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ANOTHER CHARACTERIZATION OF THE NORMING SET OF $T \in \mathcal{L}(^2 l_\infty^2)$

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Abstract. In this paper we present another characterization of the norming set of $T \in \mathcal{L}(^2 l_\infty^2)$ in terms of $\text{Norm}(T) \cap \Omega$ whose proofs are more systematic than those of Kim [6], where $\Omega = \big\{\big((1,1),(1,1)\big), \big((1,1),(1,-1)\big), \big((1,-1),(1,1)\big), \big((1,-1),(1,-1)\big)\big\}$.

1. Introduction

In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$, $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\ldots,x_n)\in S_E\times\cdots\times S_E} |T(x_1,\ldots,x_n)|$. $\mathcal{L}_s(^n E)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1,\ldots,x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1,\ldots,x_n)| = \|T\|$.

For $T \in \mathcal{L}(^n E)$, we define

$$\text{Norm}(T) = \Big\{(x_1,\ldots,x_n) \in E^n : (x_1,\ldots,x_n) \text{ is a norming point of } T\Big\}.$$

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$\text{Norm}(T)$ is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that $\text{Norm}(T) = \emptyset$ or an infinite set.

Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s({}^2 c_0).$$

We claim that $\text{Norm}(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that $\text{Norm}(T) \neq \emptyset$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \le \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i|\ |y_i| \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\text{Norm}(T) = \emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s({}^2 c_0).$$

Then,

$$\text{Norm}(T) = \Big\{\Big((\pm 1, x_2, x_3, \ldots),\ (\pm 1, y_2, y_3, \ldots)\Big) \in c_0 \times c_0 : |x_j| \le 1, \\ |y_j| \le 1 \text{ for } j \ge 2\Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}({}^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}({}^n E)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P. Notice that $\text{Norm}(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classify $\text{Norm}(P)$ for every $P \in \mathcal{P}({}^2 l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\mathrm{Norm}(T) \neq \emptyset$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* n-linear form and if $\mathrm{Norm}(P) \neq \emptyset$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* n-homogeneoue polynomial (See [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about $\mathrm{Norm}(T)$ for $T \in \mathcal{L}(^n E)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l^2_\infty = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l^2_∞ and $T \in \mathcal{L}(^n E)$, $\mathrm{Norm}(T) \neq \emptyset$ since S_E is compact. Kim ([8–10]) classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s(^2 l^2_\infty)$, $\mathcal{L}_s(^2 l_1^2)$ or $\mathcal{L}_s(^3 l_1^2)$. Kim [6] classified the norming set of $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_\infty)$ in terms of its coefficients a, b, c and d, where $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dy_1x_2$.

Let

$$\Omega = \Big\{ \Big((1,1),(1,1)\Big),\ \Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big) \Big\}.$$

In this paper we present another characterization of the norming set of $T \in \mathcal{L}(^2 l^2_\infty)$ in terms of $\mathrm{Norm}(T) \cap \Omega$.

2. Results

Let $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dy_1x_2 \in \mathcal{L}(^2 l^2_\infty)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity we denote $T = (a, b, c, d)$.

Theorem 2.1 ([6])**.** *Let $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_\infty)$ for some $a, b, c, d \in \mathbb{R}$. Then*

$$\|T\| = \max\{|a + b| + |c + d|, |a - b| + |c - d|\}.$$

Notice that if $\|T\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 1$ and $|d| \leq 1$.

Lemma 2.2 ([6])**.** *Let $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_\infty)$ for some $a, b, c, d \in \mathbb{R}$. Then there exists $T' = (a^*, b^*, c^*, d^*) \in \mathcal{L}(^2 l^2_\infty)$ such that $a^*, b^*, c^*, d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ with $a^* \geq |b^*|$ and $a^* \geq c^* \geq d^* \geq 0$ and $\|T\| = \|T'\|$.*

