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ON THE SPECTRUM AND FINE SPECTRUM OF THE UPPER TRIANGULAR DOUBLE BAND MATRIX $U(a_0, a_1, a_2; b_0, b_1, b_2)$ OVER THE SEQUENCE SPACE ℓ_p

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Abstract. The purpose of this article is to obtain the spectrum, fine spectrum, approximate point spectrum, defect spectrum and compression spectrum of the double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$, $b_0, b_1, b_2 \neq 0$ on the sequence space ℓ_p $(1 < p < \infty)$.

1. Introduction

Branches of mathematics are tools for many applied sciences. One of these tools is spectrum theory. Spectrum theory has a wide area of uses. For example, ecology, structural mechanics, quantum mechanics, electrical engineering etc. In addition, the resolvent set of the band matrix is important in solving the problems in the application areas mentioned above. As is known, there is a relationship between matrices and operators. The spectrum of an operator is a generalization of the concept of the eigenvalue of the matrix corresponding to the operator. Spectrum and spectrum decompositions of certain operators are studied on some sequence spaces. These studies mostly include Cesàro, Hölder, some difference matrix operators. For example, Gonzalez [10], Tripathy and Saikia [13] examined the spectrum of the Cesàro operator. Wenger [15] examined the spectra of Hölder summability. Yildirim [16] examined on the spectrum of the Rhaly operators.
We denote the space of all sequences with w. Also, the spaces of bounded, convergent, null, and limited variation sequences, which are Banach sequence spaces, are usually denoted by ℓ_∞, c, c_0, and bv, respectively. By w, we denote the space of all sequences. Moreover the spaces of all p-absolutely summable sequences and p-bounded variation sequences are denoted by ℓ_p, bv_p, respectively.

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In this study, the spectral decompositions of the

(1)
$$U(a_0,a_1,a_2;b_0,b_1,b_2)=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix},$$
$(b_0,b_1,b_2\neq 0)$.

band matrix over the ℓ_p sequence space were investigated.

$\ell_p=\left\{x=(x_k)\in w:\sum\limits_{k=1}|x_k|^p<\infty\right\}$, $1<p<\infty$ and $\|.\|_{\ell_p}=\left(\sum\limits_{k=1}|x_k|^p\right)^{\frac{1}{p}}$

The dual space of the ℓ_p sequence space is the

$$\ell_q=\left\{x=(x_k)\in w:\sum_{k=1}|x_k|^q<\infty\right\},\quad \frac{1}{p}+\frac{1}{q}=1.$$

The calculations of this matrix in c_0 [7] are also available.

2. Spectrum and Fine Spectrum

Let X and Y be the Banach spaces, and $T:X\to Y$ be a bounded linear operator.$R(T)$, $D(T)$, $B(X)$ are defined as $R(T)=\{y\in Y: y=Tx,\ x\in X\}$, $D(T)=\{x\in X: y=Tx\}$ and

$$B(X)=\{T:T:X\to X \text{ bounded linear operator}\},$$

respectively.

Let $T:D(T)\subset X\to X$ be a linear operator where X is a complex normed space. Let $T_\zeta:=T-\zeta I$ for $T\in B(X)$ and $\zeta\in\mathbb{C}$ where I is the identity operator. T_ζ^{-1} is the resolvent operator of T.

The resolvent set of T is the set of complex numbers ζ of T such that
(a) T_ζ^{-1} exists,
(b) T_ζ^{-1} is bounded,
(c) $T_\zeta^{-1}$is defined on a set which is dense in X, denoted by $\rho(T,X)$.

Its complement is given by $\mathbb{C}\backslash\rho(T,X)$ is called the spectrum of T, denoted by $\sigma(T,X)$.

The spectrum $\sigma(T,X)$ has three discrete decompositions. These are;
the point spectrum $\sigma_p(T,X)$ is the set which T_λ^{-1} does not exist,
the continuous spectrum $\sigma_c(T,X)$ is the set which the operator T_λ^{-1} is defined on a dense subspace of X and is unbounded,

the residual spectrum $\sigma_r(T,X)$ is the set which the operator T_λ^{-1} exists, but its domain of definition is not dense in X than in this case T_λ^{-1} may be bounded or unbounded.

