A WEIERSTRASS SEMIGROUP AT A PAIR OF INFLECTION POINTS WITH HIGH MULTIPlicITIES

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Abstract. In the previous paper [4], we classified the Weierstrass semigroups at a pair of inflection points of multiplicities $d$ and $d - 1$ on a smooth plane curve of degree $d$. In this paper, as a continuation of those results, we classify all semigroups each of which arises as a Weierstrass semigroup at a pair of inflection points of multiplicities $d$, $d - 1$ and $d - 2$ on a smooth plane curve of degree $d$.

1. INTRODUCTION AND PRELIMINARIES

Let $C$ be a smooth projective curve of genus $g \geq 2$, $\mathcal{M}(C)$ the field of rational functions on $C$ and $\mathbb{N}_0$ the set of all nonnegative integers.

For a point $P$ on $C$, there are exactly $g$ integers $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_g < 2g$ such that there is no rational function $f$ on $C$ with a pole of order $\alpha_k$ at $P$. The integer $\alpha_k$ is called a gap at $P$ and the sequence $\{\alpha_k | k = 1, 2, \cdots, g\}$ is called as the Weierstrass gap sequence at $P$. By the Riemann-Roch Theorem, we get

$$G(P) = \{\alpha \in \mathbb{N}_0 | \# f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\} = \{\alpha \in \mathbb{N}_0 | \exists \text{ holomorphic differential on } C \text{ of order } \alpha - 1 \text{ at } P\} = \{\alpha \in \mathbb{N}_0 | \exists \text{ canonical divisor on } C \text{ of order } \alpha - 1 \text{ at } P\}$$

where $(f)_\infty$ means the divisor of poles of the rational function $f$. For a smooth plane curve $C$ of degree $d \geq 4$, the canonical series is cut out by the system of all curves of degree $d - 3$. So the order sequence of canonical divisors at $P$ can be obtained as the set $\{I(C \cap f_{d-3}, P) | f_{d-3} \text{ is a polynomial of degree } d - 3\}$.

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We call that $P$ is a Weierstrass point if $G(P) \neq \{1, 2, \cdots, g\}$ or equivalently the order sequence of canonical divisors at $P$ is not $\{0, 1 \rightarrow g - 1\}$. There are only finite number of Weierstrass points on $C$, which means that the order sequence of canonical divisors at a point is exactly $\{0, 1 \rightarrow g - 1\}$ except for a finite number of points.

The non-gaps at $P$ form a semigroup under addition and we call it as the Weierstrass semigroup $H(P)$. So $H(P) = \mathbb{N}_0 \setminus G(P) = \{\alpha \in \mathbb{N}_0 | \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}$. We extend the Weierstrass semigroup at $P$ to a Weierstrass semigroup at two distinct points $P, Q \in C$ as $H(P, Q) = \{(\alpha, \beta) \in \mathbb{N}_0^2 | \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}$ and let $G(P, Q) = \mathbb{N}_0^2 \setminus H(P, Q)$.

As the cardinality of the set $G(P)$ is finite, in fact exactly $g$, the set $G(P, Q)$ is also finite, but its cardinality is dependent on the points $P$ and $Q$. In [5], the first author proved that the upper and lower bound of such sets are given as $\left(\frac{g+2}{2}\right) - 1 \leq \text{card } G(P, Q) \leq \left(\frac{g+2}{2}\right) - 1 - g + g^2$, and that $H(P, Q)$ induces a bijection $\sigma = \sigma(P, Q)$ between $G(P)$ and $G(Q)$ which is defined by $\sigma(\alpha) = \beta, := \min\{\beta \mid (\alpha, \beta) \in H(P, Q)\}$. Homma [2] obtained the same formula for the cardinality of $G(P, Q)$ using the cardinality of the set $\{(\alpha, \alpha') \mid \alpha, \alpha' \in G(P), (\alpha - \alpha')\sigma(\alpha) - \sigma(\alpha') < 0\}$ i.e., the set of pairs $(\alpha, \alpha')$ which are reversed by $\sigma$. We use the following notations;

$$\Gamma = \Gamma(P, Q) := \{(\alpha, \beta) \mid \alpha \in G(P)\} = \{(p_i, q_{\sigma(i)}) \mid i = 1, 2, \cdots, g\},$$

$$\tilde{\Gamma} = \tilde{\Gamma}(P, Q) := \Gamma(P, Q) \cup (H(P) \times \{0\}) \cup (\{0\} \times H(Q)).$$

The above set $\Gamma(P, Q)$ is called the generating subset of the Weierstrass semigroup $H(P, Q)$. Indeed, for given distinct points $P$ and $Q$, the set $\Gamma(P, Q)$ determines not only $\Gamma(P, Q)$ but also the sets $H(P, Q)$ and $G(P, Q)$ completely, as described below. We use the natural partial order on the set $\mathbb{N}_0^2$ as $(\alpha, \beta) \geq (\gamma, \delta)$ if and only if $\alpha \geq \gamma$ and $\beta \geq \delta$. Also we define the least upper bound of two elements $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ is defined as lub$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} = (\max\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\})$. In [5] and [6], the following are proved: (1) The subset $H(P, Q)$ of $\mathbb{N}_0^2$ is closed under the lub(least upper bound) operation. (2) Every element of $H(P, Q)$ is expressed as the lub of one or two elements of the set $\Gamma(P, Q)$. (3) The set $G(P, Q) = \mathbb{N}_0^2 \setminus H(P, Q)$ is expressed as $G(P, Q) = \bigcup_{l \in G(P)} \{(l, \beta) \mid \beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l)) \mid \alpha = 0, 1, \ldots, l - 1\}$.

