GOLDEN PARA-CONTACT METRIC MANIFOLDS

GHERICI BELDJILALI AND HABIB BOUIZIR

Abstract. The purpose of the present paper is to introduce a new class of almost para-contact metric manifolds namely, Golden para-contact metric manifolds. Then, we are particularly interested in a more special type called Golden para-Sasakian manifolds, where we will study their fundamental properties and we present many examples which justify their study.

1. Introduction

The Golden section or Golden mean \( \phi \) is the positive root of the polynomial equation \( x^2 - x - 1 = 0 \); i.e., \( \phi = \frac{1 + \sqrt{5}}{2} \). The negative root of the previous equation, usually denoted by \( \phi^* \), satisfies \( \phi^* = \frac{1 - \sqrt{5}}{2} \). In the last years the Golden mean can be found in many areas of mathematical and physical research.

In [3], Crasmareanu and Hretcanu introduced and studied the Golden structures and they give relationships between it and other structures (almost product, almost tangent and almost complex). As a generalization of the Golden mean appear the metallic means (see [7]), which are the positive root of the equation \( x^2 - px - q = 0 \), where \( p, q \) are positive integers.

Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds and relations between them have been studied widely in differential geometry. Recently, the author [1] introduced the notion of Golden Riemannian manifolds of type \((r, s)\) and starting from a Golden Riemannian structure, we have established many well-known structures on a Riemannian manifold. Also, he defined a new class of Golden manifolds [2].

The notion of almost para-contact manifolds (respectively, almost para-contact Riemannian manifolds) as analogue of almost contact manifolds (respectively, almost contact Riemannian manifolds) was introduced by Sato in [5].
Remarkable that an almost contact manifold is always odd dimensional but an almost para-contact manifold could be even dimensional as well.

Here we show that there is a correspondence between the Golden Riemannian structures and the almost para-contact metric structures. This text is organized in the following way: Section 2 is devoted to the background of the structures which will be used in the sequel. In Section 3, we will give the relationship between Golden Riemannian structures and almost para-contact metric structures and then employ it to extract a new class that we will call Golden para-contact metric structures where we will study their fundamental properties. Section 4 is intended to study a more special type in this class by giving their geometric properties with concrete examples.

2. Review of needed notions

In this section, we give a brief information for Golden Riemannian manifolds and almost para-contact metric manifolds.

2.1. Golden Riemannian manifold

Let \((M, g)\) be a Riemannian manifold. A Golden structure on \((M, g)\) is a non-null tensor field \(\Phi\) of type \((1, 1)\) which satisfies the equation

\[\Phi^2 = \Phi + I,\]

where \(I\) is the identity transformation.

We say that the metric \(g\) is \(\Phi\)-compatible if

\[g(\Phi X, Y) = g(X, \Phi Y)\]

for all vector fields \(X, Y\) on \(M\).

If we substitute \(\Phi X\) into \(X\) in (2), equation (2) may also written as

\[g(\Phi X, \Phi Y) = g(\Phi^2 X, Y) = g((\Phi + I)X, Y) = g(\Phi X, Y) + g(X, Y).\]

The Riemannian metric (2) is called \(\Phi\)-compatible and \((M, \Phi, g)\) is named a Golden Riemannian manifold [3].

Here, it’s the occasion to show that each Golden structure on a Riemannian manifold generates a family of Riemannian metrics.

**Proposition 2.1.** Any Golden structure \(\Phi\) on a paracompact manifold \(M\) admits a Riemannian metric \(\Phi\)-compatible.

**Proof.** Let \(h\) be any Riemannian metric on \(M\) and define \(g\) by

\[g(X, Y) = h(\Phi X, \Phi Y) + h(X, Y) = h(\Phi X, Y) + 2h(X, Y),\]

and check the details. \(\square\)

It is known that a Golden structure \(\Phi\) is integrable if the Nijenhuis tensor \(N_{\Phi}\) vanishes [3], i.e.,

\[N_{\Phi}(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] = 0.\]
We know that the integrability of $\Phi$ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla \Phi = 0$ holds [4].