Proof. If $a < 0$, taking $-T$, we assume that $a \geq 0$. If $|b| > a$, we define $T_1 = (|b|, \mathrm{sign}(b)a, \mathrm{sign}(b)d, \mathrm{sign}(b)c)$. Then, $\|T_1\| = \|T\|$. Hence, we may assume that $a \geq |b|$. If $c < 0$, we define $T_2 = (a, -b, -c, d)$. Then, $\|T_2\| = \|T\|$. Hence, we may assume that $a \geq |b|$, $c > 0$. If $d < 0$, we define $T_3 = (a, -b, c, -d)$. Then, $\|T_3\| = \|T\|$. Hence, we may assume that $a \geq |b|$, $c > 0$, $d > 0$. If $c < d$, we define $T_4 = (a, b, d, c)$. Then, $\|T_4\| = \|T\|$. Hence, we may assume that $a \geq |b|$, $c \geq d$. If $a < c$, $b \geq 0$, we define $T_5 = (c, d, a, b)$. Then, $\|T_5\| = \|T\|$. Hence, we may assume that $a \geq |b|$, $c \geq d$. If $a < c$, $b < 0$, we define $T_6 = (c, -d, a, -b)$. Then, $\|T_6\| = \|T\|$. Hence, we may assume that $a \geq |b|$ and $a \geq c \geq d \geq 0$. Therefore, we can find a bilinear form T' which satisfies the conditions of the lemma. □

Lemma 2.3. *Let $n \in \mathbb{N}$ and $w_j, t_j \in \mathbb{R}$ for $j = 1, \ldots, n$ be such that $|w_j| \leq 1$, $0 \leq t_j \leq 1$ and $\sum_{j=1}^{n} t_j = 1$. Suppose that*

$$1 = \Big| \sum_{j=1}^{n} t_j w_j \Big|.$$

If $|w_{j_0}| < 1$ for some $j_0 \in \{1, \ldots, n\}$, then $t_{j_0} = 0$.

Proof. Assume the contrary. It follows that

$$\begin{aligned} 1 = \Big| \sum_{j=1}^{n} t_j w_j \Big| &\leq t_{j_0} |w_{j_0}| + \sum_{1 \leq j \neq j_0 \leq n} t_j |w_j| \\ &< t_{j_0} + \sum_{1 \leq j \neq j_0 \leq n} t_j |w_j| \text{ (because } t_{j_0} > 0 \text{ and } |w_{j_0}| < 1) \\ &\leq t_{j_0} + \sum_{1 \leq j \neq j_0 \leq n} t_j = 1, \end{aligned}$$

which is a contradiction. Therefore, we complete the proof. □

Theorem 2.4. *Let $T \in \mathcal{L}(^2 l_\infty^2)$ with $\|T\| = 1$ and $a \geq |b|$, $a \geq c \geq d \geq 0$. Then*

$$\text{Norm}(T) \subseteq \Big\{ \Big(\pm (t(1,1) + (1-t)(1,-1)),\ \pm (s(1,1) + (1-s)(1,-1)) \Big) : 0 \leq t, s \leq 1 \Big\}.$$

Proof. Let

$$\begin{aligned} B_1 &= \{t(0,1) + (1-t)(1,1) : 0 \leq t \leq 1\}, \\ B_2 &= \{t(1,1) + (1-t)(1,-1) : 0 \leq t \leq 1\}, \\ B_3 &= \{t(1,-1) + (1-t)(0,-1) : 0 \leq t \leq 1\}. \end{aligned}$$

Let $(X, Y) \in \text{Norm}(T)$. Without loss of generality we may assume that $X, Y \in \bigcup_{1 \leq j \leq 3} B_j$.

Claim. $(X, Y) \in B_2$.

Suppose that $X \in B_1$.
If $Y \in B_1$, then

$$\begin{aligned} (*)\ 1 = |T(X,Y)| &= |T(t(0,1) + (1-t)(1,1), s(0,1) + (1-s)(1,1))| \\ &= \Big| tsT((0,1),(0,1)) + t(1-s)T((0,1),(1,1)) \\ &\quad + (1-t)sT((1,1),(0,1)) + (1-t)(1-s)T((1,1),(1,1)) \Big| \\ &\leq ts|T((0,1),(0,1))| + t(1-s)|T((0,1),(1,1))| \\ &\quad + (1-t)s|T((1,1),(0,1))| + (1-t)(1-s)|T((1,1),(1,1))| \\ &= ts|b| + t(1-s)|b+d| + (1-t)s|b+c| + (1-t)(1-s)|a+b+c+d| \\ &\leq 1. \end{aligned}$$

By Lemma 2.3, $ts = t(1-s) = (1-t)s = 0$. Hence, $t = s = 0$. Therefore, $X = Y = (1,1) \in B_2$.

