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SHARP ESTIMATES ON THE THIRD ORDER HERMITIAN-TOEPLITZ DETERMINANT FOR SAKAGUCHI CLASSES

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Abstract. In this paper, sharp lower and upper bounds on the third order Hermitian-Toeplitz determinant for the classes of Sakaguchi functions and some of its subclasses related to right-half of lemniscate of Bernoulli, reverse lemniscate of Bernoulli and exponential functions are investigated.

1. Introduction

Computing estimates on the coefficients and coefficient functionals such as Fekete-Szegö functional, Hankel determinants and Zalcman conjecture have been one of the major topics of research in geometric function theory as they give many interesting geometric properties of the functions under consideration. The area of q-calculus and the fractional q-calculus have also many applications in several areas of number theory and combinatorial analysis such as the theory of partitions [26]. Srivastava *et al.* [27] discussed the Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. Third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential functions were investigated in [28]. In the following few paragraphs we will give basic notions and history needed for further development in this paper.

Let $\mathcal{A}$ be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ and a subclass of $\mathcal{A}$ containing univalent functions be denoted by $\mathcal{S}$. There are a list of subclasses of $\mathcal{S}$ but the classes of starlike and convex functions have attracted much during 1920-1980 as these classes gave a light of hope for correctness of the Bieberbach conjecture [5]. In 1959, Sakaguchi [24] introduced the class of starlike functions with respect to

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symmetric points defined by

$$\mathcal{S}_s^* := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{2zf'(z)}{(f(z)-f(-z))} \right) > 0,\, z \in \mathbb{D} \right\}.$$

It is noted that the class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions and odd functions starlike with respect to the origin [24]. Sakaguchi [24] also proved that the n^{th} coefficient of functions in this class is bounded by 1 as in case of the convex functions. Further, Das and Singh [4] introduced the class of the convex functions with respect to symmetric points defined by

$$\mathcal{K}_s := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'} \right) > 0,\ z \in \mathbb{D} \right\}.$$

The functions in the class are convex and Das and Singh [4] found that the n^{th} coefficient of functions in this class is bounded by $1/n$, $n \geq 2$. The bounds on the second and third Hankel determinants for these classes were computed by Krishna *et al.* [30]. They proved that the non-sharp bounds for the third Hankel determinant are $|H_{3,\,1}(f)| \leq 5/2$ and $|H_{3,\,1}(f)| \leq 19/135$ for starlike and convex functions with respect to symmetric points, respectively. Recently, the bounds on the third Hankel determinant were improved by Kumar *et al.* [16].

To keep the expression brief, let us use the notations:

$$\mathcal{T}_s(f) := \frac{2zf'(z)}{(f(z)-f(-z))} \text{ and } \mathcal{T}_k(f) := \frac{(2zf'(z))'}{(f(z)-f(-z))'}.$$

Using the concept of subordination and by considering the analytic function φ with $\operatorname{Re}\varphi(z) > 0$, $z \in \mathbb{D}$ and $\varphi'(0) > 0$ which is starlike with respect to $\varphi(0) = 1$, Ravichandran [23] introduced two unified subclasses of starlike and convex functions with respect to symmetric points as follows:

$$\mathcal{S}_s^*(\varphi) := \{ f \in \mathcal{S} : \mathcal{T}_s(f) \prec \varphi(z) \} \text{ and } \mathcal{K}_s(\varphi) := \{ f \in \mathcal{S} : \mathcal{T}_k(f) \prec \varphi(z) \}.$$