We can characterize the elements of $\Gamma(P, Q)$ and $H(P, Q)$ using the dimensions of divisors. We denote dim$(\alpha, \beta) := \dim |\alpha P + \beta Q|$, the dimension of the complete linear series $|\alpha P + \beta Q|$.  

Lemma 1.1. For $\alpha \geq 1$ and $\beta \geq 1$, the pair $(\alpha, \beta)$ is an element of $\Gamma(P, Q)$ [resp. $H(P, Q)$] if and only if

$$\dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha - 1, \beta - 1) + 1$$

[resp. $\dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha - 1, \beta - 1) + 1$].

Proof. See [3]. □

Theorem 1.2. Let $m \geq 1$, $m' \geq 0$, $n' \geq n \geq 1$ and $a \geq 0$ be integers. Suppose that $\dim(s + m, t - n) = \dim(s, t) + a$ for all $s \geq m'$, $t \geq n'$. Let $\alpha \geq m' + 1$ and $\beta \geq n' + 1$. Then $(\alpha + m, \beta - n) \in \Gamma(P, Q)$ [resp. $(\alpha + m, \beta - n) \in H(P, Q)$] if and only if $(\alpha, \beta) \in \Gamma(P, Q)$ [resp. $(\alpha, \beta) \in H(P, Q)$].

Proof. It follows from Lemma 1.1. □

Theorem 1.3. Suppose that $mP$ is linearly equivalent to $mQ$. If $(\alpha, \beta), (\alpha + m, \beta') \in \Gamma(P, Q)$, then $\beta' = \beta - m$.

Proof. It follows from Theorem 1.2. □

When we prove the existence of a smooth plane curve with aligned inflection points of given intersection multiplicities, we use the following theorem. Here $\mathbb{P}_d$ denotes the set of all smooth plane curves of degree $d$, and $i(T, C; P)$ denotes the intersection multiplicity of two curves $T$ and $C$ at the point $P$.

Theorem 1.4 ([1]). Fix a line $L$ in $\mathbb{P}^2$ and different points $P_0, P_1, \ldots, P_{d-e}$ on $L$ with integers $0 \leq e \leq d$. Fix lines $T_1, \ldots, T_{d-e}$ passing through $P_1, \ldots, P_{d-e}$ different from $L$. For a sequence $m = (m_1, \ldots, m_{d-e})$ with $d \geq m_1 \geq \cdots \geq m_{d-e}$, let

$$\mathcal{P}_{(e, m)} = \{C \in \mathbb{P}_d | C \text{ is smooth, } i(L, C; P_0) = e, \quad i(T_j, C; P_j) = m_j \text{ for } 1 \leq j \leq d - e\}.$$  

Then $\mathcal{P}_{(e, m)}$ is not empty if and only if the following condition holds:

For every $j$, $1 \leq j < d - e$, if $m_{j+1} < m_j$ then $m_{j+1} \leq d - j$.

Let $C$ be a smooth plane curve of degree $d \geq 4$ and $P$ a point on $C$. From now on, $T_PC$ denotes the tangent line to $C$ at a point $P \in C$ and $T_PC \cdot C$ denotes the divisor on $C$ cut out by the line $T_PC$. Also we use the notation $i_PC = i(T_PC, C; P)$ to denote the intersection multiplicity of the tangent line and $C$ at $P$ on $C$, which satisfies that $2 \leq i_PC \leq d$. Recall that an inflection point $P$ of a curve $C$ means a simple point with $i_PC \geq 3$. 

In [4], we completed the classification of the Weierstrass semigroups each of which occurs at a pair of inflection points $P, Q$ with $i_P C \geq d - 1$ and $i_Q C \geq d - 1$.

In this paper, we will complete the classification of the Weierstrass semigroups at pairs $(P, Q)$ with $i_P C \geq d - 2$ and $i_Q C \geq d - 2$. We find all candidates of the Weierstrass semigroups at such a pair, and then prove the existence of curves and points having such semigroups as their Weierstrass semigroups.

Considering the results of [4], we only need to deal with the following cases:

1. $i_P C = d$ and $i_Q C = d - 2$.
2. $i_P C = d - 1$ and $i_Q C = d - 2$.
3. $i_P C = d - 2$ and $i_Q C = d - 2$.

Recall that, for a point $P$ with $i_P C \geq d - 2$, the Weierstrass gap sequence $G(P)$ at $P$ is uniquely determined as:

$$G(P) = \bigcup_{k=0}^{d-3} \{ k(d-t) + r \mid r = 1, \ldots, d - 2 - k \}, \quad t = 0, 1, 2,$$

where $i_P C = d - t$ (See [1]). In the following sections, to obtain $\Gamma(P, Q)$, we find a bijection between $G(P)$ and $G(Q)$. To do so, it is convenient to arrange the numbers of $G(P)$ in a triangle shape as follows:

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
1 & 2 & 3 & \cdots & \cdots & d - 3 & d - 2 \\
2 + (d - 1) & 3 + (d - 1) & \cdots & \cdots & d - 3 + (d - 1) & d - 2 + (d - 1) \\
3 + 2(d - 1) & \cdots & \cdots & \cdots & d - 3 + 2(d - 1) & d - 2 + 2(d - 1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & d - 3 + (d - 4)(d - 1) & d - 2 + (d - 4)(d - 1) \\
& & & & & d - 2 + (d - 3)(d - 1) & \\
\end{tabular}
\end{table}