If $Y \in B_2$, then

$$\begin{aligned}(*)\ 1 = |T(X,Y)| &= |T(t(0,1)+(1-t)(1,1), s(1,1)+(1-s)(1,-1))| \\ &= \Big| tsT((0,1),(1,1)) + t(1-s)T((0,1),(1,-1)) \\ &\quad + (1-t)sT((1,1),(1,1)) + (1-t)(1-s)T((1,1),(1,-1))\Big| \\ &\le ts|T((0,1),(1,1))| + t(1-s)|T((0,1),(1,-1))| \\ &\quad + (1-t)s|T((1,1),(1,1))| + (1-t)(1-s)|T((1,1),(1,-1))| \\ &= ts|b+d| + t(1-s)|-b+d| + (1-t)s|a+b+c+d| \\ &\quad + (1-t)(1-s)|a-b-c+d| \le 1.\end{aligned}$$

By Lemma 2.3, $ts = t(1-s) = 0$. Hence, $t = 0$. Therefore, $X = (1,1), Y \in B_2$.

If $Y \in B_3$, then

$$\begin{aligned}(*)\ 1 = |T(X,Y)| &= |T(t(0,1)+(1-t)(1,1), s(1,-1)+(1-s)(0,-1))| \\ &= \Big| tsT((0,1),(1,-1)) + t(1-s)T((0,1),(0,-1)) \\ &\quad + (1-t)sT((1,1),(1,-1)) + (1-t)(1-s)T((1,1),(0,-1))\Big| \\ &\le ts|T((0,1),(1,-1))| + t(1-s)|T((0,1),(0,-1))| \\ &\quad + (1-t)s|T((1,1),(1,-1))| + (1-t)(1-s)|T((1,1),(0,-1))| \\ &= ts|-b+d| + t(1-s)|b| + (1-t)s|a-b-c+d| \\ &\quad + (1-t)(1-s)|-b-c| \le 1.\end{aligned}$$

By Lemma 2.3, $ts = t(1-s) = (1-t)(1-s) = 0$. Hence, $t = 0$, $s = 1$. Therefore, $X = (1,1), Y = (1,-1) \in B_2$.

Suppose that $X \in B_2$.

If $Y \in B_1$, then

$$\begin{aligned}(*)\ 1 = |T(X,Y)| &= |T(t(1,1)+(1-t)(1,-1), s(0,1)+(1-s)(1,1))| \\ &= \Big| tsT((1,1),(0,1)) + t(1-s)T((1,1),(1,1)) \\ &\quad + (1-t)sT((1,-1),(0,1)) + (1-t)(1-s)T((1,-1),(1,1))\Big| \\ &\le ts|T((1,1),(0,1))| + t(1-s)|T((1,1),(1,1))| \\ &\quad + (1-t)s|T((1,-1),(0,1))| + (1-t)(1-s)|T((1,-1),(1,1))| \\ &= ts|b+c| + t(1-s)|a+b+c+d| + (1-t)s|-b+c| \\ &\quad + (1-t)(1-s)|a-b+c-d| \le 1.\end{aligned}$$

By Lemma 2.3, $ts = (1-t)s = 0$. Hence, $s = 0$. Therefore, $X, Y = (1,1) \in B_2$.

If $Y \in B_3$, then

$$
\begin{aligned}
(*)\ 1 = |T(X,Y)| &= |T(t(1,1)+(1-t)(1,-1), s(1,-1)+(1-s)(0,-1))| \\
&= \Big| tsT((1,1),(1,-1)) + t(1-s)T((1,1),(0,-1)) \\
&\quad + (1-t)sT((1,-1),(1,-1)) + (1-t)(1-s)T((1,-1),(0,-1)) \Big| \\
&\le ts|T((1,1),(1,-1))| + t(1-s)|T((1,1),(0,-1))| \\
&\quad + (1-t)s|T((1,-1),(1,-1))| + (1-t)(1-s)|T((1,-1),(0,-1))| \\
&= ts|a-b-c+d| + t(1-s)|-b-c| + (1-t)s|a+b-c-d| \\
&\quad + (1-t)(1-s)|b-c| \le 1.
\end{aligned}
$$

By Lemma 2.3, $t(1-s) = (1-t)(1-s) = 0$. Hence, $s = 1$. Therefore, $X, Y = (1,-1) \in B_2$.

