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ROTATIONAL HYPERSURFACES CONSTRUCTED BY DOUBLE ROTATION IN FIVE DIMENSIONAL EUCLIDEAN SPACE $\mathbb{E}^5$

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Abstract. We introduce the rotational hypersurface $\mathbf{x} = \mathbf{x}(u, v, s, t)$ constructed by double rotation in five dimensional Euclidean space $\mathbb{E}^5$. We reveal the first and the second fundamental form matrices, Gauss map, shape operator matrix of $\mathbf{x}$. Additionally, defining the i-th curvatures of any hypersurface via Cayley-Hamilton theorem, we compute the curvatures of the rotational hypersurface $\mathbf{x}$. We give some relations of the mean and Gauss-Kronecker curvatures of $\mathbf{x}$. In addition, we reveal $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is the 5×5 matrix in $\mathbb{E}^5$.

1. Introduction

The ruled (helicoidal) and rotational characters is related by Bour theorem in [6]. Do Carmo and Dajczer [14] studied the helicoidal surfaces by using Bour in Euclidean 3-space $\mathbb{E}^3$. Dillen, Pas, and Verstraelen [13] focused on the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3, 3)$, and $B \in Mat(3, 1)$, are the minimal surfaces, spheres, and circular cylinders.

Cheng and Yau [12] considered the hypersurfaces with constant scalar curvature. Lawson [26] gave the minimal submanifolds and indicated the general definition of the Laplace–Beltrami operator.

Chen [7, 8, 9, 10] studied the submanifolds of the finite-type whose immersion is into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian. Chen et al. [11] conducted an extensive investigation into 1-type submanifolds and submanifolds characterized by 1-type Gauss maps over the past four decades.

Moore [28, 29] considered the general rotational surfaces. Ganchev and Milousheva [15] gave the analogue of these surfaces in the Minkowski 4-space. Hasanis and Vlachos [25] studied the hypersurfaces with harmonic mean curvature vector field. Arslan et al. [1] considered the Vranceanu surface having pointwise 1-type Gauss map. Magid, Scharlach, and Vrancken [27] introduced

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the affine umbilical surfaces; Scharlach [30] studied the affine geometry of surfaces and hypersurfaces. Arslan et al. [2] worked the generalized rotational surfaces. Arslan et al [3] studied the tensor product surfaces having pointwise 1-type Gauss map. Güler, Magid, and Yaylı [19] introduced the helicoidal hypersurfaces in $\mathbb{E}^4$. Güler, Hacısalihoğlu, and Kim [18] served Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in $\mathbb{E}^4$. Güler [17] obtained the rotational hypersurfaces having $\Delta^I R = AR$, where $A \in Mat(4,4)$ in $\mathbb{E}^4$. He [16] also worked the fundamental form IV and the curvature formulas of the hypersphere in $\mathbb{E}^4$.

Arslan, Sütveren, and Bulca [5] considered the rotational λ-hypersurfaces in Euclidean spaces. Güler, Yaylı, and Hacısalihoğlu [20, 21, 22, 23] served the bi-rotational hypersurfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_2$, respectively.

In this work, we consider the rotational hypersurface $\mathbf{x} = \mathbf{x}(u, v, s, t)$ in Euclidean 5-space $\mathbb{E}^5$. We give some notions of five dimensional Euclidean geometry in Section 2. We reveal the first and the second fundamental form matrices, Gauss map, shape operator matrix of any hypersurface in $\mathbb{E}^5$. In Section 3, we give the definition of a rotational hypersurface in $\mathbb{E}^5$. Moreover, in Section 4, defining the i-th curvatures of any hypersurface via Cayley-Hamilton theorem, we give the curvature formulas, and compute the curvatures of the rotational hypersurface $\mathbf{x}$. We give some relations for the mean and Gauss-Kronecker curvatures of $\mathbf{x}$. In addition, in the last section, we prove the following main theorems:

Theorem 1.1. *The Laplace–Beltrami operator of the rotational hypersurface*

$$\mathbf{x}(u, v, s, t) = (f(u, v)\cos s, f(u, v)\sin s, g(u, v)\cos t, g(u, v)\sin t, h(u, v))$$

is given by $\Delta \mathbf{x} = 4\mathcal{K}_1 \mathbf{G}$, *where* $\mathcal{K}_1$ *denotes the mean curvature,* $\mathbf{G}$ *represents the Gauss map of* $\mathbf{x}$.

Theorem 1.2. *Let* $\mathbf{x} : M^4 \subset \mathbb{E}^4 \to \mathbb{E}^5$ *be an immersion given by*

$$\mathbf{x}(u, v, s, t) = (f(u, v)\cos s, f(u, v)\sin s, g(u, v)\cos t, g(u, v)\sin t, h(u, v)).$$

There exists a matrix $\mathcal{A}$ *of order* 5 *such that* $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ *if and only if* $\mathbf{x}$ *has* $\mathcal{K}_1 = 0$, *i.e., it is a minimal hypersurface.*

2. Preliminaries

We introduce the first and second fundamental forms, Gauss map $\mathbf{G}$, the shape operator matrix $\mathbf{S}$, curvature formulas: the mean curvature $\mathcal{K}_1$, and the Gauss-Kronecker curvature $\mathcal{K}_4$ of a hypersurface $\mathbf{x} = \mathbf{x}(u, v, s, t)$ in Euclidean 5-space $\mathbb{E}^5$. We identify a vector $\overrightarrow{\alpha}$ with its transpose in this work.