Table 1. $G(P)$ with $i_P C = d$

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
1 & 2 & 3 & \cdots & \cdots & d - 3 & d - 2 \\
1 + d & 2 + d & 3 + d & \cdots & \cdots & d - 3 + d & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
1 + (d - 4)d & 2 + (d - 4)d & & & & & \\
1 + (d - 3)d & & & & & & \\
\end{tabular}
\end{table}

Table 2. $G(P)$ with $i_P C = d$

Even though the shapes of arrays are different, we notice that (the set of numbers in Table 1) = (the set of numbers in Table 2), (the set of numbers in Table 3) = (the set of numbers in Table 4), (the set of numbers in Table 5) = (the set of numbers in Table 6).
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\[ 1 \quad 2 \quad 3 \quad \cdots \quad \cdots \quad d - 3 \quad d - 2 \]
\[ 2 + (d - 2) \quad 3 + (d - 2) \quad \cdots \quad \cdots \quad d - 3 + (d - 2) \quad d - 2 + (d - 2) \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ d - 3 + (d - 4)(d - 2) \quad d - 2 + (d - 4)(d - 2) \quad d - 2 + (d - 3)(d - 2) \]

Table 3. \( G(P) \) with \( i_P C = d - 1 \)

\[ 1 \quad 2 \quad 3 \quad \cdots \quad \cdots \quad d - 3 \quad d - 2 \]
\[ 1 + (d - 1) \quad 2 + (d - 1) \quad 3 + (d - 1) \quad \cdots \quad \cdots \quad d - 3 + (d - 1) \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ 1 + (d - 4)(d - 1) \quad 2 + (d - 4)(d - 1) \quad 2 + (d - 3)(d - 1) \]

Table 4. \( G(P) \) with \( i_P C = d - 1 \)

\[ 1 \quad 2 \quad 3 \quad \cdots \quad \cdots \quad d - 3 \quad d - 2 \]
\[ 2 + (d - 3) \quad 3 + (d - 3) \quad \cdots \quad \cdots \quad d - 3 + (d - 3) \quad d - 2 + (d - 3) \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ d - 3 + (d - 4)(d - 3) \quad d - 2 + (d - 4)(d - 3) \quad d - 2 + (d - 3)(d - 3) \]

Table 5. \( G(P) \) with \( i_P C = d - 2 \)

\[ 1 \quad 2 \quad 3 \quad \cdots \quad \cdots \quad d - 3 \quad d - 2 \]
\[ 1 + (d - 2) \quad 2 + (d - 2) \quad 3 + (d - 2) \quad \cdots \quad \cdots \quad d - 3 + (d - 2) \]
\[ 1 + 2(d - 2) \quad 2 + 2(d - 2) \quad 3 + 2(d - 2) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ 1 + (d - 4)(d - 2) \quad 2 + (d - 4)(d - 2) \quad 2 + (d - 3)(d - 2) \]

Table 6. \( G(P) \) with \( i_P C = d - 2 \)

2. At a Pair \((P, Q)\) with \( i_P C = d \) and \( i_Q C = d - 2 \)

Let \( i_P C = d \) and \( i_Q C = d - 2 \). Then we have \( T_{Q} C \cdot C = dP \) and \( T_{Q} C \cdot C = (d - 2)Q + R_1 + R_2 \) for some (not necessarily distinct) points \( R_1, R_2 \) different from \( Q \). There are two possibilities: either \( \{R_1, R_2\} \) contains \( P \) or not. If \( \{R_1, R_2\} \) contains \( P \), then \( T_{Q} C \cdot C = (d - 2)Q + P + R \) with \( R \neq P, Q \), since \( T_P C \neq T_Q C \).

Case 2-1. \( T_{Q} C \cdot C = (d - 2)Q + P + R \) with \( R \neq P, Q \)
In this case, we have \(|dP| = |(d-2)Q + P + R|\), which is the linear series cut out by the system of lines. Thus \(|(d-1)P| = |(d-2)Q + R|\), which we donote \((d-1)P \sim (d-2)Q + R\).

**Theorem 2.1.** (i) For \(\alpha \geq 0, \beta \geq d - 2\),
\[
\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1.
\]
(ii) For \(\alpha \geq 1, \beta \geq d - 1\),
\[
(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).
\]
(iii) Such a curve and points exist.

**Proof.** Since \((d-1)P \sim (d-2)Q + R\), we have
\[
(\alpha + (d-1))P + (\beta - (d-2))Q = \alpha P + (d-2)Q + \beta Q + R.
\]

Thus \(R\) is not a base point of \(\alpha P + \beta Q + R\). Hence \(\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1\) and (i) is proved.

By Theorem 1.2, (ii) holds.

In Theorem 1.4, let \(e = d - 2, \overline{m} = (d, d)\). Then \(\mathcal{P}_{(d-2, \overline{m})}\) is not empty and let \(C \in \mathcal{P}_{(d-2, \overline{m})}\). Then \(P = P_1, Q = P_0 \in C\) satisfy the condition. \(\square\)

**Theorem 2.2.** For \(P, Q\) as above, \(\Gamma(P, Q)\) is the set of all elements appeared in the following Table 7:

| \(1, d - 2\) | \(2, d - 3 + (d - 2)\) | \(\cdots\) | \(d - 3, 2 + (d - 4)(d - 2)\) | \(d - 2, 1 + (d - 3)(d - 2)\) | \(\cdots\) | \(d - 2 + (d - 3)(d - 1), 1\) |
|\(2 + (d - 1), d - 3\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |

Table 7. \(\Gamma(P, Q)\) when \(T_P C \ast C = dP\) and \(T_Q C \ast C = (d - 2)Q + P + R\)

**Proof.** To use Theorem 2.1 (ii), we arrange the elements of \(G(P)\) and \(G(Q)\) with \(d - 2\) columns and rows as in Table 1 and 6.