Recently, the first author [2] have defined a class of almost Golden Riemannian manifolds namely s-Golden manifolds, by:

**Definition 2.2.** Let $M$ be a differentiable manifold of dimension $n + s$. An almost $s$-Golden structure on $M$ is the data $(\Phi, (\xi_\alpha, \eta_\alpha)_s=1, g)$, where

(i) $\xi_\alpha$ is a global vector field, (called Golden vector field).
(ii) $\eta_\beta$ is a differential 1-form on $M$ such that $\eta_\beta(\xi_\alpha) = \delta_{\alpha\beta}$, where $\alpha, \beta \in \{1, \ldots, s\}$.
(iii) $g$ is a Riemannian metric such that $g(X, \xi_\alpha) = \eta_\alpha(X)$.
(iv) $\Phi$ is a tensor field of type $(1, 1)$ satisfying

$$\Phi = \phi^* I + \sqrt{5} \sum_{\alpha=1}^s \eta_\alpha \otimes \xi_\alpha$$

for all vector field $X$ on $M$.

In addition, if $\Phi$ is integrable, then $(\Phi, (\xi_\alpha, \eta_\alpha)_s=1, g)$ is an $s$-Golden structure and $(M, \Phi, (\xi_\alpha, \eta_\alpha)_s=1, g)$ is called an $s$-Golden manifold.

In this class of manifolds and for $s = 1$, there is a remarkable type called generalized $G$-Golden manifolds satisfying:

$$2\Omega(X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X = (L_\xi g)(X, Y),$$

for all vector fields $X$ and $Y$ on $M$. In particular, in an almost para-contact metric manifold we also have

$$\phi \xi = 0, \quad \eta \circ \phi = 0$$

and $\text{rank} \phi = n - 1$.

The fundamental $(0, 2)$ symmetric tensor of the almost para-contact metric structure is defined by

$$\Omega(X, Y) = g(X, \varphi Y)$$

for any vector fields $X, Y$ on $M$. Such a manifold is said to be a para-contact metric manifold if [8]

$$2\Omega(X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X = (L_\xi g)(X, Y),$$

where $L$ is the operator of Lie differentiation.
On the other hand, the almost para-contact metric structure of $M$ is said to be normal if
\[ N_{\varphi}(X, Y) = [\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi = 0 \]
for any $X, Y$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of $\varphi$, given by
\[ [\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \]

An almost para-contact metric manifold $(M, \varphi, \xi, \eta, g)$ on $M$ is said to be a para-Sasakian manifold if
\[ (\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi. \]

For more background on almost para-contact metric manifolds, we recommend the reference [5], [6] and [8].

3. Golden para-contact metric manifold

In this section, we will associate and combine the structures Golden with the structures para-contact defined on the same manifold $M$ and extract a new structure we will call it Golden para-contact metric structure.

**Theorem 3.1.** Every almost para-contact metric structure $(\varphi, \xi, \eta, g)$ on a Riemannian manifold $(M, g)$ induces only two Golden Riemannian structures on $(M, g)$, given as follows:

\[ \Phi = \frac{1}{2}I + \frac{\sqrt{5}}{2}(\varepsilon \varphi + \eta \otimes \xi), \]

where $\xi$ is the unique eigenvector of $\Phi$ associated with $\varphi$ and $\varepsilon = \pm 1$.

**Proof.** We try to write the Golden structure $\Phi$ defined on $(M, g)$, using almost para-contact metric structure $(\varphi, \xi, \eta, g)$, in the form $\Phi = a\varphi + bI + c\eta \otimes \xi$, where $a, b, c \in \mathbb{R}^+$. Thus
\[ \Phi^2 = 2ab\varphi + (a^2 + b^2)I + (c^2 - a^2 + 2bc)\eta \otimes \xi, \]
and using formula (1) we obtain the formulas (8). Moreover, we have
\[ g(\Phi X, Y) = g(X, \Phi Y) \Leftrightarrow g(\varphi X, Y) = g(X, \varphi Y) \]
for every tangent vector fields $X$ and $Y$ on $M$, which completes the demonstration. \[\square\]

Regarding expressions (8) and (3) with $s = 1$, one can ask if it’s possible to get an explicit and direct expression for $\varphi$ without $\Phi$? The answer is positive and this gives the following definition:

**Definition 3.2.** An almost Golden almost para-contact metric manifold is the quintuple $(M, \varphi, \xi, \eta, g)$ which satisfies:

\[ \eta(\xi) = 1, \quad \eta(X) = g(\xi, X), \quad \varphi X = \varepsilon(-X + \eta(X)\xi) \]

for all vector field $X$ on $M$ and $\varepsilon = \pm 1$. 
Remark 3.3. We can easily check that any almost Golden almost para-contact metric manifold is an almost para-contact metric manifold.