We assume that $\mathbf{x} = \mathbf{x}(u, v, s, t)$ is an immersion from $M^4 \subset \mathbb{E}^4$ to $\mathbb{E}^5$.

Definition 2.1. *A Euclidean dot product of* $\overrightarrow{x^1} = (x_1^1, ..., x_5^1)$, $\overrightarrow{x^2} = (x_1^2, ..., x_5^2)$ *of* $\mathbb{E}^5$ *is given by*

$$\overrightarrow{x^1} \cdot \overrightarrow{x^2} = \sum_{i=1}^{5} x_i^1 x_i^2.$$

Definition 2.2. *A quadruple vector product of* $\overrightarrow{x^1}, ..., \overrightarrow{x^4}$ *of* $\mathbb{E}^5$ *is defined by*

$$\overrightarrow{x^1} \times \overrightarrow{x^2} \times \overrightarrow{x^3} \times \overrightarrow{x^4} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 & x_5^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \end{pmatrix},$$

where e_i, $i = 1, ..., 5$, *are the base elements of* $\mathbb{E}^5$.

Definition 2.3. *For a hypersurface* $\mathbf{x}(u, v, s, t)$ *in 5-space, the first and second fundamental form matrices, respectively, are given by*

$$\mathbf{I} = \begin{pmatrix} E & F & A & D \\ F & G & B & J \\ A & B & C & Q \\ D & J & Q & S \end{pmatrix}, \quad \mathbf{II} = \begin{pmatrix} L & M & P & X \\ M & N & T & Y \\ P & T & V & Z \\ X & Y & Z & I \end{pmatrix},$$

with

$$\begin{aligned} \det \mathbf{I} &= \left(EG - F^2\right)\left(CS - Q^2\right) + \left(J^2 - GS\right)A^2 + \left(D^2 - ES\right)B^2 \\ &\quad +2\left((CF - AB)\,DJ + (EB - FA)\,JQ + (GA - FB)\,DQ\right) \\ &\quad -\left(EJ^2 + GD^2\right)C + 2FABS, \\ \det \mathbf{II} &= \left(LN - M^2\right)\left(IV - Z^2\right) + \left(Y^2 - IN\right)P^2 + \left(X^2 - IL\right)T^2 \\ &\quad +2\left((VM - PT)\,XY + (LT - MP)\,YZ + (NP - MT)\,XZ\right) \\ &\quad -\left(LY^2 + NX^2\right)V + 2MIPT. \end{aligned}$$

Here, the components of the matrices are described by

$$\begin{array}{lllll} E = \mathbf{x}_u \cdot \mathbf{x}_u, & F = \mathbf{x}_u \cdot \mathbf{x}_v, & A = \mathbf{x}_u \cdot \mathbf{x}_s, & D = \mathbf{x}_u \cdot \mathbf{x}_t, & G = \mathbf{x}_v \cdot \mathbf{x}_v, \\ B = \mathbf{x}_v \cdot \mathbf{x}_s, & J = \mathbf{x}_v \cdot \mathbf{x}_t, & C = \mathbf{x}_s \cdot \mathbf{x}_s, & Q = \mathbf{x}_s \cdot \mathbf{x}_t, & S = \mathbf{x}_t \cdot \mathbf{x}_t, \\ L = \mathbf{x}_{uu} \cdot \mathbf{G}, & M = \mathbf{x}_{uv} \cdot \mathbf{G}, & P = \mathbf{x}_{us} \cdot \mathbf{G}, & X = \mathbf{x}_{ut} \cdot \mathbf{G}, & N = \mathbf{x}_{vv} \cdot \mathbf{G}, \\ T = \mathbf{x}_{vs} \cdot \mathbf{G}, & Y = \mathbf{x}_{vt} \cdot \mathbf{G}, & V = \mathbf{x}_{ss} \cdot \mathbf{G}, & Z = \mathbf{x}_{st} \cdot \mathbf{G}, & I = \mathbf{x}_{tt} \cdot \mathbf{G}, \end{array}$$

$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$, $\mathbf{x}_{uv} = \frac{\partial^2 \mathbf{x}}{\partial u \partial v}$, $\mathbf{x}_{vv} = \frac{\partial^2 \mathbf{x}}{\partial v^2}$, *etc., and*

$$\mathbf{G} = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_s \times \mathbf{x}_t}{\left\| \mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_s \times \mathbf{x}_t \right\|}$$

is the Gauss map of $\mathbf{x}$.