Note that the lengths of columns in the array in each of Table 1 and 6 are all different. Also note that the sequence in each column of \(G(P)\) is increasing by \(d - 1\) and the sequence in each column of \(G(Q)\) is increasing by \(d - 2\).

By Theorem 2.1 (ii), \((\alpha + (d-1), \beta - (d-2)) \in \Gamma(P, Q)\) if and only if \((\alpha, \beta) \in \Gamma(P, Q)\). It means \(\{\alpha, \alpha + (d - 1), \cdots, \alpha + k(d - 1)\} \subset G(P)\) if and only if \(\{\beta, \beta - \)
Thus if \((\alpha, \beta) \in \Gamma(P, Q)\) then \(\alpha\) and \(\beta\) should belong to the columns of same length in Table 1 and 6. Hence \(\Gamma(P, Q)\) is determined as Table 7. \(\square\)

**Case 2-2.** \(TQC \ast C = (d - 2)Q + R_1 + R_2\) with \(R_1 + R_2 \nsubseteq P\)

**Theorem 2.3.** (i) For \(\alpha \geq 0\) and \(\beta \geq d - 2\),
\[
\dim(\alpha + d, \beta - (d - 2)) = \dim(\alpha, \beta) + 2.
\]
(ii) For \(\alpha \geq 1\) and \(\beta \geq d - 1\),
\[
(\alpha + d, \beta - (d - 2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).
\]
(iii) Such a curve and points exist.

**Proof.** Note that \(R_1\) and \(R_2\) need not be distinct. When \(R_1 \neq R_2\) then let \(L_1\) be a line passing through \(R_1\) but not containing \(R_2\) so \(L_1 \neq TQC\). When \(R_1 = R_2\) then let \(L_1\) be a line passing through \(R_1\) such that \(L_1 \neq TQC\). In both cases, we have \(L_1 \ast C = R_1 + S_2 + \cdots + S_d\) for points \(S_2, \ldots, S_d \in C\) with \(R_2 \neq S_j\) for all \(j\). Since \(dP \sim (d - 2)Q + R_1 + R_2 \sim L_1 \ast C\), we have
\[
(\alpha + d)P + (\beta - (d - 2))Q = \alpha P + \beta Q + R_1 + R_2
\]
\[
= \alpha P + (\beta - (d - 2))Q + L_1 \ast C.
\]

Thus \(R_1\) is not a base point of the linear series \(|\alpha P + \beta Q + R_1 + R_2|\) and \(R_2\) is not a base point of the linear series \(|\alpha P + \beta Q + R_2| = |\alpha P + (\beta - (d - 2))Q + S_2 + \cdots + S_d|\). Hence
\[
\dim(\alpha + d, \beta - (d - 2)) = \dim(\alpha P + \beta Q + R_1 + R_2)
\]
\[
= \dim(\alpha P + \beta Q + R_2) + 1 = \dim(\alpha, \beta) + 2.
\]

Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let \(e = 0, \underline{m} = (d, d - 2, \cdots, d - 2)\). Then \(\mathcal{P}_{(0, \underline{m})}\) is not empty and \(C \in \mathcal{P}_{(0, \underline{m})}\) contains \(P_1, P_2, \cdots, P_d\). Then \(P = P_1, Q = P_2 \in C\) satisfy the condition. Therefore we get the result (iii). \(\square\)
**Theorem 2.4.** For \( P, Q \) as above, \( \Gamma(P, Q) \) is the set of all elements appeared in the following Table 8:

<table>
<thead>
<tr>
<th>(1,1 + (d-3)(d-2))</th>
<th>(2,2 + (d-4)(d-2))</th>
<th>\cdots</th>
<th>(d-3,d-3 + (d-2))</th>
<th>(d-2,d-2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1+d,1+(d-4)(d-2))</td>
<td>(2+d,2+(d-5)(d-2))</td>
<td>\cdots</td>
<td>(d-3+d,d-3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>\vdots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1 + (d-4)d, 1 + (d-2))</td>
<td>(2 + (d-4)d, 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1 + (d-3)d, 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8. \( \Gamma(P, Q) \) when \( T_P C \cdot C = dP \) and \( T_Q C \cdot C = (d-2)Q + R_1 + R_2 \) with \( R_1 + R_2 \not\leq P \)

**Proof.** To use Theorem 2.3 (ii), we rearrange the elements of \( G(P) \) and \( G(Q) \) with \( d-2 \) columns and rows such that the sequence in each column of \( G(P) \) is increasing by \( d \) and the sequence in each column of \( G(Q) \) is increasing by \( d-2 \). Then \( G(P) \) and \( G(Q) \) can be represented as Table 2 and 6.

Note that the lengths of columns in the array in each of Table 2 and 6 are all different. So in view of Theorem 2.3 (ii), if \( (\alpha, \beta) \in \Gamma(P, Q) \) then \( \alpha \) and \( \beta \) should belong to the columns of same length in Table 2 and 6. The proof is similar to that of Theorem 2.2 and \( \Gamma(P, Q) \) is determined as Table 8. \( \square \)

3. **AT A PAIR \((P, Q)\) WITH \( i_P C = d - 1 \) AND \( i_Q C = d - 2 \)**

In this case, there are points \( R_1, R_2, R_3 \in C \) such that \( T_P C \cdot C = (d-1)P + R_1 \) with \( R_1 \neq P \) and \( T_Q C \cdot C = (d-2)Q + R_2 + R_3 \) with \( R_2 + R_3 \not\leq Q \). There are 4 possible cases for points \( P, Q, \) and \( R_i \)'s.