That's to say $\sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X) = \sigma(T,X)$ and $\sigma_p(T,X) \cap \sigma_c(T,X) = \emptyset$, $\sigma_p(T,X) \cap \sigma_r(T,X) = \emptyset$, $\sigma_r(T,X) \cap \sigma_c(T,X) = \emptyset$ from definitions.

Many researchers have studied the spectrum and fine spectrum of linear operators on some sequence spaces. In [8], Fathi studied on the fine spectrum of generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c. In [11], Srivastava and Kumar studied fine spectrum of the generalized difference operator Δ_v on sequence space ℓ_1. In [7], Durna and Kılıç studied spectra and fine spectra for the upper triangular band matrix $U(a_0,a_1,a_2;b_0,b_1,b_2)$ over the sequence space c_0. In [3], Das studied on the spectrum and fine spectrum of the upper triangular matrix $U(r_1,r_2;s_1,s_2)$ over the sequence space c_0.

Lemma 2.1 (Stieglitz and Tietz [12])**.** *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in (\ell_p;\ell_p)$ from ℓ_p to itself if and only if*
(i) $\sup_n \sum\limits_k |a_{nk}| < \infty$*, for each k,*
(ii) $\sup_k \sum\limits_n |a_{nk}| < \infty$*, for each n.*

Theorem 2.2. *$U(a_0,a_1,a_2;b_0,b_1,b_2) : \ell_p \to \ell_p$ is a bounded linear operator.*

Proof. The linearity of $U(a_0,a_1,a_2;b_0,b_1,b_2)$ is clear, so we omit that proof. Now we consider boundedness of operator. If the conditions of lemma 2.1 are calculated for the $U(a_0,a_1,a_2;b_0,b_1,b_2)$ operator, we obtain

$$\sup_n \sum_k |a_{nk}| = \max\{|a_0|+|b_0|, |a_1|+|b_1|, |a_2|+|b_2|\}$$
$$\sup_k \sum_n |a_{nk}| = \max\{|a_1|+|b_0|, |a_2|+|b_1|, |a_0|+|b_2|\}.$$

Thus since $\sup_n \sum\limits_k |a_{nk}| < \infty$ and $\sup_k \sum\limits_n |a_{nk}| < \infty$, we get $(x_n) \in \ell_p$. Hence $U(a_0,a_1,a_2;b_0,b_1,b_2)$ is bounded. □

Lemma 2.3 (Golberg [9, p.59])**.** *T has a dense range if and only if T^* is 1-1.*

Lemma 2.4 (Golberg [9, p.60])**.** *T^{-1} is bounded if and only if T^* is onto.*

Notation. During this paper, it will be denoted as

$$S = \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}.$$

Hence the boundary of S and the interior of S will be denoted by ∂S, $\mathring{S}$ respectively.

Theorem 2.5. *$\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2), \ell_p) = \mathring{S}$.*

Proof. Let ζ be an eigenvalue of the operator $U(a_0,a_1,a_2;b_0,b_1,b_2)$. Then there exists $x\neq\theta=(0,0,0,...)$ in ℓ_p such that $U(a_0,a_1,a_2;b_0,b_1,b_2)x=\zeta x$. Then we obtain

$$\left\{\begin{array}{lcl} x_{3n} & = & d^n x_0, \\ x_{3n+1} & = & \dfrac{\zeta-a_0}{b_0}d^n x_0, \\ x_{3n+2} & = & \dfrac{(\zeta-a_0)(\zeta-a_1)}{b_0b_1}d^n x_0, \end{array}\right. ,\ n\geq 0$$

where $d=\dfrac{(\zeta-a_0)(\zeta-a_1)(\zeta-a_2)}{b_0b_1b_2}$. Thus we get

$$\sum_{n=0}^{\infty}|x_{3n}|^p=\sum_{n=0}^{\infty}|d^n x_0|^p=|x_0|^p\sum_{n=0}^{\infty}|d^n|^p$$

$\sum\limits_{n=0}^{\infty}|d^n|^p<\infty$ if and only if $|d|<1$.

