K'$ be a lightlike submersion from K_1 onto K', then*

$$\begin{aligned}\bar{K}(Y_1,V_1) &= \|T_{Y_1}V_1\|^2-g_1(L(V_1,V_1),\tilde{\nabla}^{K'}_{Y_1}Y_1)+g_1(L([Y_1,Y_1],V_1),Y_1)\\ &\quad +g_2(h^s(V_1,V_1),h^s(Y_1,Y_1)),\end{aligned}$$

where $Y_1\in\Gamma(D)$ and $V_1\in\Gamma(D')$.

Proof. For $Y_1 \in \Gamma(D)$ and $V_1 \in \Gamma(D')$, put $Y_2 = Y_1, V_2 = V_1$ in Eq. (61), we get

$$\begin{aligned}\bar{R}(Y_1, V_1, Y_1, V_1) &= g_1(T_{Y_1}V_1, P(\nabla_{V_1}Y_1)) - g_1(\mathcal{A}_{V_1}Y_1, Q\nabla_{Y_1}V_1) \\ &\quad - g_1(L(V_1, V_1), \tilde{\nabla}^{K'}_{Y_1}Y_1) + g_1(L([Y_1, Y_1], V_1), Y_1) \\ &\quad + g_1(\hat{\nabla}_{V_1}C(Y_1, Y_1), V_1) + g_2(h^s(V_1, V_1), h^s(Y_1, Y_1)).\end{aligned}$$

Using Eq. (33) and skew-symmetric property of tensor C, the result follows. □

References

[1] M. Barros and A. Romero, *Indefinite Kaehler manifolds*, Math. Ann. **261** (1982), 55–62.

[2] J. P. Bourguinon and H. B. Lawson, *Stability and isolation phenomena for Yang-Mills fields*, Commun. Math. Phys. **79** (1981), 189–230.

[3] B. Dogan, B. Sahin and E. Yasar, *Screen generic lightlike submanifold*, Mediterr. J. Math. **16** (2019), 040001–21.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands 1996.

[5] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. **26** (2003), 49–68.

[6] K. L. Duggal and D. H. Jin, *Generic lightlike submanifolds of an indefinite sasakian manifold*, Int. Electron. J. Geom. **5** (2012), no. 1, 108–119.

[7] T. Fatima, M. A. Akyol, and A. A. Alzulaibani, *On a submersion of generic submanifold of a nearly Kaehler manifold*, Int. J. Geom. Methods Mod. **19** (2022), no. 4, 2250048–62.

[8] A. Gray, *Pseudo-Riemannian almost product manifold and submersion*, J. Math. Mech. **16** (1967), 715–737.

[9] R. S. Gupta and A. Sharfuddin, *Screen transversal lightlike submanifolds of indefinite cosymplectic manifolds*, Rend. Semin. Mat. Univ. Padova. **124** (2010), 145–156.

[10] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalised Hopf manifolds*, Class. Quantam Gravity **4** (1987), 1317–1325.

[11] S. Kobayashi, *Submersions of CR submanifolds*, Tohoku Math. J. **9** (1987), 95–100.

[12] S. Kumar, *Geometry warped product lightlike submanifolds of indefinite nearly Kaehler manifolds*, J. Geom. **21** (2018), 010001–18.

[13] M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. **41** (2000), 6918–6929.

[14] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York-London, 1983.

[15] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

[16] B. Sahin, *On a submersion between Reinhart lightlike manifolds and semi-Riemannian manifolds*, Mediterr. J. Math. **5** (2008), 273–284.

[17] B. Sahin and Yilmaz Gündüzalp, *Submersion from semi-Riemannian manifolds onto lightlike manifolds*, Hacet. J. Math. Stat **39** (2010), 41–53.

[18] G. Sharma, S. Kumar, and M. Kumar, *On lightlike submersion of radical transversal lightlike submanifolds of a Kaehler manifold*, ECS Trans. **107** (2022), no. 1, 10069–10084.

Gaurav Sharma
Department of Mathematics,
Maharaja Agrasen University,
Baddi-174103, Himachal Pradesh, India.

SDWG Government College Beetan,
Una-176601, Himachal pradesh, India.
E-mail: g.gauravsh@gmail.com

Sangeet Kumar
Department of Mathematics,
Sri Guru Teg Bahadur Khalsa College,
Sri Anandpur Sahib-140118, Punjab, India.
E-mail: sp7maths@gmail.com

Dinesh Kumar Sharma
Department of Mathematics,
Maharaja Agrasen University,
Baddi-174103, Himachal Pradesh, India.
E-mail: dksharma200513@gmail.com