- **Case 3-1.** \( R_1 = Q \) (Then \( R_2 + R_3 \not\leq P \) since \( T_P C \neq T_Q C \).
- **Case 3-2.** \( R_1 \neq Q, R_3 = P \)
- **Case 3-3.** \( R_1 \neq Q, R_1 = R_3 \neq P \)
- **Case 3-4.** \( R_1 \neq Q, R_2 + R_3 \not\leq P, R_2 + R_3 \not\leq R_1 \)

We find \( \Gamma(P, Q) \) for each cases through this section.

**CASE 3-1.** \( T_P C \cdot C = (d-1)P + Q \) and \( T_Q C \cdot C = (d-2)Q + R_2 + R_3 \) with \( R_2 + R_3 \not\leq P \)

**Theorem 3.1.** (i) For \( \alpha \geq 0, \beta \geq d-2 \),

\[
\dim(\alpha + (d-1), \beta - (d-3)) = \dim(\alpha, \beta) + 2.
\]
(ii) For $\alpha \geq 1, \beta \geq d - 1,$

$$(\alpha + (d - 1), \beta - (d - 3)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) Such a curve and points exist.

Proof. Let $L_1$ be general line passing through $R_2$ but not containing $Q$ and $L_1 \cdot C = R_2 + S_2 + \cdots + S_d$ with $R_2 \neq S_j$ and $R_3 \neq S_j$ for all $j$. Since $(d - 1)P \sim (d - 3)Q + R_2 + R_3$

$$(\alpha + (d - 1))P + (\beta - (d - 3))Q
\sim \alpha P + \beta Q + R_2 + R_3
= \alpha P + (\beta - (d - 2))Q + ((d - 2)Q + R_2 + R_3)
\sim \alpha P + (\beta - (d - 2))Q + R_2 + S_2 + \cdots + S_d.$$

Thus $\dim(\alpha + (d - 1), \beta - (d - 3)) = \dim(\alpha, \beta) + 2$ and (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = d - 1, m = (d - 2)$. Then $P_{(d - 1, m)}$ is not empty and $C \in P_{(d - 1, m)}$ contains $P = P_0, Q = P_1$ which satisfy the condition. Therefore we get the result (iii). $\square$

Theorem 3.2. For $P, Q$ as above, $\Gamma(P, Q)$ is the set of all elements appeared in the following Table 9:

$$\begin{array}{cccc}
(1, d - 2 + (d - 3)(d - 3)) & (2, d - 3 + (d - 4)(d - 3)) & \cdots & (d - 3, 2 + (d - 3)) (d - 2, 1) \\
(1 + (d - 1), d - 2 + (d - 4)(d - 3)) & (2 + (d - 1), d - 3 + (d - 5)(d - 3)) & \cdots & (d - 3 + (d - 1), 2) \\
\vdots & \vdots & \ddots & \vdots \\
(1 + (d - 4)(d - 1), d - 2 + (d - 3)) & (2 + (d - 4)(d - 1), d - 3) & \cdots & (1 + (d - 3)(d - 1), d - 2) \\
\end{array}$$

Table 9. $\Gamma(P, Q)$ when $T_P C \cdot C = (d - 1)P + Q$ and $T_Q C \cdot C = (d - 2)Q + R_2 + R_3$ with $R_2 + R_3 \not\sim P$

Proof. We rearrange the elements of $G(P)$ and $G(Q)$ with $d - 2$ columns and rows such that the sequence in each column of $G(P)$ is increasing by $d - 1$ and the sequence in each column of $G(Q)$ is increasing by $d - 3$. Then $G(P)$ and $G(Q)$ can be represented as Table 4 and 5.

Note that the lengths of columns in the array in each of Table 4 and 5 are all different. In view of Theorem 3.1 (ii), if $(\alpha, \beta) \in \Gamma(P, Q)$ then $\alpha$ and $\beta$ should belong to the columns of same length in Table 4 and 5. Hence $\Gamma(P, Q)$ is determined as Table 9. $\square$
Case 3.2. $T_P C \cdot C = (d - 1)P + R_1$ and $T_Q C \cdot C = (d - 2)Q + R_2 + P$ with $R_1 \neq R_2$

Theorem 3.3. (i) For $\alpha \geq 0, \beta \geq d - 2,$

\[
\dim(\alpha + (d - 2), \beta - (d - 2)) = \dim(\alpha, \beta).
\]

(ii) For $\alpha \geq 1, \beta \geq d - 1,$

\[
(\alpha + (d - 2), \beta - (d - 2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).
\]

(iii) Such a curve and points exist.

Proof. Since $(d - 2)P + R_1 \sim (d - 2)Q + R_2,$

\[
(\alpha + (d - 2))P + (\beta - (d - 2))Q + R_1 \sim \alpha P + \beta Q + R_2.
\]

Thus neither $R_1$ nor $R_2$ is a base point of the linear series

\[
|(\alpha + (d - 2))P + (\beta - (d - 2))Q + R_1| = |\alpha P + \beta Q + R_2| - 1.
\]

Hence $\dim(\alpha + (d - 2), \beta - (d - 2)) = \dim(\alpha, \beta) = \dim|\alpha P + \beta Q + R_2| - 1.$

Thus (i) is proved and by Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = d - 2, m = (d - 1, d - 1).$ Then $P_{(d-2,m)}$ is not empty and $C \in P_{(d-2,m)}$ contains $Q = P_0, P = P_1$ which satisfy the condition. Therefore we get the result (iii).