Suppose that $X \in B_3$.
If $Y \in B_1$, then

$$
\begin{aligned}
(*)\ 1 = |T(X,Y)| &= |T(t(0,-1)+(1-t)(1,-1), s(0,1)+(1-s)(1,1))| \\
&= \Big| tsT((0,-1),(0,1)) + t(1-s)T((0,-1),(1,1)) \\
&\quad + (1-t)sT((1,-1),(0,1)) + (1-t)(1-s)T((1,-1),(1,1)) \Big| \\
&\le ts|T((0,-1),(0,1))| + t(1-s)|T((0,-1),(1,1))| \\
&\quad + (1-t)s|T((1,-1),(0,1))| + (1-t)(1-s)|T((1,-1),(1,1))| \\
&= ts|b| + t(1-s)|b+d| + (1-t)s|-b+c| \\
&\quad + (1-t)(1-s)|a-b+c-d| \le 1.
\end{aligned}
$$

By Lemma 2.3, $ts = t(1-s) = (1-t)s = 0$. Hence, $t = s = 0$. Therefore, $X = (1,-1)$, $Y = (1,1) \in B_2$.
If $Y \in B_2$, then

$$
\begin{aligned}
(*)\ 1 = |T(X,Y)| &= |T(t(0,-1)+(1-t)(1,-1), s(1,1)+(1-s)(1,-1))| \\
&= \Big| tsT((0,-1),(1,1)) + t(1-s)T((0,-1),(1,-1)) \\
&\quad + (1-t)sT((1,-1),(1,1)) + (1-t)(1-s)T((1,-1),(1,-1)) \Big| \\
&\le ts|T((0,-1),(1,1))| + t(1-s)|T((0,-1),(1,-1))| \\
&\quad + (1-t)s|T((1,-1),(1,1))| + (1-t)(1-s)|T((1,-1),(1,-1))| \\
&= ts|b+d| + t(1-s)|b-d| + (1-t)s|a-b+c-d| \\
&\quad + (1-t)(1-s)|a+b-c-d| \le 1.
\end{aligned}
$$

By Lemma 2.3, $ts = t(1-s) = 0$. Hence, $t = 0$. Therefore, $X = (1,-1), Y \in B_2$.
If $Y \in B_3$, then

$$
(*)\ 1 = |T(X,Y)| = |T(t(0,-1)+(1-t)(1,-1), s(1,-1)+(1-s)(0,-1))|
$$

$$= \Big|tsT((0,-1),(1,-1)) + t(1-s)T((0,-1),(0,-1)) + (1-t)sT((1,-1),(1,-1)) + (1-t)(1-s)T((1,-1),(0,-1))\Big|$$
$$\leq ts|T((0,-1),(1,-1))| + t(1-s)|T((0,-1),(0,-1))| + (1-t)s|T((1,-1),(1,-1))| + (1-t)(1-s)|T((1,-1),(0,-1))|$$
$$= ts|b-d| + t(1-s)|b| + (1-t)s|a+b-c-d| + (1-t)(1-s)|b-c| \leq 1.$$

By Lemma 2.3, $ts = t(1-s) = (1-t)(1-s) = 0$. Hence, $t = 0$, $s = 1$. Therefore, $X = Y = (1,-1) \in B_2$. We complete the proof. □

Let

$$l_1 = T((1,1),(1,1)),\ l_2 = T((1,-1),(1,1)),$$
$$l_3 = T((1,1),(1,-1)),\ l_4 = T((1,-1),(1,-1)).$$

If $T \in \mathcal{L}(^2l^2_\infty)$ with $\|T\| = 1$ and $a \geq |b|$, $a \geq c \geq d \geq 0$, then $l_j \geq 0$ for $j = 1, 2$.

Kim [6] classified the norming set of $T = (a,b,c,d) \in \mathcal{L}(^2l^2_\infty)$ in terms of its coefficients a, b, c and d, where $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dy_1x_2$. We present an another characterization of Norm(T) whose proofs are more systematic than those of [6].

We are in position to prove the main result of this paper.