Definition 2.4. *Computing* $\mathbf{I}^{-1}\mathbf{II}$, *the shape operator matrix* $\mathbf{S} = \frac{1}{\det \mathbf{I}} \begin{pmatrix} s_{ij} \end{pmatrix}_{4\times 4}$ *is found with the following components*

$$\begin{aligned} s_{11} &= AJ^2P - CJ^2L - B^2LS + B^2XD + CJMD - BJPD - BMQD \\ &\quad -CGXD + GPQD + ABMS - ABJX - AJMQ + BJLQ \\ &\quad +BJLQ - CFMS + CGLS - AGPS + BFPS + CFJX \\ &\quad +AGQX - BFQX - FJPQ + FMQ^2 - GLQ^2, \end{aligned}$$

$$\begin{aligned} s_{12} &= AJ^2T - CJ^2M - B^2MS + B^2YD + CJND - BJTD - BNQD \\ &\quad -CGYD + GQTD + ABNS - ABJY - AJNQ + BJMQ \\ &\quad +BJMQ - CFNS + CGMS + CFJY - AGST + BFST \\ &\quad +AGQY - BFQY + FNQ^2 - GMQ^2 - FJQT, \end{aligned}$$

$$\begin{aligned} s_{13} &= AJ^2V - CJ^2P - B^2PS + B^2ZD + CJTD - BJVD - CGZD \\ &\quad -BQTD + GQVD - ABJZ + ABST + BJOP + BJPQ \\ &\quad +CFJZ + CGPS - AJOT - CFST + AGQZ - AGSV \\ &\quad -BFQZ + BFSV - FJQV - GPQ^2 + FTQ^2, \end{aligned}$$

$$\begin{aligned} s_{14} &= AJ^2Z - CJ^2X - B^2SX + B^2DI - BJZD + CJYD - BQYD \\ &\quad +GQZD - ABJI + CFJI + AGQI - BFQI - CGDI \\ &\quad +ABSY - AJQY + BJQX - AGSZ + BFSZ + BJQX \\ &\quad -CFSY + CGSX - FJQZ + FQ^2Y - GQ^2X, \end{aligned}$$

$$\begin{aligned} s_{21} &= -A^2MS + A^2JX - CMD^2 + BPD^2 + CJLD - ABXD \\ &\quad -AJPD + AMQD - BLQD + AMQD + CFXD + CMSE \\ &\quad -BPSE - CJXE - FPQD + BQXE + JPQE - MQ^2E \\ &\quad +ABLS - AJLQ - CFLS + AFPS - AFQX + FLQ^2, \end{aligned}$$

$$\begin{aligned} s_{22} &= A^2JY - A^2NS - CND^2 + BTD^2 + CJMD - ABYD \\ &\quad +ANQD - BMQD - AJTD + ANQD + CFYD + CNSE \\ &\quad -CJYE - BSTE + BQYE - FQTD - NQ^2E + JQTE \\ &\quad +ABMS - AJMQ - CFMS + AFST - AFQY + FMQ^2, \end{aligned}$$

$$\begin{aligned} s_{23} &= A^2JZ - A^2ST - CTD^2 + BVD^2 - ABZD + CJPD \\ &\quad -AJVD - BQPD + CFZD + AQTD + AQTD - CJZE \\ &\quad +CSTE + BQZE - BSVE - FQVD + JQVE - Q^2TE \\ &\quad +ABPS - AJPQ - CFPS - AFQZ + AFSV + FPQ^2, \end{aligned}$$

$$
\begin{aligned}
s_{24} &= -A^2SY+A^2JI-CYD^2+BZD^2-AJZD+CJXD \\
&\quad +AQYD-BQXD+AQYD-BSZE+CSYE-FQZD \\
&\quad +JQZE-Q^2YE-AFQI-ABDI+CFDI-CJEI \\
&\quad +BQEI+ABSX+AFSZ-AJQX-CFSX+FQ^2X,
\end{aligned}
$$

$$
\begin{aligned}
s_{31} &= AJ^2L-F^2PS+F^2QX+BMD^2-GPD^2-J^2PE-AJMD \\
&\quad -BJLD+AGXD-BFXD+2FJPD-FMQD+GLQD \\
&\quad -BMSE+BJXE+JMQE+GPSE-GQXE+AFMS \\
&\quad -AGLS+BFLS-AFJX-FJLQ,
\end{aligned}
$$

$$
\begin{aligned}
s_{32} &= AJ^2M-F^2ST+F^2OY+BND^2-GTD^2-J^2TE-AJND \\
&\quad -BJMD+AGYD-BFYD-FNQD+GMQD-BNSE \\
&\quad +2FJTD+BJYE+JNQE+GSTE-GQYE+AFNS \\
&\quad -AGMS+BFMS-AFJY-FJMQ,
\end{aligned}
$$

$$
\begin{aligned}
s_{33} &= AJ^2P+F^2QZ-F^2SV+BTD^2-GVD^2-J^2VE-BJPD \\
&\quad -AJTD+AGZD-BFZD+2FJVD+GQPD+BJZE \\
&\quad -FQTD-BSTE+JQTE-GQZE+GSVE-AFJZ \\
&\quad -AGPS+BFPS+AFST-FJQP,
\end{aligned}
$$

$$
\begin{aligned}
s_{34} &= AJ^2X-F^2SZ+BYD^2-GZD^2-J^2ZE+F^2QI-AJYD \\
&\quad -BJXD+2FJZD-FQYD+GQXD-BSYE+JQYE \\
&\quad +GSZE-AFJI+AGDI-BFDI+BJEI-GQEI \\
&\quad +AFSY-AGSX+BFSX-FJQX,
\end{aligned}
$$