These classes include many subclasses of $\mathcal{S}_s^*$ and $\mathcal{K}_s$. For $0 \leq \alpha < 1$, if we take $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$, the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{K}_s(\varphi)$ reduce to following two subclasses

$$\mathcal{S}_s^*(\alpha) = \{ f \in \mathcal{S} : \operatorname{Re}\mathcal{T}_s(f) > \alpha \} \text{ and } \mathcal{K}_s(\alpha) = \{ f \in \mathcal{S} : \operatorname{Re}\mathcal{T}_k(f) > \alpha \}$$

consisting of the starlike and convex functions of order α with respect to symmetric points, respectively. Further details related to these classes are available in [1, 6, 22, 29]. Some other subclasses of starlike functions with respect to symmetric points associated with lemniscate of Bernoulli, exponential function and reverse lemniscate are obtained by taking $\varphi(z) = \sqrt{1+z}$, e^z and $\sqrt{2} - (\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}$, denoted by $\mathcal{S}_{s,\,L}^*$, $\mathcal{S}_{s,\,e}^*$ and $\mathcal{S}_{s,\,RL}^*$, respectively. These classes are analogues to the corresponding classes of starlike functions studied in [20, 21, 25]. Sharp estimates on the initial coefficients of the functions belonging to these classes were also investigated in [9].

In the same line it is interesting to investigate the bound on the Hermitian-Toeplitz determinant. For $q, n \in \mathbb{N}$, the Hermitian-Toeplitz determinant (see [7, 12]) of order n associated with sequence $\langle a_k \rangle_{k\geq 1}$ of the coefficients of the function $f \in \mathcal{A}$, is defined by

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{vmatrix}. \tag{1.1}$$

A simple commutation using (1.1) yields the second and third order Hermitian-Toeplitz determinants as

$$\begin{aligned} &T_{2,1}(f) = 1 - |a_2|^2 \quad \text{and} \\ &T_{3,1}(f) := 2\operatorname{Re}(a_2^2\,\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1, \end{aligned} \tag{1.2}$$

respectively. The study of Hermitian-Toeplitz determinants for the subclasses of normalized analytic functions was investigated in [3, 10] and then continued in [8]. Further recent results in this direction were obtained in [18] and [11]. Cudna *et al.* [3] investigated the sharp lower and upper bounds for the second and third Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order α. Later, Kumar *et al.* [18] investigated the sharp bounds on the second and third Hankel determinants for the classes of Janowski starlike and convex functions and thus generalised the results in [3]. Some recent results and development can be found in [2, 13–15, 17].

The above work motivates us to investigate the sharp bounds on third order Hermitian-Toeplitz determinants for the classes $\mathcal{K}_s(\alpha)$, $\mathcal{S}_s^*(\alpha)$, $\mathcal{S}^*_{s,\,L}$, $\mathcal{S}^*_{s,\,e}$ and $\mathcal{S}^*_{s,\,RL}$.

2. The classes $\mathcal{K}_s(\alpha)$ and $\mathcal{S}_s^*(\alpha)$

This section provides sharp estimates on the second and third order Hermitian–Toeplitz determinants for the classes $\mathcal{K}_s(\alpha)$ and $\mathcal{S}_s^*(\alpha)$. The class $\mathcal{P}$ of analytic functions with positive real part in the unit disk $\mathbb{D}$ plays a vital role while investigating the bounds on coefficient functionals. We now recall the following result due to Libera and Zlotkiewicz which will be used in the proof of main results.

Lemma 2.1 ([19, Lemma 3, p. 254])**.** *Let $\mathcal{P}$ be the class of analytic functions having the Taylor series of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.1}$$

satisfying the condition $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. *Then* $2p_2 = p_1^2 + (4 - p_1^2)\xi$ *for some* $\xi \in \overline{\mathbb{D}}$.

For any function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}_s^*(\alpha)$, there is a function $p\in\mathcal{P}$ of the form (2.1) such that

$$\mathcal{T}_s(f)=(1-\alpha)p(z)+\alpha,\quad z\in\mathbb{D}. \tag{2.2}$$

On comparison of the coefficients of like power terms, we get

$$a_2=\frac{(1-\alpha)p_1}{2}\quad\text{and}\quad a_3=\frac{1-\alpha}{2}p_2. \tag{2.3}$$