Theorem 2.5. *Let $T \in \mathcal{L}(^2l^2_\infty)$ with $\|T\| = 1$ and $a \geq |b|$, $a \geq c \geq d \geq 0$. Let* $\Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big)\Big\}$.

We consider four cases:

Case 1. $|\text{Norm}(T) \cap \Omega| = 1$

$$\text{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \text{Norm}(T) \cap \Omega\}.$$

Case 2. $|\text{Norm}(T) \cap \Omega| = 2$

Suppose that

$$\text{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,1),(1,-1)\Big)\Big\}.$$

If $l_3 = 1$, then

$$\text{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(1, 2t-1)\Big) : 0 \leq t \leq 1\Big\}.$$

If $l_3 = -1$, then

$$\text{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(1,1)\Big), \Big(\pm(1,1), \pm(1,-1)\Big)\Big\}.$$

Suppose that

$$\text{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,-1),(1,1)\Big)\Big\}.$$

Then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1, 2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that

$$\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big)\Big\}$$
$$\textit{or } \Big\{\Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,1)\Big)\Big\}.$$

Then

$$\mathrm{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \mathrm{Norm}(T)\cap\Omega\}.$$

Suppose that

$$\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,-1)\Big)\Big\}.$$

If $l_3 = l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1, 2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 \neq l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,-1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big)\Big\}.$$

Suppose that

$$\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,-1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big)\Big\}.$$

If $l_4 = 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,2t-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_4 = -1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big)\Big\}.$$

Case 3. $|\,\mathrm{Norm}(T)\cap\Omega| = 3$

Suppose that $\Big((1,1),(1,1)\Big) \notin \mathrm{Norm}(T)\cap\Omega$.

If $l_3 l_4 = 1 = l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = 1 = -l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big),$$
$$\Big(\pm (1,2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = -1 = l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,2t-1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big),$$

$$\Big(\pm (1,1), \pm(1,-1)\Big)\Big\}.$$

If $l_3 l_4 = -1 = -l_4$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big(\pm (1,-1), \pm(1,1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big)\Big\}.$$

Suppose that $\Big((1,1),(1,-1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

If $l_4 = 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_4 = -1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,-1), \pm(1,1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that $\Big((1,-1),(1,1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

If $l_3 l_4 = 1 = l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = 1 = -l_4$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big(\pm (1,1), \pm(1,1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big), \Big(\pm (1,2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = -1 = -l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big)\Big\}.$$

If $l_3 l_4 = -1 = l_4$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big(\pm (1,1), \pm(1,2t-1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big)\Big\}.$$

Suppose that $\Big((1,-1),(1,-1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

If $l_3 = 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 = -1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Case 4. $|\,\mathrm{Norm}(T) \cap \Omega| = 4$

If $l_j = 1$ *for* $j = 3, 4$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,2t-1),\ \pm(1,2s-1)\Big) : 0 \le t, s \le 1\Big\}.$$

If $l_3 = -1 = -l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{ \pm \Big((1,1),(1,-1)\Big), \Big(\pm (1,-1),\ \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 = 1 = -l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{ \pm \Big((1,-1),(1,-1)\Big), \Big(\pm (1,1),\ \pm(1,2t-1)\Big), \Big(\pm (1,2t-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 = -1 = l_4$, *then*

$$\mathrm{Norm}(T) = \Big\{ \pm \Big((1,1),(1,1)\Big), \pm\Big((1,-1),(1,1)\Big), \Big(\pm (1,2t-1), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

Proof. Let $(X,Y) \in \mathrm{Norm}(T)$. By Theorem 2.4, we may assume that $X = t(1,1) + (1-t)(1,-1)$ and $Y = s(1,1) + (1-s)(1,-1)$ for $0 \le t, s \le 1$. It follows that

$$\begin{aligned}(*)\ 1 &= |T(X,Y)| = |T(t(1,1)+(1-t)(1,-1), s(1,1)+(1-s)(1,-1))| \\ &= \Big| tsT((1,1),(1,1)) + t(1-s)T((1,1),(1,-1)) \\ &\quad + (1-t)sT((1,-1),(1,1)) + (1-t)(1-s)T((1,-1),(1,-1))\Big| \\ &\le ts|T((1,1),(1,1))| + t(1-s)|T((1,1),(1,-1))| \\ &\quad + (1-t)s|T((1,-1),(1,1))| + (1-t)(1-s)|T((1,-1),(1,-1))| \le 1.\end{aligned}$$