$$
\begin{aligned}
s_{41} &= A^2JM-A^2GX-CF^2X+F^2PQ+B^2LD-B^2XE-ABMD \\
&\quad +CFMD-CGLD+AGPD-BFPD-CJME+BJPE \\
&\quad +BMQE+CGXE-GPQE-ABJL+CFJL+2ABFX \\
&\quad -AFJP-AFMQ+AGLQ-BFLQ,
\end{aligned}
$$

$$
\begin{aligned}
s_{42} &= A^2JN-A^2GY-CF^2Y+F^2QT+B^2MD-B^2YE-ABND \\
&\quad +CFND-CGMD-CJNE+AGTD-BFTD+BJTE \\
&\quad +BNQE+CGYE-GQTE-ABJM+CFJM+2ABFY \\
&\quad -AFJT-AFNQ+AGMQ-BFMQ,
\end{aligned}
$$

$$
\begin{aligned}
s_{43} &= A^2JT-A^2GZ-CF^2Z+F^2QV+B^2PD-B^2ZE-ABTD \\
&\quad -CGPD+CFTD+AGVD-BFVD-CJTE+BJVE \\
&\quad +CGZE+BQTE-GQVE-ABJP+2ABFZ+CFJP \\
&\quad -AFJV+AGPQ-BFPQ-AFQT,
\end{aligned}
$$

$$\begin{aligned} s_{44} &= A^2JY + F^2QZ + B^2XD - A^2GI - CF^2I - B^2EI - ABYD \\ &\quad +AGZD - BFZD + CFYD - CGXD + BJZE - CJYE \\ &\quad +BQYE - GQZE + 2ABFI + CGEI - ABJX - AFJZ \\ &\quad +CFJX - AFQY + AGQX - BFQX. \end{aligned}$$

Definition 2.5. *The formulas of the mean and the Gauss-Kronecker curvatures, respectively, are given by*

$$\mathcal{K}_1 = \frac{1}{4} tr(\mathbf{S}), \tag{1}$$

and

$$\mathcal{K}_4 = \det(\mathbf{S}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}, \tag{2}$$

where

$$\begin{aligned} tr(\mathbf{S}) &= (EN + GL - 2FM)\left(CS - Q^2\right) + \left(EG - F^2\right)(SV + IC) \\ &\quad - (GI + NS)A^2 - (LS + EI)B^2 - (CN + GV)D^2 - (EV + CL)J^2 \\ &\quad +2(A^2JY + B^2XD + D^2BT + J^2AP + F^2QZ + CJMD - ABYD \\ &\quad -BJPD + ANQD - AJTD - BMQD + AGZD - BFZD + CFYD \\ &\quad -AGPS - CGXD + FJVD + GQPD + BJZE - CJYE + BFPS \\ &\quad -BSTE - FQTD + BQYE + JQTE + AGQX - BFQX - GQZE \\ &\quad +ABFI - FJPQ + AFST - AFQY + ABMS - ABJX - AJMQ \\ &\quad +BJLQ + CFJX - AFJZ)/\det \mathbf{I}, \end{aligned}$$

and

$$\begin{aligned} \det \mathbf{I} &= \left(EG - F^2\right)\left(CS - Q^2\right) + \left(J^2 - GS\right)A^2 + \left(D^2 - ES\right)B^2 \\ &\quad +2\left((CF - AB)DJ + (EB - FA)JQ + (GA - FB)DQ\right) \\ &\quad - \left(EJ^2 + GD^2\right)C + 2FABS, \\ \det \mathbf{II} &= \left(LN - M^2\right)\left(IV - Z^2\right) + \left(Y^2 - IN\right)P^2 + \left(X^2 - IL\right)T^2 \\ &\quad +2\left((VM - PT)XY + (LT - MP)YZ + (NP - MT)XZ\right) \\ &\quad - \left(LY^2 + NX^2\right)V + 2MIPT. \end{aligned}$$

A hypersurface $\mathbf{x}$ *is* j*-minimal if* $\mathcal{K}_j = 0$ *identically on* $\mathbf{x}$.

Definition 2.6. *In* $\mathbb{E}^5$, *the curvature formulas* $\mathcal{K}_i$, *where* $i = 0, ..., 4$, *are obtained by the characteristic polynomial of* $\mathbf{S}$*:*

$$\sum_{k=0}^{4} (-1)^k s_k \lambda^{n-k} = P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda \mathcal{I}_4) = 0, \tag{3}$$

$\mathcal{I}_4$ *describes the identity matrix of order* 4. *Hence, we reveal the curvature formulas* $\binom{n}{i}\mathcal{K}_i = s_i$. *Here,* $\binom{4}{0}\mathcal{K}_0 = s_0 = 1$ *(by definition),* $\binom{4}{1}\mathcal{K}_1 = s_1, \ldots, \binom{4}{4}\mathcal{K}_4 = s_4$, *and* $\mathcal{K}_1$ *is the mean curvature,* $\mathcal{K}_4$ *is the Gauss-Kronecker curvature, and* $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

See [24] for details. See also [18, 19, 20, 21] for details of dimension 4.