□

Theorem 3.4. For $P, Q$ as above, $\Gamma(P, Q)$ is the set of all elements appeared in the following Table 10:

\[
\begin{array}{cccccccc}
(1, d-2) & (2, d-3+(d-2)) & \cdots & \cdots & (d-3, 2+(d-4)(d-2)) & (d-2, 1+(d-3)(d-2)) \\
(2, d-2, d-3) & \cdots & \cdots & (d-3+(d-2), 1+(d-5)(d-2)) & (d-2+(d-2), 1+(d-4)(d-2)) \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & (d-3+(d-4)(d-2), 2) & (d-2+(d-4)(d-2), 1+(d-2)) & (d-2+(d-3)(d-2), 1)
\end{array}
\]

Table 10. $\Gamma(P, Q)$ when $T_P C \cdot C = (d - 1)P + R_1$ and $T_Q C \cdot C = (d - 2)Q + R_2 + P, R_1 \neq R_2$

Proof. We use the array in Table 3 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 3 [resp. Table 6] is increasing by $(d - 2)$ [resp. $(d - 2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.3 (ii), we obtain $\Gamma(P, Q)$.

□
Case 3-3. $T_P C \cdot C = (d - 1)P + R_1$ and $T_Q C \cdot C = (d - 2)Q + R_1 + R_2$

Theorem 3.5. (i) For $\alpha \geq 0, \beta \geq d - 2$,
\[
\dim(\alpha + (d - 1), \beta - (d - 2)) = \dim(\alpha, \beta) + 1.
\]
(ii) For $\alpha \geq 1, \beta \geq d - 1$,
\[
(\alpha + (d - 1), \beta - (d - 2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).
\]
(iii) Such a curve and points exist.

Proof. Since $(d - 1)P \sim (d - 2)Q + R_2$,
\[
(\alpha + (d - 1))P + (\beta - (d - 2))Q \sim \alpha P + \beta Q + R_2.
\]
Since $R_2$ is not a base point of $|\alpha P + \beta Q + R_2|$, $\dim(\alpha + (d - 1), \beta - (d - 2)) = \dim(\alpha, \beta) + 1$ holds. Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

Modifying the idea in [1], we construct a desired polynomial of degree $d$. Consider a linear system $\{ag^{d-2}(y+x)z+b\prod_{n=0}^{d-1}(x-nz) \mid (a, b) \in \mathbb{P}^1\}$. By Bertini’s theorem, a general element in this system is smooth. In fact, easy calculation shows that $C := ag^{d-2}(y+x)z+b\prod_{n=0}^{d-1}(x-nz)$ is smooth and for $P = (0, 0, 1)$ and $Q = (1, 0, 1)$, $T_PC = \{x = 0\}$ and $T_PQ = \{x = z\}$ satisfy the conditions. Note that $R_1 = (0, 1, 0)$ is contained in all of $C$, $T_PC$ and $T_QC$. Therefore we get the result (iii). \hfill $\square$

Theorem 3.6. For $P, Q$ as above, $\Gamma(P, Q)$ is the set of all elements appeared in the following Table 11.

<table>
<thead>
<tr>
<th>$d-4$</th>
<th>$d-5$</th>
<th>$d-6$</th>
<th>$d-7$</th>
<th>$d-8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1 + (d - 3)(d - 2))</td>
<td>(2, 2 + (d - 4)(d - 2))</td>
<td>\ldots</td>
<td>(d - 3, d - 3 + (d - 2))</td>
<td>(d - 2, d - 2)</td>
</tr>
<tr>
<td>(1 + (d - 1), 1 + (d - 4)(d - 2))</td>
<td>(2 + (d - 1), 2 + (d - 5)(d - 2))</td>
<td>\ldots</td>
<td>(d - 3 + (d - 1), d - 3)</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>(1 + (d - 4)(d - 1), 1 + (d - 2))</td>
<td>(2 + (d - 4)(d - 1), 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1 + (d - 3)(d - 1), 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 11. $\Gamma(P, Q)$ when $T_PC \cdot C = (d - 1)P + R_1$ and $T_QC \cdot C = (d - 2)Q + R_1 + R_2$

Proof. We use the array in Table 4 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 4 [resp. Table 6] is increasing by $(d - 1)$ [resp. $(d - 2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.5 (ii), we obtain $\Gamma(P, Q)$. \hfill $\square$
CASE 3-4. \( T_PC \cdot C = (d-1)P + R_1, R_1 \neq Q \) and \( T_QC \cdot C = (d-2)Q + R_2 + R_3 \) with \( R_2 + R_3 \not\subseteq R_1, P \)

**Theorem 3.7.** (i) For \( \alpha \geq 0, \beta \geq d-2 \),
\[
\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1.
\]

(ii) For \( \alpha \geq 1, \beta \geq d-1 \),
\[
(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).
\]

(iii) Such a curve and points exist.