**Proposition 3.4.** Let \((M, \varphi, \xi, \eta, g)\) be an almost Golden almost para-contact metric manifold and the set \(\{\xi, e_i\}_{1 \leq i \leq n-1}\) of vector fields where \(\varphi e_i = -e_i\). Then we may easily check that \(\{\xi, e_i\}\) is an orthonormal basis on \(M\).

We refer to this basis as G-basis.

**Lemma 3.5.** Let \((M, \varphi, \xi, \eta, g)\) be an almost Golden almost para-contact metric manifold. If \(\nabla\) is the Levi-Civita connection, then for all vector fields \(X\) and \(Y\) on \(M\) we have
\[
(\nabla_X \varphi)Y = \varepsilon (g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi).
\]

**Proof.** Knowing that
\[
(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y,
\]
and using formulas (9), the proof is direct. □

**Proposition 3.6.** Let \((M, \varphi, \xi, \eta, g)\) be an almost Golden almost para-contact metric manifold. If \(\nabla\) is the Levi-Civita connection, then
\[
(\nabla_X \varphi)Y = 0 \iff \nabla_X \xi = 0
\]
for all vector fields \(X\) and \(Y\) on \(M\).

**Proof.** Suppose that \(\nabla \varphi = 0\), from Lemma 3.5 we have
\[
g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi = 0,
\]
taking \(Y = \xi\) we obtain \(\nabla_X \xi = 0\). The inverse is direct. □

Now, for the tensor \(N_{\varphi}\), using (7) and (9), we can check that is very simply as follows:
\[
N_{\varphi}(X, Y) = -2d\eta(\varphi^2 X, \varphi^2 Y)\xi - 2d\eta(X, Y)\xi,
\]
which give the following proposition:

**Proposition 3.7.** The almost Golden almost para-contact metric structure \((\varphi, \xi, \eta, g)\) is normal if and only if \(\eta\) is closed.

**Proof.** Suppose that \(N_{\varphi} = 0\), from formula (10), we get
\[
2d\eta(X, Y) - \eta(Y)d\eta(X, \xi) - \eta(X)d\eta(\xi, Y) = 0
\]
for \(Y = \xi\) we obtain
\[
d\eta(X, \xi) = 0,
\]
using (12) in (11) we get \(d\eta = 0\). The inverse is direct. □

**Remark 3.8.** An almost Golden almost para-contact metric structure \((\varphi, \xi, \eta, g)\) is called an almost Golden para-contact metric structure if it is normal.
In addition, we say that \((M, \varphi, \xi, \eta, g)\) is an almost Golden para-contact metric manifold if the condition (6) is satisfies. Based on these facts, we give the following definition:

**Definition 3.9.** We say that \((\varphi, \xi, \eta, g)\) is a Golden para-contact structure if and only if

\[
d\eta = 0 \quad \text{and} \quad 2\Omega = L_{\xi}g.
\]

In this case \((M, \varphi, \xi, \eta, g)\) is a Golden para-contact metric manifold.

**Theorem 3.10.** Every almost Golden almost para-contact metric structure \((\varphi, \xi, \eta, g)\) is a Golden para-contact metric structure if and only if

\[
\nabla_X\xi = \varphi X.
\]

**Proof.** The proof is direct, just use the formulas (13). \(\square\)

### 4. Golden \(\alpha\)-para-Sasakian manifolds

In this section, a generalization of para-Sasakian manifolds is included. First of all, we’re going to give the concept of Golden para-Sasakian manifolds.

**Theorem 4.1.** Every Golden para-contact metric manifold \((M, \varphi, \xi, \eta, g)\) is a para-Sasakian manifold.

**Proof.** Suppose that \((M, \varphi, \xi, \eta, g)\) is a Golden para-contact metric manifold. Thus we have

\[
\varphi X = \varepsilon(-X + \eta(X)\xi) \quad \text{and} \quad \nabla_X\xi = \varphi X.
\]

Knowing that \(\nabla_X\varphi Y = (\nabla_X\varphi)Y + \varphi\nabla_XY\), then

\[
(\nabla_X\varphi)Y = \varepsilon((\nabla_X\eta)(Y)\xi + \eta(Y)\nabla_X\xi)
\]

\[
= \varepsilon(g(\nabla_X\xi, Y)\xi + \eta(Y)\nabla_X\xi)
\]

\[
= -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.
\]

Which completes the proof. \(\square\)

The notion of \(\alpha\)-para-Sasakian manifolds as analogue of \(\alpha\)-Sasakian manifolds [9]. It gives its definition as follows:

**Definition 4.2.** An \(\alpha\)-para-Sasakian manifold is an almost para-contact metric manifold \((M, \varphi, \xi, \eta, g)\) which satisfies:

\[
(\nabla_X\varphi)Y = \alpha \left(-g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi\right),
\]

where \(\alpha\) is a function on \(M\).