3. Rotational Hypersurface in $\mathbb{E}^5$

In this section, we define the rotational hypersurface in $\mathbb{E}^5$.

Definition 3.1. *For an open interval $\grave{I}$, let $\gamma : \grave{I} \subset \mathbb{R}^2 \longrightarrow \Pi \subset \mathbb{R}^5$ be a surface in $\mathbb{E}^5$, and let ℓ be a straight line in Π. A rotational hypersurface in $\mathbb{E}^5$ is defined as a hypersurface obtained by rotating a surface (i.e., profile surface γ) around a line (i.e., axis ℓ).*

We may suppose that ℓ is the line spanned by the vector $(0,0,0,0,1)^t$. The rotation matrix $\mathcal{R} = \mathcal{R}(s,t)$ of $\mathbb{E}^5$ is given by

$$\mathcal{R} = \begin{pmatrix} \cos s & -\sin s & 0 & 0 & 0 \\ \sin s & \cos s & 0 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t & 0 \\ 0 & 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad s,t \in [0, 2\pi), \tag{4}$$

where $\mathcal{R}\ell = \ell$, $\mathcal{R}^t\mathcal{R} = \mathcal{R}\mathcal{R}^t = \mathcal{I}_5$, $\det \mathcal{R} = 1$. When the axis of rotation is ℓ, there is a Euclidean transformation by which the axis ℓ is transformed to the x_5-axis of $\mathbb{E}^5$. Parametrization of the profile surface is given by $\gamma(u,v) = (f, 0, g, 0, h)$, where $f, g, h : \grave{I} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are the differentiable functions depending on $u, v \in \grave{I}$. In $\mathbb{E}^5$, the rotational hypersurface $\mathbf{x}$ spanned by the vector $(0,0,0,0,1)$, is given by $\mathbf{x} = \mathcal{R}.\gamma^t$, where $u, v \in \grave{I}$, $s,t \in [0, 2\pi)$. Therefore, the rotational hypersurface is given by

$$\mathbf{x}(u,v,s,t) = (f(u,v)\cos s, f(u,v)\sin s, g(u,v)\cos t, g(u,v)\sin t, h(u,v)). \tag{5}$$

4. Curvatures in $\mathbb{E}^5$

In this section, we obtain the curvature formulas of a hypersurface $\mathbf{x}$ having four parameters in $\mathbb{E}^5$.

Theorem 4.1. *In $\mathbb{E}^5$, a hypersurface $\mathbf{x}$ having four parameters has the following curvature formulas, $\mathcal{K}_0 = 1$ by definition,*

$$4\mathcal{K}_1 = -\frac{\mathfrak{b}}{\mathfrak{a}}, \; 6\mathcal{K}_2 = \frac{\mathfrak{c}}{\mathfrak{a}}, \; 4\mathcal{K}_3 = -\frac{\mathfrak{d}}{\mathfrak{a}}, \; \mathcal{K}_4 = \frac{\mathfrak{e}}{\mathfrak{a}}, \tag{6}$$

where $P_{\mathbf{S}}(\lambda) = \mathfrak{a}\lambda^4 + \mathfrak{b}\lambda^3 + \mathfrak{c}\lambda^2 + \mathfrak{d}\lambda + \mathfrak{e} = 0$ is the characteristic polynomial of shape operator matrix $\mathbf{S}$, $\mathfrak{a} = \det \mathbf{I}$, $\mathfrak{e} = \det \mathbf{II}$, and $\mathbf{I}$, $\mathbf{II}$ are the first, and the second fundamental form matrices, respectively.

Proof. The product matrix $\mathbf{I}^{-1}\mathbf{II}$ supplies the shape operator matrix $\mathbf{S}$ of the hypersurface $\mathbf{x}$ in 5-space. Computing the curvature formula $\mathcal{K}_i$, where $i = 0, 1, \ldots, 4$, we find the characteristic polynomial $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_4) = 0$ of $\mathbf{S}$. Then, we find the following curvatures in 5-space:

$$\begin{aligned}
\binom{4}{0}\mathcal{K}_0 &= 1, \\
\binom{4}{1}\mathcal{K}_1 &= \sum_{i=1}^{4} k_i = -\frac{\mathfrak{b}}{\mathfrak{a}}, \\
\binom{4}{2}\mathcal{K}_2 &= \sum_{1=i_1<i_2}^{4} k_{i_1}k_{i_2} = \frac{\mathfrak{c}}{\mathfrak{a}}, \\
\binom{4}{3}\mathcal{K}_3 &= \sum_{1=i_1<i_2<i_3}^{4} k_{i_1}k_{i_2}k_{i_3} = -\frac{\mathfrak{d}}{\mathfrak{a}}, \\
\binom{4}{4}\mathcal{K}_4 &= \prod_{i=1}^{4} k_i = \frac{\mathfrak{e}}{\mathfrak{a}}.
\end{aligned}$$

Here, k_i, $i = 1, ..., 4$, denote the principal curvatures of the hypersurface $\mathbf{x}$. □

See [16, 18, 19, 20, 21] for case $\mathbb{E}^4$.