Now $T_{2,1}(f)=1-|a_2|^2=1-(1-\alpha)^2p_1^2/4$. Since $|p_n|\leq 2$ (see [5]), it follows that $(2-\alpha)\alpha\leq T_{2,1}(f)\leq 1$. The upper bound is sharp for the function f_0 satisfying

$$\mathcal{T}_s(f_0)=\frac{1+(1-2\alpha)z^3}{1-z^3},\ z\in\mathbb{D}, \tag{2.4}$$

whereas the lower bound attained for the function f_1 satisfying

$$\mathcal{T}_s(f_1)=\frac{1+(1-2\alpha)z}{1-z},\ z\in\mathbb{D}. \tag{2.5}$$

Theorem 2.2. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be a function in the class $\mathcal{S}_s^*(\alpha)$. Then the following estimates hold*

$$(3-2\alpha)\alpha^2\leq T_{3,1}(f)\leq 1.$$

The estimates are sharp.

Proof. The well-known fact is that the class of functions with positive real part $\mathcal{P}$ is invariant under rotation and $|p_1|\leq 2$, where we are not going to loose anything in considering $0\leq p_1\leq 2$. A computation using the expressions (1.2) and (2.3) for some $\xi\in\overline{\mathbb{D}}$ yields

$$\begin{aligned}
T_{3,1}(f) &:= 1+\frac{1}{8}(1-\alpha)^3p_1^2\operatorname{Re}(2\overline{p_2})-\frac{1}{2}(1-\alpha)^2p_1^2-\frac{1}{16}(1-\alpha)^2|2p_2|^2\\
&= 1+\frac{1}{8}(1-\alpha)^3p_1^2(p_1^2+(4-p_1^2)\operatorname{Re}(\xi))-\frac{1}{2}(1-\alpha)^2p_1^2\\
&\quad -\frac{1}{16}(1-\alpha)^2|p_1^2+(4-p_1^2)\operatorname{Re}(\xi)|^2\\
&= 1+\frac{1}{16}(1-2\alpha)(1-\alpha)^2p_1^4-\frac{1}{16}(1-\alpha)^2(4-p_1^2)^2|\xi|^2-\frac{1}{2}(1-\alpha)^2p_1^2\\
&\quad -\frac{1}{8}(1-\alpha)^2\alpha(4-p_1^2)p_1^2\operatorname{Re}(\xi)\\
&= \Psi(p_1^2,|\xi|,\operatorname{Re}(\xi)).
\end{aligned} \tag{2.6}$$

Since $-\operatorname{Re}\xi\leq|\xi|$ and $0\leq\alpha<1$, it follows that

$$\Psi(p_1^2,|\xi|,\operatorname{Re}(\xi))\leq\Psi(p_1^2,|\xi|,-|\xi|).$$

The well-known fact is that the class of functions with positive real part $\mathcal{P}$ is invariant under rotation and $|p_1|\leq 2$, where we are not going to loose anything in considering $0 \le p_1 \le 2$. With this consideration, and settings $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, we can write

$$\begin{aligned}\Psi(p_1^2, |\xi|, -|\xi|) &= H(x,\, y)\\ &:= 1 + \frac{1}{16}(1-2\alpha)(1-\alpha)^2x^2 - \frac{1}{16}(1-\alpha)^2(4-x)^2y^2\\ &\quad - \frac{1}{2}(1-\alpha)^2x + \frac{1}{8}(1-\alpha)^2\alpha(4-x)xy.\end{aligned}$$

The function H needs to be maximize for upper bound over the rectangular region $[0,4]\times[0,1]$. On the boundaries, we see that

$$H(0,\, y) = 1-(1-\alpha)^2y^2 \le 1, \quad H(4,\, y) = (3-2\alpha)\alpha^2,$$

$$H(x,\, 0) = 1 - \frac{1}{2}(1-\alpha)^2x + \frac{1}{16}(1-2\alpha)(1-\alpha)^2x^2 = k_1(x),$$

and

$$H(x,\, 1) = 1-(1-\alpha)^2 + \frac{1}{2}\alpha(1-\alpha)^2x - \frac{1}{4}(1-\alpha)^2\alpha x^2 = k_2(x)$$