Thus $(x_{3n})\in\ell_p$ if and only if $|d|<1$. Similarly (x_{3n+1}) and (x_{3n+2}) are also convergent. Hence the subsequences $(x_{3n}),(x_{3n+1}),(x_{3n+2})$ of $x=(x_n)$ are in ℓ_p if and only if $|\zeta-a_0||\zeta-a_1||\zeta-a_2|<|b_0||b_1||b_2|$. Thus $x=(x_n)\in\ell_p$ if and only if $|\zeta-a_0||\zeta-a_1||\zeta-a_2|<|b_0||b_1||b_2|$. So

$$\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)=\mathring{S}.$$

$\square$

Let $T:\ell_p\longmapsto\ell_p$ be a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^*:\ell_p^*\longmapsto\ell_p^*$ is a bounded linear operator and A^t is its matrix representation. Where the dual space ℓ_p^* of ℓ_p is isomorphic to l_q with $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.6. $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,\ell_p^*\approx l_q)=\emptyset$.

Proof. It is calculated similarly to the proof of Theorem 2 in [7] $\square$

Theorem 2.7. $\sigma_r(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)=\emptyset$.

Proof. Owing to $\sigma_r(A,X)=\sigma_p(A^*,X^*)\backslash\sigma_p(A,X)$ proof is clear. $\square$

Theorem 2.8. $\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)=S$.

Proof. First, we prove that $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)^{-1}$ exists and is in (ℓ_p,ℓ_p) for $\lambda\notin\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|\}$ and then we have to show that the operator $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)$ is not invertible for $|\zeta-a_0||\zeta-a_1||\zeta-a_2|>|b_0||b_1||b_2|$.

Let $\lambda\notin\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|\}$. Since $b_n\neq 0$, $n=0,1,2$ we get $a_n\neq\zeta$, $n=0,1,2$. Hence, since $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)$ is a upper triangle, $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)^{-1}$ exists.

$$\begin{bmatrix} a_0-\lambda & b_0 & 0 & 0 & \cdots \\ 0 & a_1-\lambda & b_1 & 0 & \cdots \\ 0 & 0 & a_2-\lambda & b_2 & \cdots \\ 0 & 0 & 0 & a_0-\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \cdots \\ 0 & c_{11} & c_{12} & c_{13} & \cdots \\ 0 & 0 & c_{22} & c_{23} & \cdots \\ 0 & 0 & 0 & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then we get

$$c_{nk} = \begin{cases} 0 & , \quad n>k \\ \prod\limits_{v=0}^{k-n-1} \frac{b_{k-1-v}}{a_{k-v}-\lambda} & , \quad n<k \\ \frac{1}{a_n-\lambda} & , \quad n=k \end{cases}$$

Now, we have to show $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)^{-1} \in (\ell_p,\ell_p)$.
Firstly, if arrange the $\left|\prod\limits_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda}\right|$ multiplication then we have

$$\left|\prod_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda}\right| = B_t \left|q^{k-n}\right|$$

where $q=\frac{b_0b_1b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}$ and

$$B_t = \left\{ \begin{array}{c|ccc} & k=3i & k=3i-1 & k=3i-2 \\ \hline n=3j & 1 & \left|\frac{a_0-\lambda}{b_2}\right| & \left|\frac{(a_0-\lambda)(a_2-\lambda)}{b_2b_1}\right| \\ n=3j-1 & \left|\frac{b_2}{a_0-\lambda}\right| & 1 & \left|\frac{a_2-\lambda}{b_1}\right| \\ n=3i-2 & \left|\frac{b_2b_1}{(a_0-\lambda)(a_2-\lambda)}\right| & \left|\frac{b_1}{a_2-\lambda}\right| & 1 \end{array} \right.$$

So from the 1st condition of Lemma 2.1 we get, for each k

$$
\begin{aligned}
(2)\quad \sup_n \sum_k |c_{nk}| &= \sup_n \sum_k \left| \frac{1}{a_n-\lambda} \prod_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda} \right| \\
&\leq \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_n \sum_k \left| \prod_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda} \right| \\
(3)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_n \sum_k |q|^{k-n} \\
(4)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_n |q|^{-n} \sum_k |q|^{k} \\
(5)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_n \frac{1}{|q|^n (1-|q|)}
\end{aligned}
$$

therefore, since

$$\sup_n \frac{1}{|q|^n (1-|q|)} = \begin{cases} \frac{1}{1-|q|} & |q| < 1 \\ \infty & |q| \geq 1 \end{cases}$$

we get

$$(6)\qquad \sup_n \sum_k |c_{nk}| = \begin{cases} \text{convergent} & , \quad |q| < 1 \\ \text{divergent} & , \quad |q| \geq 1 \end{cases}.$$