Theorem 4.2. *A hypersurface* $\mathbf{x} = \mathbf{x}(u, v, s, t)$ *in* $\mathbb{E}^5$ *has the following relation*

$$\mathcal{K}_0\mathbf{V} - 4\mathcal{K}_1\mathbf{IV} + 6\mathcal{K}_2\mathbf{III} - 4\mathcal{K}_3\mathbf{II} + \mathcal{K}_4\mathbf{I} = 0,$$

where $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathbf{V}$ *are the fundamental form matrices having order* 4×4 *of the hypersurface.*

Proof. Considering $n = 4$ in (3), it is clear. □

Using the first derivatives of (5) with respect to u, v, s, t, we get the following first quantities

$$\mathbf{I} = \begin{pmatrix} f_u^2 + g_u^2 + h_u^2 & f_uf_v + g_ug_v + h_uh_v & 0 & 0 \\ f_uf_v + g_ug_v + h_uh_v & f_v^2 + g_v^2 + h_v^2 & 0 & 0 \\ 0 & 0 & f^2 & 0 \\ 0 & 0 & 0 & g^2 \end{pmatrix}, \tag{7}$$

where $\det \mathbf{I} = f^2g^2W$, $W = \mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2$, $\mathfrak{A} = g_uh_v - g_vh_u$, $\mathfrak{B} = f_ug_v - f_vg_u$, $\mathfrak{C} = f_uh_v - f_vh_u$.

The Gauss map of the rotational hypersurface (5) is described by

$$\mathbf{G} = \frac{1}{W^{1/2}}\left(-\mathfrak{A}\cos s, -\mathfrak{A}\sin s, \mathfrak{C}\cos t, \mathfrak{C}\sin t, -\mathfrak{B}\right). \tag{8}$$

By taking the second derivatives with respect to u, v, s, t, and using them with (8) of $\mathbf{x}$, we have the following second fundamental form matrix

$$\mathbf{II} = \begin{pmatrix} \frac{-\mathfrak{A}f_{uu}+\mathfrak{C}g_{uu}-\mathfrak{B}h_{uu}}{W^{1/2}} & \frac{-\mathfrak{A}f_{uv}+\mathfrak{C}g_{uv}-\mathfrak{B}h_{uv}}{W^{1/2}} & 0 & 0 \\ \frac{-\mathfrak{A}f_{uv}+\mathfrak{C}g_{uv}-\mathfrak{B}h_{uv}}{W^{1/2}} & \frac{-\mathfrak{A}f_{vv}+\mathfrak{C}g_{vv}-\mathfrak{B}h_{vv}}{W^{1/2}} & 0 & 0 \\ 0 & 0 & \frac{f\mathfrak{A}}{W^{1/2}} & 0 \\ 0 & 0 & 0 & \frac{g\mathfrak{C}}{W^{1/2}} \end{pmatrix}.$$

The matrix $\mathbf{I}^{-1}{\cdot}\mathbf{II}$ gives the shape operator matrix $\mathbf{S}$ of the hypersurface $\mathbf{x}$. Then, we compute the mean curvature $\mathcal{K}_1$ and Gauss-Kronecker curvature $\mathcal{K}_4$.

Therefore, the following holds.

Theorem 4.3. *The mean and Gauss-Kronecker curvatures of the rotational hypersurface* (5) *are given by, respectively, as follows*

$$\mathcal{K}_1 = \frac{\left\{ \begin{array}{c} fg\left(f_v^2+g_v^2+h_v^2\right)\left(-\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}-\mathfrak{C}g_{uu}\right) \\ +\left(f\mathfrak{C}+\mathfrak{A}g\right)\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right) \\ -fg\left(f_u^2+g_u^2+h_u^2\right)\left(\mathfrak{A}f_{vv}+\mathfrak{C}g_{vv}+\mathfrak{B}h_{vv}\right) \end{array} \right\}}{4W^{3/2}},$$

$$\mathcal{K}_4 = \frac{f^2g^2\mathfrak{A}\mathfrak{C}\left\{ \begin{array}{c} \left(\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}+\mathfrak{C}g_{uu}\right)\left(\mathfrak{A}f_{vv}+\mathfrak{B}h_{vv}+\mathfrak{C}g_{vv}\right) \\ -\left(\mathfrak{A}f_{uv}+\mathfrak{B}h_{uv}+\mathfrak{C}g_{uv}\right)^2 \end{array} \right\}}{W^3},$$

where $W=\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2$, $\mathfrak{A}=g_uh_v-g_vh_u$, $\mathfrak{B}=f_ug_v-f_vg_u$, $\mathfrak{C}=f_uh_v-f_vh_u$ *and* $f=f(u,v)$, $g=g(u,v)$, $h=h(u,v)$, $f_u=\frac{\partial f}{\partial u}$, $f_{uv}=\frac{\partial^2 f}{\partial u \partial v}$, *etc.*

Proof. By using the Cayley-Hamilton theorem, we reveal the following characteristic polynomial of $\mathbf{S}$:

$$\mathcal{K}_0\lambda^4-4\mathcal{K}_1\lambda^3+6\mathcal{K}_2\lambda^2-4\mathcal{K}_3\lambda+\mathcal{K}_4=0.$$