for all $x \in [0,\, 4]$ and $y \in [0,\, 1]$. We notice that $k_1''(x) > 0$ and $k_2''(x) < 0$ for all $x \in (0,\, 4)$. Therefore, the function k_1 has no maximum in interval $(0,\, 4)$ and the function k_2 may have maximum in interval $(0,\, 4)$. We find that $k_2'(x) = 0$ has only one root namely 1 root in $(0,4)$ and $k_2(1) = k_2(1) = (3-\alpha)^2\alpha/4$, $k_2(4) = (3-2\alpha)\alpha^2$, $k_2(0) = 1-(1-\alpha)^2$. Further calculations give $k_1(0) = 1$ and $k_1(4) = (3-2\alpha)\alpha^2$. Now we examine the function H in interior of $[0,4]\times[0,1]$. For this, we observe that

$$\frac{\partial H(x,y)}{\partial y} = \frac{1}{8}((1-\alpha)^2\alpha(4-x)x - (1-\alpha)^2(4-x)^2y) = 0 \Leftrightarrow y = y_1 = \frac{\alpha x}{4-x}$$

and

$$\frac{\partial H(x,y_1)}{\partial x} = \frac{1}{8}(1-\alpha)^2(-4+x(1-\alpha)^2) = 0 \Leftrightarrow x = x' = \frac{4}{(1-\alpha)^2}$$

and so $y_1 = 1/(\alpha-2)$. Clearly we see that $y_1 < 0$ for $0 \le \alpha < 1$. So there is no maxima of the function H inside $(0,\, 4)\times(0,1)$. Therefore,

$$T_{3,1}(f) \le \max\{H(0,\, 0),\, H(0,\, 1),\, H(1,\, 0),\, H(1,\, 1)\} = 1.$$

From (2.6), we note that

$$\begin{aligned}\Psi(p_1^2, |\xi|, \mathrm{Re}(\xi)) &\ge \Psi(p_1^2, |\xi|, |\xi|)\\ &\ge \Psi(p_1^2, 1, 1) = \Psi(p_1^2).\end{aligned}$$

With settings $p^2 =: x \in [0,4]$, we can write $\Psi(p_1^2) = G(x)$, where

$$\begin{aligned}G(x) = 1 &+ \frac{1}{16}(1-2\alpha)(1-\alpha)^2x^2 - \frac{1}{16}(1-\alpha)^2(4-x)^2\\ &- \frac{1}{2}(1-\alpha)^2x + \frac{1}{8}(1-\alpha)^2\alpha(4-x)x.\end{aligned}$$

Since $G''(x) < 0$ for all x, it follows that the minimum may attain only at the end points of [0, 4]. Note that $G(0) = 1 - (1-\alpha)^2$ and $G(4) = (3-2\alpha)\alpha^2$. Therefore,

$$T_{3,1}(f) \geq \min\{G(0),\, G(1)\} = G(4).$$

The lower and upper bounds are sharp for the function f_0 and f_1 defined by (2.4) and (2.5), respectively. □

If $\alpha = 0$, Theorem 2.2 yields the following sharp bound on Hermitian-Toeplitz determinant of the third order of starlike functions with respect to symmetric points.

Corollary 2.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{S}_s^*$. Then*

$$0 \leq T_{3,1}(f) \leq 1.$$

For any function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{K}_s(\alpha)$, there is a $p \in \mathcal{P}$ of the form (2.1) such that we have

$$\mathcal{T}_k(f) = (1-\alpha)p(z) + \alpha,\ z \in \mathbb{D}.$$

On comparison of the coefficients of like power terms, we get

$$a_2 = \frac{(1-\alpha)p_1}{2} \quad \text{and} \quad a_3 = \frac{1-\alpha}{6}p_2. \tag{2.7}$$

Now $T_{2,1}(f) = 1 - |a_2|^2 = 1 - (1-\alpha)^2 p_1^2/4$ and thus we have $(2-\alpha)\alpha \leq T_{2,1}(f) \leq 1$. The upper bound is sharp in case of the function $\tilde{f}_0$ for which

$$\mathcal{T}_k(\tilde{f}_0) = \frac{1 + (1-2\alpha)z^3}{1 - z^3},\ z \in \mathbb{D}, \tag{2.8}$$

whereas the lower bound attained for the function $\tilde{f}_1$ satisfying

$$\mathcal{T}_k(\tilde{f}_1) = \frac{1 + (1-2\alpha)z}{1 - z},\ z \in \mathbb{D}. \tag{2.9}$$