From the 2st condition of Lemma 2.1 we get

$$
\begin{aligned}
(7)\quad \sup_k \sum_n |c_{nk}| &= \sup_k \sum_n \left| \frac{1}{a_n-\lambda} \prod_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda} \right| \\
&\leq \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_k \sum_n \left| \prod_{v=0}^{k-n-1} (-1)^{k-n} \frac{b_{k-1-v}}{a_{k-v}-\lambda} \right| \\
(8)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_k \sum_n |q|^{k-n} \\
(9)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_k |q|^{k} \sum_n |q|^{-n} \\
(10)\quad &= B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right| \sup_k |q|^{k} \left(\frac{1-\frac{1}{|q|^{k+1}}}{1-|q|} \right) \\
(11)\quad &= \frac{B_t \max_{m=0}^{2} \left| \frac{1}{a_m-\lambda} \right|}{1-|q|} \sup_k \frac{|q|^{k+1}-1}{|q|}
\end{aligned}
$$

Therefore since

$$\sup_i \frac{|q|^{i+1}-1}{|q|} = \begin{cases} \frac{B_t \max_{m=0}^{2}\left|\frac{1}{a_m-\lambda}\right|}{1-|q|} & , \quad |q|<1 \\ \infty & , \quad |q|\geq 1 \end{cases}$$

we get

$$\sup_k \sum_n |c_{nk}| = \begin{cases} \text{convergent} & , \quad |q|<1 \\ \text{divergent} & , \quad |q|\geq 1 \end{cases} \tag{12}$$

Hence $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)^{-1} \in (\ell_p,\ell_p)$ if and only if $|q|<1$ from (6) and (12). Thus λ is spectral value while

$$\lambda \in \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \leq |b_0|\,|b_1|\,|b_2|\}.$$

So $\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = S$. □

Theorem 2.9. $\sigma_c(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \partial S$

Proof. Owing to $\sigma(T,X)$ is the disjoint union of $\sigma_p(T,X)$, $\sigma_r(T,X)$ and $\sigma_c(T,X)$, thence

$$\sigma_c(T,X) = \sigma(T,X) \backslash (\sigma_p(T,X) \cup \sigma_r(T,X)).$$

By Theorem 2.5 and Theorem 2.7 we get required result. □

3. Subdivision of the Spectrum

The spectrum $\sigma(T,X)$ is partitioned into three sets which are not necessarily disjoint as follows:

a) The set $\sigma_{ap}(T,X) := \{\zeta \in \mathbb{C}$:there exists a Weyl sequence for $T-\zeta I\}$ the approximate point spectrum of T. Herein if there exists a sequence (x_n) in X such that $\|x_n\|=1$ and $\|Tx_n\| \to 0$ as $n \to \infty$ then (x_n) is called Weyl sequence for T.

b)The set $\sigma_\delta(T,X) := \{\zeta \in \sigma(T,X) : T-\zeta I$ is not surjective$\}$ is called defect spectrum of T.

c)The set $\sigma_{co}(T,X) = \{\zeta \in \mathbb{C} : \overline{R(T-\zeta I)} \neq X\}$ is called compression spectrum in the literature.

Proposition 3.1 ([1], p. 28)**.** *The spectrum and the subspectrum of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*) = \sigma(T)$,
(b) $\sigma_c(T^*) \subseteq \sigma_{ap}(T)$,
(c) $\sigma_{ap}(T^*) = \sigma_\delta(T)$,
(d) $\sigma_\delta(T^*) = \sigma_{ap}(T)$,
(e) $\sigma_p(T^*) = \sigma_{co}(T)$,
(f) $\sigma_{co}(T^*) \supseteq \sigma_p(T)$,
(g) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*) = \sigma_p(T) \cup \sigma_{ap}(T^*)$.

Goldberg's Classification of Spectrum

If $T \in B(X)$, then there are three cases for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$, (III) $\overline{R(T)} \neq X$
and three cases for T^{-1}:

(1) T^{-1} exists, bounded, (2) T^{-1} exists, unbounded, (3) T^{-1} doesn't exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [9]).

$\sigma(T,X)$ can be divided into subdivisions $I_2\sigma(T,X) = \emptyset$, $I_3\sigma(T,X)$, $II_2\sigma(T,X)$, $II_3\sigma(T,X)$, $III_1\sigma(T,X)$, $III_2\sigma(T,X)$, $III_3\sigma(T,X)$. For example, if $T_\zeta = \zeta I - T$ is in a given state, III_1 (say), then we write $\zeta \in III_1\sigma(T,X)$.