Case 1. $|\,\mathrm{Norm}(T) \cap \Omega| = 1$

Suppose that $\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big)\Big\}$.
By $(*)$ and Lemma 2.3,

$$t(1-s) = (1-t)s = (1-t)(1-s) = 0.$$

Hence, $t = s = 1$ and

$$\mathrm{Norm}(T) = \Big\{(\pm(1,1), \pm(1,1))\Big\}.$$

Suppose that $\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,-1)\Big)\Big\}$.
By $(*)$ and Lemma 2.3,

$$ts = (1-t)s = (1-t)(1-s) = 0.$$

Hence, $t = 1$, $s = 0$ and

$$\mathrm{Norm}(T) = \Big\{(\pm(1,1), \pm(1,-1))\Big\}.$$

Suppose that $\mathrm{Norm}(T) \cap \Omega = \Big\{\Big((1,-1),(1,1)\Big)\Big\}$.

By $(*)$ and Lemma 2.3,

$$ts = t(1-s) = (1-t)(1-s) = 0.$$

Hence, $t = 0$, $s = 1$ and

$$\text{Norm}(T) = \Big\{(\pm(1,-1), \pm(1,1))\Big\}.$$

Suppose that $\text{Norm}(T) \cap \Omega = \Big\{\Big((1,-1),(1,-1)\Big)\Big\}$.
By $(*)$ and Lemma 2.3,

$$ts = t(1-s) = (1-t)s = 0.$$

Hence, $t = s = 0$ and

$$\text{Norm}(T) = \Big\{(\pm(1,-1), \pm(1,-1))\Big\}.$$

Case 2. $|\,\text{Norm}(T) \cap \Omega| = 2$

Suppose that

$$\text{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,1),(1,-1)\Big)\Big\}.$$

By $(*)$ and Lemma 2.3,

$$(1-t)s = (1-t)(1-s) = 0.$$

Hence, $t = 1$ and

$$\text{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(t(1,1) + (1-t)\,\text{sign}(l_3)(1,-1))\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 = 1$, then

$$\text{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(t(1,1) + (1-t)(1,-1))\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 = -1$, then

$$\text{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(1,1)\Big), \Big(\pm(1,1), \pm(1,-1)\Big)\Big\}.$$

Suppose that

$$\text{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,-1),(1,1)\Big)\Big\}.$$

By $(*)$ and Lemma 2.3,

$$t(1-s) = (1-t)(1-s) = 0.$$

Hence, $s = 1$ and

$$\text{Norm}(T) = \Big\{\Big(\pm(t + (1-t)(1,-1), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that

$$\text{Norm}(T) \cap \Omega = \Big\{\Big((1,1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big)\Big\} \quad \text{or } \Big\{\Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,1)\Big)\Big\}.$$

By $(*)$ and Lemma 2.3,

$$t(1-s) = (1-t)s = 0.$$

If $t = 0$, then $((1,-1),(1,-1)) \in \mathrm{Norm}(T)$. If $s = 1$, then $((1,1),(1,1)) \in \mathrm{Norm}(T)$. Hence,

$$\mathrm{Norm}(T) = \{(\pm(1,1), \pm(1,1)), (\pm(1,-1), \pm(1,-1))\}.$$

Suppose that

$$\mathrm{Norm}(T) \cap \Omega = \Big\{ \Big((1,1),(1,-1)\Big),\ \Big((1,-1),(1,-1)\Big) \Big\}.$$

By $(*)$ and Lemma 2.3,

$$ts = (1-t)s = 0.$$

If $t = 0$, then $((1,-1),(1,-1)) \in \mathrm{Norm}(T)$. If $s = 0$, then

$$\Big(t\,\mathrm{sign}(l_3)(1,1) + (1-t)\,\mathrm{sign}(l_4)(1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

If $l_3 = l_4$, then

$$\mathrm{Norm}(T) = \Big\{ \Big(\pm(t(1,1) + (1-t)(1,-1), \pm(1,-1) \Big) : 0 \leq t \leq 1 \Big\}.$$