The curvatures $\mathcal{K}_i$ of the rotational hypersurface $\mathbf{x}$ are also found by the above equation. □

Theorem 4.4. *The rotational hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^5$ *has the umbilical point iff the following holds*

$$\begin{aligned} &\left\{ \begin{array}{c} f^2g^2\left(f_v^2+g_v^2+h_v^2\right)\left(-\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}-\mathfrak{C}g_{uu}\right) \\ +fg\left(\mathfrak{C}f+\mathfrak{A}g\right)\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right) \\ -f^2g^2\left(f_u^2+g_u^2+h_u^2\right)\left(\mathfrak{A}f_{vv}+\mathfrak{C}g_{vv}+\mathfrak{B}h_{vv}\right) \end{array} \right\}^4 \\ &= 256W^3fg\mathfrak{A}\mathfrak{C}\left\{ \begin{array}{c} \left(\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}+\mathfrak{C}g_{uu}\right)\left(\mathfrak{A}f_{vv}+\mathfrak{B}h_{vv}+\mathfrak{C}g_{vv}\right) \\ -\left(\mathfrak{A}f_{uv}+\mathfrak{B}h_{uv}+\mathfrak{C}g_{uv}\right)^2 \end{array} \right\}. \end{aligned}$$

Proof. Hypersurface $\mathbf{x}$ has the umbilical point, then it has the equation $(\mathcal{K}_1)^4=\mathcal{K}_4$. □

Open Problem 4.5. *Find the* $h=h(u,v)$ *solutions of the 2nd-order partial differential equation in Theorem 4.4.*

Corollary 4.6. *Let* $\mathbf{x}: M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$ *be an immersion given by* (5). $\mathbf{x}$ *has zero mean curvature iff the following holds*

$$\begin{gathered}(\mathfrak{C}f+\mathfrak{A}g)\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right)\\ +fg\left(f_v^2+g_v^2+h_v^2\right)\left(-\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}-\mathfrak{C}g_{uu}\right)\\ -fg\left(f_u^2+g_u^2+h_u^2\right)\left(\mathfrak{A}f_{vv}+\mathfrak{C}g_{vv}+\mathfrak{B}h_{vv}\right)=0,\end{gathered}$$

where $f, g, h \neq 0$.

Open Problem 4.7. *Find the* $h = h(u, v)$ *solutions of the 2nd-order partial differential equation in Corollary 4.6.*

Corollary 4.8. *Let* $\mathbf{x}: M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$ *be an immersion given by* (5). $\mathbf{x}$ *has zero Gauss-Kronecker curvature iff the following holds*

$$f^2g^2\mathfrak{A}\mathfrak{C}\left\{\begin{array}{c}(\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}+\mathfrak{C}g_{uu})(\mathfrak{A}f_{vv}+\mathfrak{B}h_{vv}+\mathfrak{C}g_{vv})\\ -(\mathfrak{A}f_{uv}+\mathfrak{B}h_{uv}+\mathfrak{C}g_{uv})^2\end{array}\right\}=0,$$

where $f, g, h \neq 0$.

Open Problem 4.9. *Find the* $h = h(u, v)$ *solutions of the 2nd-order partial differential equation in Corollary 4.8.*

5. Rotational Hypersurface Supplying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}^5$

In this section, we present the proof of the theorems in the Introduction section. We also give the Laplace–Beltrami operator depending on the first fundamental form of a smooth function in $\mathbb{E}^5$. Then, we calculate it by using the rotational hypersurface determined by (5).

Definition 5.1. *The Laplace–Beltrami operator of a smooth function* $\phi = \phi(x^1, x^2, x^3, x^4)\,|_{\mathbf{D}}$ *of class* C^4 *within a constrained domain* $\mathbf{D} \subset \mathbb{R}^4$ *depending on the first fundamental form is the operator defined by*

$$\Delta\phi=\frac{1}{\mathbf{g}^{1/2}}\sum_{i,j=1}^{4}\frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2}\mathbf{g}^{ij}\frac{\partial\phi}{\partial x^j}\right), \tag{9}$$

where $\left(\mathbf{g}^{ij}\right) = (\mathbf{g}_{kl})^{-1}$ *and* $\mathbf{g} = \det(\mathbf{g}_{ij})$.

Hence, the Laplace–Beltrami operator depending on the first fundamental form of the rotational hypersurface $\mathbf{x} = \mathbf{x}(u, v, s, t)$ is given by

(10)

$$\Delta\mathbf{x}=\frac{1}{\mathbf{g}^{1/2}}\left\{\begin{array}{c}\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{11}\frac{\partial\mathbf{x}}{\partial u}\right)+\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{12}\frac{\partial\mathbf{x}}{\partial v}\right)+\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{13}\frac{\partial\mathbf{x}}{\partial s}\right)+\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{14}\frac{\partial\mathbf{x}}{\partial t}\right)\\ +\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{21}\frac{\partial\mathbf{x}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{22}\frac{\partial\mathbf{x}}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{23}\frac{\partial\mathbf{x}}{\partial s}\right)+\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{24}\frac{\partial\mathbf{x}}{\partial t}\right)\\ +\frac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{31}\frac{\partial\mathbf{x}}{\partial u}\right)+\frac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{32}\frac{\partial\mathbf{x}}{\partial v}\right)+\frac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{33}\frac{\partial\mathbf{x}}{\partial s}\right)+\frac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{34}\frac{\partial\mathbf{x}}{\partial t}\right)\\ +\frac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{41}\frac{\partial\mathbf{x}}{\partial u}\right)+\frac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{42}\frac{\partial\mathbf{x}}{\partial v}\right)+\frac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{43}\frac{\partial\mathbf{x}}{\partial s}\right)+\frac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{44}\frac{\partial\mathbf{x}}{\partial t}\right)\end{array}\right\},$$