Theorem 2.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{K}_s(\alpha)$. Then the best possible bounds on Hermitian-Toeplitz are given by*

$$\frac{1}{9}(-4 + 20\alpha - \alpha^2 - 6\alpha^3) \leq T_{3,1}(f) \leq 1.$$

Proof. Using the expressions (1.2) and (2.7) for some $\xi \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} T_{3,1}(f) := {} & 1 + \frac{1}{144}(5-6\alpha)(1-\alpha)^2 p_1^4 - \frac{1}{144}(1-\alpha)^2(4-p_1^2)^2|\xi|^2 \\ & - \frac{1}{2}(1-\alpha)^2 p_1^2 + \frac{1}{72}(1-\alpha)^2(2-3\alpha)(4-p_1^2)p_1^2 \operatorname{Re}(\xi) \\ = {} & \Gamma(p_1^2, |\xi|, \operatorname{Re}(\xi)). \end{aligned}$$

We consider two cases.

Case (I). Let $0 \le \alpha \le 2/3$. Then $\Gamma(p_1^2, |\xi|, \mathrm{Re}(\xi)) \le \Gamma(p_1^2, |\xi|, |\xi|)$. As before, we set $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$ to arrive at

$$\begin{aligned}\Gamma(p_1^2, |\xi|, -|\xi|) = 1 &+ \frac{1}{144}(5-6\alpha)(1-\alpha)^2x^2 - \frac{1}{144}(1-\alpha)^2(4-x)^2y^2 \\ &- \frac{1}{2}(1-\alpha)^2x + \frac{1}{72}(1-\alpha)^2(2-3\alpha)(4-x)xy =: S(x,\, y).\end{aligned}$$

Here the function S is defined on the rectangular region $[0,4] \times [0,1]$. On the boundary of this region, we have

$$S(0,\, y) = 1 - \frac{1}{9}(1-\alpha)^2y^2 \le 1 \quad \text{and} \quad S(4,\, y) = \frac{1}{9}(-4 + 20\alpha - \alpha^2 - 6\alpha^3).$$

Also, for all $x \in [0,4]$, we have

$$S(x,\, 0) = 1 - \frac{1}{2}(1-\alpha)^2x + \frac{1}{144}(1-\alpha)^2(5-6\alpha)x^2 =: s_1(x)$$

and

$$S(x,\, 1) = 1 - \frac{1}{9}(1-\alpha)^2 - \frac{1}{6}(1-\alpha)^2(2+\alpha)x =: s_2(x). \tag{2.10}$$

We note that $s_1'' = (5-6\alpha)(1-\alpha)^2/72 > 0$ for all $\alpha \in [0,\, 2/3]$. So, the function s_1 has no maximum in interval $(0,\, 4)$ and $s_1(0) = 1$, $s_1(4) = (-4 + 20\alpha - \alpha^2 - 6\alpha^3)/9$. Further, it is easy to see that $s_2(x) \le 1 - (1-\alpha)^2/9$. Further computation shows that the function $S(x,\, y)$ has no maximum in the interior of $[0,4] \times [0,1]$. Therefore, for $0 \le \alpha \le 2/3$,

$$T_{3,1}(f) \le \max\{S(0,\, y),\, S(4,\, y)\} = 1.$$

Next we determine the lower bound on $T_{3,1}(f)$. For $0 \le \alpha \le 2/3$ and $p^2 =: x \in [0,4]$, we can write $\Gamma(p_1^2, |\xi|, \mathrm{Re}(\xi)) \ge \Gamma(p_1^2, |\xi|, -|\xi|) \ge \Gamma(p_1^2, 1, -1) = s_2(x)$, where s_2 is defined by (2.10). Note that $s_2'(x) = 0$ if and only if $x = x_4 = (10-3\alpha)/(2-3\alpha)$ and we observe that $x_4 \notin (0.4)$. Further, $s_2(0) = 1 - (1-\alpha)^2/9$ and $s_2(4) = (-4 + 20\alpha - \alpha^2 - 6\alpha^3)/9$. Thus,

$$T_{3,1}(f) \ge \max\{s_2(0),\, s_2(4)\} = \frac{-4 + 20\alpha - \alpha^2 - 6\alpha^3}{9}.$$