If $l_3 \neq l_4$, then

$$\mathrm{Norm}(T) = \Big\{ \Big(\pm(1,1), \pm(1,-1) \Big), \Big(\pm(1,-1), \pm(1,-1) \Big) \Big\}.$$

Suppose that

$$\mathrm{Norm}(T) \cap \Omega = \Big\{ \Big((1,-1),(1,1)\Big),\ \Big((1,-1),(1,-1)\Big) \Big\}.$$

By $(*)$ and Lemma 2.3,

$$ts = t(1-s) = 0.$$

Hence, $t = 0$ and

$$\mathrm{Norm}(T) = \Big\{ \Big(\pm(1,-1), \pm(t(1,1) + (1-t)\,\mathrm{sign}(l_4)(1,-1)) \Big) : 0 \leq t \leq 1 \Big\}.$$

If $l_4 = 1$, then

$$\mathrm{Norm}(T) = \Big\{ \Big(\pm(1,-1), \pm(t(1,1) + (1-t)(1,-1)) \Big) : 0 \leq t \leq 1 \Big\}.$$

If $l_4 = -1$, then

$$\mathrm{Norm}(T) = \Big\{ \Big(\pm(1,-1), \pm(1,1) \Big), \Big(\pm(1,-1), \pm(1,-1) \Big) \Big\}.$$

Case 3. $|\,\mathrm{Norm}(T) \cap \Omega| = 3$

Suppose that $\Big((1,1),(1,1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

By $(*)$ and Lemma 2.3, $ts = 0$. If $t = 0$, then

$$\Big((1,-1), t(1,1) + (1-t)\,\mathrm{sign}(l_4)(1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 0$, then

$$\Big(t\,\mathrm{sign}(l_3)(1,1) + (1-t)\,\mathrm{sign}(l_4)(1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

If $l_3 l_4 = 1 = l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(t(1,1) + (1-t)(1,-1))\Big), \Big(\pm(t(1,1) + (1-t)(1,-1)), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = 1 = -l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(1,1)\Big), \Big(\pm(1,-1), \pm(1,-1)\Big), \Big(\pm(t(1,1) + (1-t)(1,-1)), \pm(1,-1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_3 l_4 = -1 = l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(t(1,1) + (1-t)(1,-1))\Big), \Big(\pm(1,-1), \pm(1,-1)\Big), \Big(\pm(1,1), \pm(1,-1)\Big)\Big\}.$$

If $l_3 l_4 = -1 = -l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(1,1)\Big)\Big(\pm(1,-1), \pm(1,-1)\Big), \Big(\pm(1,1), \pm(1,-1)\Big)\Big\}.$$

Suppose that $\Big((1,1), (1,-1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

If $l_4 = 1$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(t(1,1) + (1-t)(1,-1))\Big), \Big(\pm(t(1,1) + (1-t)(1,-1)), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

If $l_4 = -1$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,-1), \pm(1,1)), \Big(\pm(1,-1), \pm(1,-1)\Big), \Big(\pm(t(1,1) + (1-t)(1,-1)), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that $\Big((1,-1), (1,1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

By $(*)$ and Lemma 2.3, $(1-t)s = 0$. If $t = 1$, then

$$\Big((1,1), t(1,1) + (1-t)\,\mathrm{sign}(l_3)(1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 0$, then

$$\Big(t\,\mathrm{sign}(l_3)(1,1) + (1-t)\,\mathrm{sign}(l_4)(1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

If $l_3 l_4 = 1 = l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,1), \pm(t(1,1) + (1-t)(1,-1))\Big),$$

$$\Big(\pm (t(1,1) + (1-t)(1,-1)), \pm(1,-1)\Big) : 0 \leq t \leq 1\Big\}.$$

If $l_3 l_4 = 1 = -l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big),$$
$$\Big(\pm (t(1,1) + (1-t)(1,-1)), \pm(1,-1)\Big) : 0 \leq t \leq 1\Big\}.$$

If $l_3 l_4 = -1 = -l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,1)\Big)\Big(\pm (1,-1), \pm(1,-1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big)\Big\}.$$

If $l_3 l_4 = -1 = l_4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(t(1,1) + (1-t)(1,-1))\Big),$$
$$\Big(\pm (1,1), \pm(1,-1)\Big), \Big(\pm (1,-1), \pm(1,-1)\Big)\Big\}.$$

Suppose that $\Big((1,-1),(1,-1)\Big) \notin \mathrm{Norm}(T) \cap \Omega$.