A table was created in [4] with the help of the above definitions. According to this table, some results are as follows;
a) $\sigma_p(T,X) = I_3\sigma(T,X) \cup II_3\sigma(T,X) \cup III_3\sigma(T,X)$,
b) $\sigma_r(T,X) = III_1\sigma(T,X) \cup III_2\sigma(T,X)$,
c) $\sigma_{ap}(T,X) = \sigma(T,X)\backslash III_1\sigma(T,X)$,
d) $\sigma_\delta(T,X) = \sigma(T,X)\backslash I_3\sigma(T,X)$,
e) $\sigma_{co}(T,X) = III_1\sigma(T,X) \cup III_2\sigma(T,X) \cup III_3\sigma(T,X)$.

In chapter 2, decompositions defined by Goldberg were examined. In this section, the definitions given above will be examined. Some studies on this subject are as follows: Subdivisions of spectra for factorable matrices in [4], Subdivisions of spectra for generalized difference operator in [2] were examined. In [14], the fine spectrum of the upper triangular matrix $U(r,0,0,s)$ over the sequence spaces c_0 and c was studied. In [5] partition of the spectra for the generalized difference operator $B(r,s)$ on the sequence space cs was studied, in [6], subdivision of spectra for some lower triangular double-band matrices as operators on c_0 was studied.

Theorem 3.2. *$I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2), \ell_p) = \emptyset$.*

Proof. For $\zeta \in I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2), \ell_p)$, we should show that $U(a_0,a_1,a_2;b_0,b_1,b_2) - \zeta I$ is onto. Let $y = (y_n) \in \ell_p$ be such that

$$(U(a_0,a_1,a_2;b_0,b_1,b_2) - \zeta I)x = y$$

for $x = (x_n)$. Then

$$\begin{aligned}
(a_0 - \zeta)x_0 + b_0x_1 &= y_0 \\
(a_1 - \zeta)x_1 + b_1x_2 &= y_1 \\
(a_2 - \zeta)x_2 + b_2x_3 &= y_2 \\
&\vdots \\
(a_n - \zeta)x_n + b_nx_{n+1} &= y_n
\end{aligned}$$

Calculating x_k, we get
(13)
$$x_n = \frac{1}{b_{n-1}}\left(y_{n-1} + \sum_{k=0}^{n-2} y_k \prod_{u=1}^{n-k-1} \frac{\zeta - a_{n-u}}{b_{n-u-1}}\right) + x_0 \prod_{u=1}^{n} \frac{\zeta - a_{n-u}}{b_{n-u}}, \quad n = 1,2,3,\ldots$$
We have to show that $x = (x_k) \in \ell_p$. $\frac{1}{p} + \frac{1}{q} = 1$ and setting
$$d := \frac{(\zeta - a_0)(\zeta - a_1)(\zeta - a_2)}{b_0 b_1 b_2},$$
since $\prod_{u=1}^{n-k-1} \frac{\zeta - a_{n-u}}{b_{n-u-1}} = Md^{\frac{k-n-1}{3}}$ and $\prod_{u=1}^{n} \frac{\zeta - a_{n-u}}{b_{n-u}} = Nd^{\frac{n}{3}}$, where
$$M = \begin{cases} \dfrac{\zeta - a_2}{b_1} & , \quad n-k = 3t \\ 1 & , \quad n-k = 3t-1 \\ \dfrac{b_2}{\zeta - a_0} & , \quad n-k = 3t-2 \end{cases} \text{ and } N = \begin{cases} 1 & , \quad n = 3t \\ \dfrac{b_1}{\zeta - a_1} & , \quad n = 3t-1 \\ \dfrac{b_1}{\zeta - a_1}\dfrac{b_2}{\zeta - a_2} & , \quad n = 3t-2, \end{cases}$$
are constants then we get
$$x_n = \frac{1}{b_{n-1}} y_{n-1} + \frac{1}{b_{n-1}} \sum_{k=0}^{n-2} y_k M d^{\frac{k-n-1}{3}} + x_0 N d^{\frac{n}{3}}, \quad n = 1,2,3,\ldots.$$

Now suppose $y = (e_{n-1}) = (0,0,\ldots,0,1,0,\ldots)$ then we get
$$x_n = \frac{1}{b_{n-1}}(1+M) + x_0 N d^{\frac{n}{3}}$$
$$x_n = \frac{1}{b_{n-1}}(1+M) + x_0 N d^3 \longrightarrow \frac{1}{\lim_{n\longrightarrow\infty} b_{n-1}}(1+M) \neq 0.$$
Hence $\sum |x_n|^p$ divergent so $(x_n) \notin \ell_p$. Therefore ζ doesn't satisfies Golberg's condition I. So $I\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), \ell_p) = \emptyset$. □

Corollary 3.3. $II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \mathring{S}$.