**Proof.** Let \( L_1 \) be a line passing through \( R_2 \) differnet from \( T_QC \) and \( L_1 \mid C \sim R_2 + S_2 + \cdots + S_d \) with \( R_2 \neq S_j \) for all \( j \). Then
\[
(\alpha + (d-1))P + (\beta - (d-2))Q + R_1
\]
\[
\sim \alpha P + \beta Q + R_2 + R_3
\]
\[
\sim \alpha P + (\beta - (d-2))Q + L_1 \cdot C
\]
\[
\sim \alpha P + (\beta - (d-2))Q + R_2 + S_2 + \cdots + S_d.
\]

Thus \( R_1 \) is not a base point of \( |\alpha P + \beta Q + R_2 + R_3| \) and
\[
\dim(\alpha + (d-1), \beta - (d-2))
\]
\[
= \dim(|(\alpha + (d-1))P + (\beta - (d-2))Q + R_1| - 1
\]
\[
= \dim|\alpha P + \beta Q + R_2 + R_3| - 1
\]
\[
= \dim(\alpha, \beta) + 1
\]
since \( R_2 \) is not a base point of
\[
|\alpha P + \beta Q + R_2 + R_3| = |(\alpha + (d-1))P + (\beta - (d-2))Q + R_1|,
\]
and \( R_3 \) is not a base point of
\[
|\alpha P + \beta Q + R_3| = |\alpha P + (\beta - (d-2))Q + S_2 + \cdots + S_d|.
\]

Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let \( e = 0, m = (d-1, d-2, \cdots, d-2) \). Choose three lines \( T_1, T_2, T_3 \) which are not concurrent. Then \( \mathcal{P}_{(0,m)} \) is not empty and take \( C \in \mathcal{P}_{(0,m)} \) which satisfy \( T_1 \cap T_2 \not\subseteq C \) or \( T_1 \cap T_3 \not\subseteq C \), since \( T_1 \) meet \( C \) at only one more point other than \( P_1 \). We may assume \( T_1 \cap T_2 \not\subseteq C \). Then \( P = P_1 \) and \( Q = P_2 \in C \) satisfy the condition. Therefore we get the result (iii). \( \square \)
Theorem 3.8. For $P, Q$ as above, $\Gamma(P, Q)$ is the set of all elements appeared in the Table 11 of Theorem 3.6.

Proof. We use the array in Table 4 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 4 [resp. Table 6] is increasing by $(d - 1)$ [resp. $(d - 2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.7 (ii), we obtain $\Gamma(P, Q)$.

\[ \square \]

4. At a Pair $(P, Q)$ with $i_P C = d - 2$ and $i_Q C = d - 2$

In this case, $T_P \cdot C = (d - 2)P + R_1 + R_2$ and $T_Q \cdot C = (d - 2)Q + S_1 + S_2$.

There are 3 possible cases for points $P, Q, R_i'$s and $S_i'$s:

Case 4-1. $R_2 = Q$ (Then $S_1 + S_2 \notin P$ since $T_P \neq T_Q C$.)

Case 4-2. $R_1, R_2, S_1, S_2 \notin \{P, Q\}, R_2 = S_2$

Case 4-3. $R_1, R_2, S_1, S_2 \notin \{P, Q\}, R_1, R_2 \notin \{S_1, S_2\}$ (maybe $R_1 = R_2$ or $S_1 = S_2$)

Case 4-1. $T_P \cdot C = (d - 2)P + R_1 + Q$ and $T_Q \cdot C = (d - 2)Q + S_1 + S_2$

Theorem 4.1. (i) For $\alpha \geq 0, \beta \geq d - 2$,

$$\dim (\alpha + (d - 2), \beta - (d - 3)) = \dim (\alpha, \beta) + 1.$$

(ii) For $\alpha \geq 1, \beta \geq d - 1$,

$$(\alpha + (d - 2), \beta - (d - 3)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) Such a curve and points exist.

Proof. Let $L_1$ be a line passing through $S_1$ different from $T_Q C$ and $L_1 \cdot C = S_1 + U_2 + \cdots + U_d$ with $S_1 \neq U_j$ for all $j$.

Since $(d - 2)P + R_1 \sim (d - 3)Q + S_1 + S_2$, we have

$$(\alpha + (d - 2))P + (\beta - (d - 3))Q + R_3 \sim \alpha P + \beta Q + S_1 + S_2 \sim \alpha P + (\beta - (d - 2))Q + S_1 + U_2 + \cdots + U_d.$$

Hence

$$\dim |(\alpha + (d - 2))P + (\beta - (d - 3))Q| + 1$$

$$= \dim |(\alpha + (d - 2))P + (\beta - (d - 3))Q + R_1|$$

$$= \dim |\alpha P + \beta Q + S_1 + S_2|$$

$$= \dim (\alpha, \beta) + 2.$$
Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let \( e = d - 2, m = (d - 2, d - 2) \). Then \( P_{(d-2,m)} \) is not empty and \( C \in P_{(d-2,m)} \) contains \( P = P_0, Q = P_1 \) which satisfy the condition. Therefore we get the result (iii).

\[ \text{□} \]

**Theorem 4.2.** For \( P, Q \) as above, \( \Gamma(P, Q) \) is the set of all elements appeared in the following Table 12:

\[
\begin{align*}
(1, d - 2 + (d - 3)(d - 3)) & \quad (2, d - 3 + (d - 4)(d - 3)) & \quad \cdots & \quad (d - 3, 2 + (d - 3)) & \quad (d - 2, 1) \\
(1 + (d - 2), d - 2 + (d - 4)(d - 3)) & \quad (2 + (d - 2), d - 3 + (d - 5)(d - 3)) & \quad \cdots & \quad (d - 3 + (d - 2), 2) & \\
\vdots & \quad \vdots & \quad \vdots & & \\
(1 + (d - 4)(d - 2), d - 2 + (d - 3)) & \quad (2 + (d - 4)(d - 2), d - 3) & & & \\
(1 + (d - 3)(d - 2), d - 2) & & & & 
\end{align*}
\]

\[ \text{Table 12. } \Gamma(P, Q) \text{ when } T_P C \circ C = (d - 2)P + R_1 + Q \text{ and } T_Q C \circ C = (d - 2)Q + S_1 + S_2 \]

\[ \text{Proof. } \] The proof is similar to the proof of Theorem 3.2. In this proof, we use Table 6 for \( G(P) \) and Table 5 for \( G(Q) \). Then we obtain \( \Gamma(P, Q) \). \[ \text{□} \]