By $(*)$ and Lemma 2.3, $(1-t)(1-s) = 0$. If $t = 1$, then

$$\Big((1,1), t(1,1) + (1-t)\,\mathrm{sign}(l_3)(1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 1$, then

$$\Big(t(1,1) + (1-t)(1,-1), (1,1)\Big) \in \mathrm{Norm}(T).$$

If $l_3 = 1$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(t(1,1) + (1-t)(1,-1))\Big),$$
$$\Big(\pm (t(1,1) + (1-t)(1,-1)), \pm(1,1)\Big) : 0 \leq t \leq 1\Big\}.$$

If $l_3 = -1$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (1,1), \pm(1,1)\Big), \Big(\pm (1,1), \pm(1,-1)\Big),$$
$$\Big(\pm (t(1,1) + (1-t)(1,-1)), \pm(1,1)\Big) : 0 \leq t \leq 1\Big\}.$$

Case 4. $|\,\mathrm{Norm}(T) \cap \Omega| = 4$

By $(*)$ and Lemma 2.3, if $l_j = 1$ for $j = 3, 4$, then

$$\mathrm{Norm}(T) = \Big\{\Big(\pm (t(1,1) + (1-t)(1,-1)),\ \pm(s(1,1) + (1-s)(1,-1))\Big) :$$
$$0 \leq t, s \leq 1\Big\}.$$

Suppose that $l_3 = -1 = -l_4$. Then $t(1-s) = 0$ or 1. Suppose that $t(1-s) = 0$. If $t = 0$, then

$$\Big((1,-1), t(1,1) + (1-t)(1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 1$, then

$$\Big(t(1,1) + (1-t)(1,-1), (1,1)\Big) \in \mathrm{Norm}(T).$$

Suppose that $t(1-s) = 1$. Then $t = 1$, $s = 0$ and

$$\Big((1,1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

Hence,

$$\mathrm{Norm}(T) = \Big\{ \pm\Big((1,1),(1,-1)\Big), \Big(\pm(1,-1),\ \pm(t(1,1)+(1-t)(1,-1))\Big), \Big(\pm(t(1,1)+(1-t)(1,-1)), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that $l_3 = 1 = -l_4$.

Notice that $(1-t)(1-s) = 0$ or 1. Suppose that $(1-t)(1-s) = 0$. If $t = 1$, then

$$\Big((1,1), t(1,1) + (1-t)(1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 1$, then

$$\Big(t(1,1) + (1-t)(1,-1), (1,1)\Big) \in \mathrm{Norm}(T).$$

Suppose that $(1-t)(1-s) = 1$. Then $t = s = 0$ and

$$\Big((1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

Hence,

$$\mathrm{Norm}(T) = \Big\{ \pm\Big((1,-1),(1,-1)\Big), \Big(\pm(1,1),\ \pm(t(1,1)+(1-t)(1,-1))\Big), \Big(\pm(t(1,1)+(1-t)(1,-1)), \pm(1,1)\Big) : 0 \le t \le 1\Big\}.$$

Suppose that $l_3 = -1 = l_4$.

Notice that $ts + (1-t)s = 0$ or 1. Suppose that $ts + (1-t)s = 0$. Then $ts = (1-t)s = 0$. If $t = 0$, then

$$\Big((1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

If $s = 0$, then

$$\Big((1,1), (1,-1)\Big), \Big((1,-1), (1,-1)\Big) \in \mathrm{Norm}(T).$$

Suppose that $ts + (1-t)s = 1$. Then $t(1-s) = (1-t)(1-s) = 0$. If $t = 0$, then

$$\Big((1,-1), (1,1)\Big) \in \mathrm{Norm}(T).$$

If $s = 1$, then

$$\Big((1,1), (1,-1)\Big), \Big((1,1), (1,1)\Big) \in \mathrm{Norm}(T).$$

Hence,

$$\text{Norm}(T) = \Big\{ \pm\Big((1,1),(1,1)\Big), \pm\Big((1,-1),(1,1)\Big), \Big(\pm(t(1,1)+(1-t)(1,-1)), \pm(1,-1)\Big) : 0 \le t \le 1 \Big\}.$$

We complete the proof. □

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