Case (II). Consider $2/3 < \alpha \le 1$. Then $\Gamma(p_1^2, |\xi|, \mathrm{Re}(\xi)) \le \Gamma(p_1^2, |\xi|, -|\xi|)$. Settings $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, we write

$$\begin{aligned}\Gamma(p_1^2, |\xi|, -|\xi|) = 1 &+ \frac{1}{144}(5-6\alpha)(1-\alpha)^2x^2 - \frac{1}{144}(1-\alpha)^2(4-x)^2y^2 \\ &- \frac{1}{2}(1-\alpha)^2x - \frac{1}{72}(1-\alpha)^2(2-3\alpha)(4-x)xy =: T(x,\, y).\end{aligned}$$

A simple calculation gives

$$T(0,\, y) = 1 - \frac{1}{9}(1-\alpha)^2y^2 \le 1 \quad \text{and} \quad T(4,\, y) = \frac{1}{9}(-4 + 20\alpha - \alpha^2 - 6\alpha^3).$$

Also for all $x \in [0,4]$, we have

$$T(x,\,0) = 1 - \frac{1}{4}(1-\alpha)^2 x + \frac{1}{144}(1-\alpha)^2(5-6\alpha)x^2 = t_1(x),$$

and

$$T(x,\,1) = 1+\frac{1}{18}(1-\alpha)^2(-10+3\alpha)x-\frac{1}{36}(1-\alpha)^2(-2+3\alpha)x^2-\frac{1}{9}(1-\alpha)^2 = t_2(x).$$

We note that $t_1'(x) = 0$ if and only if $x = x_3 = 36/(5-6\alpha) \notin (0,\,4)$. Therefore, we need to check at the end point of $[0,4]$ and so $t_1(0) = 1$ and $t_1(4) = (-4+20\alpha-\alpha^2-6\alpha^3)/9$. Further, $t_2'(x) = 0$ if and only if $x = x_4 = (10-3\alpha)/(2-3\alpha) \notin (0,\,4)$. A simple calculation gives

$$\frac{\partial T}{\partial y} = 0 = \frac{\partial T}{\partial x} \text{ if and only if } x = x_5 = \frac{2}{(1-\alpha)^2},\; y = y_5 = \frac{3\alpha-2}{1-4\alpha+2\alpha^2}$$

but $y_5 \notin (0,\,1)$. Therefore, for $2/3 \le \alpha \le 1$, $T_{3,1}(f) \le \max\{T(0,\,0),\, T(4,\,0)\} = 1$. Thus, for all $\alpha \in [0,\,1]$, we get the desired upper estimate.

Next we determine the lower bound on $T_{3,1}(f)$. For $2/3 \le \alpha \le 1$, we have

$$\begin{aligned}\Gamma(p_1^2, |\xi|, \mathrm{Re}(\xi)) &\ge \Gamma(p_1^2, |\xi|, |\xi|)\\ &\ge \Gamma(4,1,1)\\ &= \frac{(-4+20\alpha-\alpha^2-6\alpha^3)}{9}.\end{aligned}$$

Combining the discussions of the above two cases, we conclude the desired estimates for all $\alpha \in [0,\,1]$. The lower and upper bounds are sharp for the function $\tilde{f}_0$ and $\tilde{f}_1$ defined by (2.8) and (2.10), respectively. □

If $\alpha = 0$, Theorem 2.4 yields the following sharp bound on Hermitian–Toeplitz determinant of third order of convex functions with respect to symmetric points.