**Case 4-2.** \( T_P C \circ C = (d - 2)P + R_1 + R_2 \) and \( T_Q C \circ C = (d - 2)Q + S_1 + R_2 \)

\[ \text{Theorem 4.3.} \]

(i) For \( \alpha \geq 0, \beta \geq d - 2 \),

\[ \dim(\alpha + (d - 2), \beta - (d - 2)) = \dim(\alpha, \beta). \]

(ii) For \( \alpha \geq 1, \beta \geq d - 1 \),

\[ (\alpha + (d - 2), \beta - (d - 2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q). \]

(iii) Such a curve and points exist.

\[ \text{Proof. } \] Since \( (d - 2)P + R_1 \sim (d - 2)Q + S_1 \),

\[ (\alpha + (d - 2))P + (\beta - (d - 2))Q + R_1 \sim \alpha P + \beta Q + S_1. \]

Thus \( \dim(\alpha + (d - 2), \beta - (d - 2)) = \dim(\alpha, \beta). \)

By Theorem 1.2, (ii) is proved.

Consider a generic smooth curve \( C \) given in the proof of Theorem 3.5. Then \( P = (1, 0, 1), Q = (2, 0, 1), R_1 = (1, -1, 1), S_1 = (2, -2, 1), R_2 = (0, 1, 0) \) on \( C \) satisfy the condition and (iii) is proved.

\[ \text{□} \]

**Theorem 4.4.** For \( P, Q \) as above, \( \Gamma(P, Q) \) is the set of all elements appeared in the following Table 13:
The proof is similar to the proof of Theorem 3.2. In this proof, we use Table 13.

\[\begin{array}{cccc}
(1, 1 + (d - 3)(d - 2)) & (2, 2 + (d - 4)(d - 2)) & \cdots & (d - 3, d - 3 + (d - 2)) \\
(1 + (d - 2), 1 + (d - 4)(d - 2)) & (2 + (d - 2), 2 + (d - 5)(d - 2)) & \cdots & (d - 3 + (d - 2), d - 3) \\
\vdots & \vdots & & \vdots \\
(1 + (d - 4)(d - 2), 1 + (d - 2)) & (2 + (d - 4)(d - 2), 2) & & (1 + (d - 3)(d - 2), 1) \\
\end{array}\]

Table 13. $\Gamma(P, Q)$ when $T_pC \cdot C = (d - 2)P + R_1 + R_2$ and $T_qC \cdot C = (d - 2)Q + S_1 + R_2$

**Proof.** The proof is similar to the proof of Theorem 3.2. In this proof, we use Table 6 for both $G(P)$ and $G(Q)$. Then we obtain $\Gamma(P, Q)$. \(\square\)

**Case 4-3.** $T_pC \cdot C = (d - 2)P + R_1 + R_2$ and $T_qC \cdot C = (d - 2)Q + S_1 + S_2$

**Theorem 4.5.**

(i) For $\alpha \geq 0, \beta \geq d - 2$,

\[\dim(\alpha + (d - 2), \beta - (d - 2)) = \dim(\alpha, \beta).\]

(ii) For $\alpha \geq 1, \beta \geq d - 1$,

\[(\alpha + (d - 2), \beta - (d - 2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).\]

(iii) *Such a curve and points exist.*

**Proof.** Let $L_1$ be a line passing through $R_1$ different from $T_pC$ and $L_1 \cdot C \sim R_1 + R_2' + \cdots + R_d'$ with $R_1 \neq R_j'$ for all $j$. Let $L_2$ be a line passing through $S_1$ different from $T_qC$ and $S_1 \cdot C \sim S_1 + S_2' + \cdots + S_d'$ with $S_1 \neq S_j'$ for all $j$.

Since $(d - 2)P + R_1 + R_2 \sim (d - 2)Q + S_1 + S_2$, we have

\[\alpha P + (\beta - (d - 2))Q + R_1 + R_2' + \cdots + R_d' \sim (\alpha + (d - 2))P + (\beta - (d - 2))Q + R_1 + R_2 \sim \alpha P + \beta Q + S_1 + S_2 \sim \alpha P + (\beta - (d - 2))Q + S_1 + S_2' + \cdots + S_d'.\]

Hence

\[\dim((\alpha + (d - 2), \beta - (d - 2)) = \dim((\alpha + (d - 2))P + (\beta - (d - 2))Q + R_1 + R_2) - 2 = \dim(\alpha P + \beta Q + S_1 + S_2) - 2 = \dim(\alpha, \beta).\]

By Theorem 1.2, (ii) is proved.
In Theorem 1.4, let $e = 0$, $m = (d - 2, d - 2, \ldots, d - 2)$. Then $P_{(0, m)}$ is not empty and take $C \in P_{(0, m)}$. Then $P = P_1, Q = P_2 \in C$ satisfy the condition. Therefore we get the result (iii).

\textbf{Theorem 4.6.} For $P, Q$ as above, $\Gamma(P, Q)$ is the same table as that in Theorem 4.4.

\textit{Proof.} The proof is same as that of Theorem 4.4.

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