Proof. $(U(a_0,a_1,a_2;b_0,b_1,b_2) - \zeta I)^*$ is injective from Theorem 2.6. So from Lemma 2.3 $U(a_0,a_1,a_2;b_0,b_1,b_2)_\zeta$ has a dense range. Thus for
$$\zeta \in \sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p),$$
$\zeta \in I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$ or $\zeta \in II\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$.
$\zeta \in II\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$ is gotten from Theorem 3.2.
Also, if $|\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| < |b_0|\,|b_1|\,|b_2|$, then
$$\zeta \in 3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$$
from Theorem 3.2. Hence $II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \mathring{S}$ □

Corollary 3.4.

$III_1\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = III_2\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \emptyset.$

Proof. Owing to $\sigma_r(T,X) = III_1\sigma(T,X) \cup III_2\sigma(T,X)$, the required result is obtained by Theorem 2.7. □

Corollary 3.5.

$$I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = III_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \emptyset.$$

Proof. Owing to $\sigma_p(T,X) = I_3\sigma(T,X) \cup II_3\sigma(T,X) \cup III_3\sigma(T,X)$, the required result is obtained by Theorem 2.5 and Corollary 3.3. □

Theorem 3.6. *The following spectral decompositions are valid:*
(a)

$$\begin{aligned}&\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)\\&= \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}\,.\end{aligned}$$

(b)

$$\begin{aligned}&\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)\\&= \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}\,.\end{aligned}$$

(c) $\sigma_{co}(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p) = \emptyset$.

Proof. (a) Owing to $\sigma_{ap}(T,X) = \sigma(T,X)\backslash III_1\sigma(T,X)$,

$$\begin{aligned}&\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)\\&= \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}\end{aligned}$$

from Corollary 3.4.

(b) Owing to

$$\sigma_\delta(T,X) = \sigma(T,X)\backslash I_3\sigma(T,X),$$

using Theorem 2.8 and 3.2, the required result is gotten.

(c) By Proposition 3.1 (e), we obtain

$$\sigma_p(T^*,X^*) = \sigma_{co}(T,X).$$

Using Theorem 2.6, the required result is gotten. □

Corollary 3.7. *The following spectral decompositions are valid:*
(a)

$$\begin{aligned}&\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,\ell_p^* \cong l_q)\\&= \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}\,.\end{aligned}$$

(b)

$$\begin{aligned}&\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,\ell_p^* \cong l_q)\\&= \{\zeta \in \mathbb{C} : |\zeta - a_0|\,|\zeta - a_1|\,|\zeta - a_2| \le |b_0|\,|b_1|\,|b_2|\}\,.\end{aligned}$$

Proof. By Proposition 3.1 (c) and (d), we obtain

$$\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,\ell_p^*\cong l_q)=\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$$

and

$$\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,\ell_p^*\cong l_q)=\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p).$$

from Theorem 3.6 (a) and (b), the required results are gotten. □

4. Results

We can generalize our operator

$$U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})$$
$$=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix},$$

where $b_0,b_1,\ldots,b_{n-1}\neq 0$.

Similar to the results in the previous sections, the spectrum and fine spectrum of the n-repeated double band matrix are as follows.

Theorem 4.1. *The following are valid, where*

$$M=\left\{\zeta\in\mathbb{C}:\prod_{k=0}^{n-1}\left|\frac{\zeta-a_k}{b_k}\right|\leq 1\right\},$$

$\mathring{M}$ *is the interior of the set M, and ∂M is the boundary of the set M:*

1. $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),\ell_p)=\mathring{M}$.
2. $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})^*,\ell_p^*\cong\ell_q)=\emptyset$.
3. $\sigma_r(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),\ell_p)=\emptyset$.
4. $\sigma_c(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),\ell_p)=\partial M$.
5. $\sigma(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),\ell_p)=M$.

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