Corollary 2.5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{K}_s$. Then*

$$-\frac{4}{9} \le T_{3,1}(f) \le 1.$$

3. The classes $\mathcal{S}^*_{s,\,L}$, $\mathcal{S}^*_{s,\,e}$ and $\mathcal{S}^*_{s,\,RL}$

Theorem 3.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{S}^*_{s,\,L}$. Then the following sharp bounds hold:*

$$\frac{221}{256} \le T_{3,1}(f) \le 1.$$

Proof. Let $f \in \mathcal{S}^*_{s,\,L}$. Then, for some $p \in \mathcal{P}$ of the form (2.1), we have

$$\mathcal{T}_s(f) = \sqrt{\frac{2p(z)}{p(z)+1}},\; z \in \mathbb{D}.$$

On comparison of the coefficients of like power terms, we get

$$a_2 = \frac{p_1}{8} \quad \text{and} \quad a_3 = \frac{1}{64}(-5p_1^2 + 8p_2). \tag{3.1}$$

Using Lemma 2.1 and expression (3.1), a computation gives

$$T_{3,1}(f) := 1 + \frac{1}{4096}(-3p_1^4 - 128p_1^2 - 16(4-p_1^2)^2|\xi|^2 + 16(4-p_1^2)p_1^2\,\mathrm{Re}(\xi))$$

for some $\xi \in \overline{\mathbb{D}}$. As before, with the settings $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, we have

$$T_{3,1}(f) \le 1 + \frac{1}{4096}(-3x^2 - 128x - 16(4-x)^2y^2 + 16(4-x)xy) =: \Upsilon_1(x,y)$$

and

$$T_{3,1}(f) \ge 1 + \frac{1}{4096}(-3x^2 - 128x - 16(4-x)^2y^2 - 16(4-x)xy) =: \Upsilon_2(x,y).$$

Proceeding as in the proof of Theorem 2.4, we arrive at

$$\Upsilon_1(x,y) \le 1 \text{ and } \Upsilon_2(x,y) \ge \frac{221}{258}$$

for all $(x,y) \in [0,4] \times [0,1]$ which gives the desired bounds on $T_{3,1}(f)$. The upper bound is sharp for the function f_2 satisfying $\mathcal{T}_s(f_2) = \sqrt{1+z^3}$, that is

$$f_2(z) = z + \frac{1}{8}z^4 + \cdots$$

and the lower bound is sharp for the function f_3 satisfying $\mathcal{T}_s(f_3) = \sqrt{1+z}$, that is

$$f_3(z) = z + \frac{1}{4}z^2 - \frac{1}{16}z^3 + \cdots .$$

This ends the proof. □

Theorem 3.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{S}^*_{s,\,e}$. Then the best possible bounds on Hermitian–Toeplitz are given by*

$$\frac{9}{16} \le T_{3,1}(f) \le 1.$$

Proof. Let $f \in \mathcal{S}^*_{s,\,e}$. Then, for some $p \in \mathcal{P}$ of the form (2.1), we have

$$\mathcal{T}_s(f) = e^{\left(\frac{p(z)-1}{p(z)+1}\right)},\ z \in \mathbb{D}.$$

On comparison of the coefficients of like power terms, we get

$$a_2 = \frac{p_1}{4} \quad \text{and} \quad a_3 = \frac{1}{16}(-p_1^2 + 4p_2). \tag{3.2}$$

As before, using Lemma 2.1 and expression (3.2), a computation gives

$$T_{3,1}(f) : = 1 + \frac{1}{256}(p_1^4 - 32p_1^2 - 4(4-p_1^2)^2|\xi|^2)$$

for some $\xi \in \overline{\mathbb{D}}$. With the settings $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, we have

$$T_{3,1}(f) = 1 + \frac{1}{256}(x^2 - 32x - 4(4-x)^2y^2) =: \Xi(x,\, y).$$

By making use of second derivative test for maximum and minimum value of function $\Xi(x,\, y)$, we see that

$$\max_{x,y\in S} \Xi(x,\, y) = 1 \quad \text{and} \quad \min_{x,y\in S} \Xi(x,\, y) = \frac{9}{16}.$$