where

$$
\begin{aligned}
\mathbf{g}^{11} &= \left(-CJ^2 - B^2 S - GQ^2 + 2BJQ + CGS\right)/\mathbf{g}, \\
\mathbf{g}^{12} &= \left(FQ^2 + CJD - BQD + ABS - AJQ - CFS\right)/\mathbf{g} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= \left(AJ^2 - BJD + GQD - AGS + BFS - FJQ\right)/\mathbf{g} = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= \left(B^2 D - CGD - ABJ + CFJ + AGQ - BFQ\right)/\mathbf{g} = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \left(-A^2 S - CD^2 - Q^2 E + 2AQD + CSE\right)/\mathbf{g}, \\
\mathbf{g}^{23} &= \left(BD^2 - AJD - BSE - FQD + JQE + AFS\right)/\mathbf{g} = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= \left(A^2 J - ABD + CFD - CJE + BQE - AFQ\right)/\mathbf{g} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \left(-F^2 S - GD^2 - J^2 E + 2FJD + GSE\right)/\mathbf{g}, \\
\mathbf{g}^{34} &= \left(F^2 Q + AGD - BFD + BJE - GQE - AFJ\right)/\mathbf{g} = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \left(-A^2 G - CF^2 - B^2 E + CGE + 2ABF\right)/\mathbf{g},
\end{aligned}
$$

and $\mathbf{g} = \det \mathbf{I}$. By using the inverse matrix of (7):

$$
\begin{aligned}
\mathbf{g}^{11} &= \frac{f^2 g^2 \left(f_v^2 + g_v^2 + h_v^2\right)}{\det \mathbf{I}}, \\
\mathbf{g}^{12} &= -\frac{f^2 g^2 \left(f_u f_v + g_u g_v + h_u h_v\right)}{\det \mathbf{I}} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= 0 = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= 0 = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \frac{f^2 g^2 \left(f_u^2 + g_u^2 + h_u^2\right)}{\det \mathbf{I}}, \\
\mathbf{g}^{23} &= 0 = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= \frac{\left(f_u f_v + g_u g_v\right) h_u}{\det \mathbf{I}} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \frac{g^2 \left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right)}{\det \mathbf{I}}, \\
\mathbf{g}^{34} &= 0 = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \frac{f^2 \left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right)}{\det \mathbf{I}},
\end{aligned}
$$

and by derivating the functions in (10) with respect to u, v, s, t, respectively, we then present the following proof of Theorem 1.1.

Proof. By directly computing (10), we obtain $\Delta \mathbf{x}$. □

Next, we give the proof of Theorem 1.2.

Proof. We obtain $4\mathcal{K}_1\mathbf{G} = \mathcal{A}\mathbf{x}$, and then we have the following

$$\begin{pmatrix} a_{11}f\cos s + a_{12}f\sin s + a_{13}g\cos t + a_{14}g\sin t + a_{15}h \\ a_{21}f\cos s + a_{22}f\sin s + a_{23}g\cos t + a_{44}g\sin t + a_{25}h \\ a_{31}f\cos s + a_{32}f\sin s + a_{33}g\cos t + a_{34}g\sin t + a_{35}h \\ a_{41}f\cos s + a_{42}f\sin s + a_{43}g\cos t + a_{44}g\sin t + a_{45}h \\ a_{51}f\cos s + a_{52}f\sin s + a_{53}g\cos t + a_{54}g\sin t + a_{55}h \end{pmatrix} = \begin{pmatrix} -\Phi fg\mathfrak{A}\cos s \\ -\Phi fg\mathfrak{A}\sin s \\ \Phi fg\mathfrak{C}\cos t \\ \Phi fg\mathfrak{C}\sin t \\ -\Phi fg\mathfrak{B} \end{pmatrix},$$

where $\mathcal{A}$ is the 5×5 matrix $\Phi = 4\mathcal{K}_1(\det\mathbf{I})^{-1/2}$. Derivativing above ODEs twice with respect to s, we get the following

$$a_{15} = a_{25} = a_{35} = a_{45} = a_{55} = 0,\ \Phi = 0.$$

Then, we have

$$a_{i1}f\cos s + a_{i2}f\sin s = 0,$$

where $i = 1, ..., 5$. The functions sin and cos are linearly independent on s, then all the components of the matrix $\mathcal{A}$ are 0. Since $\Phi = 4\mathcal{K}_1(\det\mathbf{I})^{-1/2}$, then $\mathcal{K}_1 = 0$. This means, $\mathbf{x}$ is a minimal rotational hypersurface with double rotations. □

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