The equality in case of the lower bound holds for the function f_4 satisfying $\mathcal{T}_s(f_4) = e^z$, $z \in \mathbb{D}$ that is for

$$f_4(z) = z + \frac{1}{4}z^4 + \cdots,$$

whereas the upper bound is sharp for the function f_5 satisfying $\mathcal{T}_s(f_5) = e^{z^3}$, that is for

$$f_5(z) = z + \frac{1}{2}z^2 + \frac{1}{4}z^3 + \cdots.$$

This concludes the proof. □

Theorem 3.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the class $\mathcal{S}^*_{s,\,RL}$. Then the best possible bounds on Hermitian–Toeplitz is given by*

$$\frac{1}{256}(863 - 444\sqrt{2}) \leq T_{3,1}(f) \leq 1.$$

Proof. Let $f \in \mathcal{S}^*_{s,\,RL}$. Then, for some $p \in \mathcal{P}$ of the form (2.1), we have

$$\mathcal{T}_s(f) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{2p(z)}{2(\sqrt{2}-1)p(z) + 3 - 2\sqrt{2}}}, \; z \in \mathbb{D}.$$

Comparison of the coefficients gives

$$a_2 = \frac{(5-3\sqrt{2})p_1}{8} \quad \text{and} \quad a_3 = \frac{(51-39\sqrt{2})p_1^2 + 8(5-3\sqrt{2})p_2}{64}. \tag{3.3}$$

Using Lemma 2.1 and expression (3.3), we have

$$\begin{aligned} T_{3,1}(f) = 1 + \frac{1}{4096}\Big(&(1983 - 1404\sqrt{2})p_1^4 - 128(43 - 30\sqrt{2})p_1^2 \\ &-(688 - 480\sqrt{2})(4 - p_1^2)^2|\xi|^2 \\ &-56(38 - 27\sqrt{2})(4 - p_1^2)p_1^2 \operatorname{Re}(\xi)\Big) \end{aligned}$$

for some $\xi \in \overline{\mathbb{D}}$. In view of the fact $-|z| \leq \operatorname{Re} z \leq |z|$ and settings $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$, we have

$$\begin{aligned} T_{3,1}(f) \leq 1 + \frac{1}{4096}\Big(&(1983 - 1404\sqrt{2})x^2 - 128(43 - 30\sqrt{2})x \\ &-(688 - 480\sqrt{2})(4 - x)^2y^2 \\ &+56(38 - 27\sqrt{2})(4 - x)xy\Big) = \boldsymbol{\eta}_1(x,y) \end{aligned}$$

and

$$T_{3,1}(f) \geq 1 + \frac{1}{4096}\Big((1983 - 1404\sqrt{2})x^2 - 128(43 - 30\sqrt{2})x$$
$$-(688 - 480\sqrt{2})(4-x)^2y^2$$
$$-56(38 - 27\sqrt{2})(4-x)xy\Big) = \boldsymbol{\eta}_2(x,y).$$

Using second derivative test, we see that the maximum value of the function $\boldsymbol{\eta}_1(x,y)$ and minimum value of the function $\boldsymbol{\eta}_2(x,y)$ in rectangular domain $S = [0,4] \times [0,1]$ are given by

$$\max_{x,y\in S} \boldsymbol{\eta}_1(x,y) = 1 \quad \text{and} \quad \min_{x,y\in S} \boldsymbol{\eta}_2(x,y) = \frac{1}{256}(863 - 444\sqrt{2}).$$

The upper bound is sharp in case of the function f_6 satisfying

$$\mathcal{T}_s(f_6) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z^3}{1+2(\sqrt{2}-1)z^3}}, \; z \in \mathbb{D}$$

that is for

$$f_6(z) = z + \frac{5-3\sqrt{2}}{8}z^4 + \cdots.$$

The lower bound is sharp in case of the function f_7 satisfying

$$\mathcal{T}_s(f_7) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}, \; z \in \mathbb{D}$$

that is for

$$f_7(z) = z + \frac{5-3\sqrt{2}}{4}z^2 + \frac{71-51\sqrt{2}}{4}z^3 + \cdots.$$

This ends the proof. □

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