**Example 4.3.** We denote the Cartesian coordinates in a 3-dimensional Euclidean space \(E^3\) by \((x, y, z)\) and define a symmetric tensor field \(g\) by

\[
g = \begin{pmatrix}
    f e^{2z} & 0 & 0 \\
    0 & f e^{-2z} & 0 \\
    0 & 0 & f^2
\end{pmatrix},
\]
where \( f = f(z) \) is a positive function on \( E^3 \).

Further, we define an almost para-contact metric structure \((\varphi, \xi, \eta)\) on \( E^3 \) by

\[
\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{f} \end{pmatrix}, \quad \eta = (0, 0, f).
\]

Using Koszul’s formula for the metric \( g \) for the local orthonormal basis \( \{ \partial_i = \frac{\partial}{\partial x^i} \} \),

\[
g(\nabla_{\partial_i} \partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_i, \partial_k) - \partial_k g(\partial_i, \partial_j),
\]

we get

\[
\nabla_{\partial z} \partial x = -\frac{e^{2z}}{2f^2} (f' + 2f) \partial z, \quad \nabla_{\partial z} \partial y = 0, \quad \nabla_{\partial z} \partial z = \frac{1}{2f^2} (f' + 2f) \partial x,
\]

\[
\nabla_{\partial y} \partial x = 0, \quad \nabla_{\partial y} \partial y = -\frac{e^{-2z}}{2f^2} (f' - 2f) \partial z, \quad \nabla_{\partial y} \partial z = \frac{1}{2f^2} (f' - 2f) \partial y,
\]

\[
\nabla_{\partial z} \partial x = \frac{1}{2f^2} (f' + 2f) \partial x, \quad \nabla_{\partial z} \partial y = \frac{1}{2f^2} (f' - 2f) \partial z, \quad \nabla_{\partial z} \partial z = \frac{f'}{f^2} \partial z,
\]

where \( f' = \frac{\partial f}{\partial z} \).

Knowing that

\[
(\nabla_{\partial_i} \varphi) \partial_j = -\varphi \nabla_{\partial_i} \varphi_j - \varphi \nabla_{\partial_i} \partial_j,
\]

one can easily check that

\[
(\nabla_{\partial_i} \varphi) \partial_j = \frac{2f^2}{f' + 2f} \left( -g(\partial_i, \partial_j) \varphi - \eta(\partial_j) \partial_i + 2\eta(\partial_i) \eta(\partial_j) \varphi \right),
\]

with \( f \neq e^{-2z} \), i.e., \( (E^3, \varphi, \xi, \eta) \) is an \( \alpha \)-para-Sasakian manifold, where \( \alpha = \frac{2f^2}{f' + 2f} \).

**Proposition 4.4.** For every \( \alpha \)-para-Sasakian manifold \((M, \Phi, \xi, \eta, g)\) we have

\[
(15) \quad \nabla_X \xi = \alpha \varphi X
\]

for all vector field \( X \) on \( M \).

**Proof.** Putting \( Y = \xi \) in formula (14), we get

\[
(\nabla_X \varphi) \xi = \alpha (X + \eta(X) \xi) \iff \varphi \nabla_X \xi = \alpha (X - \eta(X) \xi)
\]

\[
\iff \nabla_X \xi = \alpha \varphi X. \quad \square
\]

The famous Eq. (15) gives important informations about the curvature properties of \( \alpha \)-para-Sasakian manifold. We start with the first proposition:

**Proposition 4.5.** Let \((M, \Phi, \xi, \eta, g)\) be an \( n \)-dimensional \( \alpha \)-para-Sasakian manifold. Then we have

\[
(16) \quad R(X, Y) \xi = \alpha^2 (\eta(Y) X - \eta(Y) X) + X(\alpha) \varphi Y - Y(\alpha) \varphi X,
\]

\[
R(X, \xi) Y = \alpha^2 (g(X, Y) \xi - \eta(Y) X) + g(X, \varphi Y) \varphi X - Y(\alpha) \varphi X,
\]
\[ S(X, \xi) = \alpha^2 (1 - n) \eta(X) + \varphi X(\alpha) - X(\alpha) \operatorname{tr}_g \varphi \]

for all vector fields \( X \) and \( Y \) on \( M \) and \( S \) denotes the Ricci curvature and \( R \) is the curvature tensor defined by:

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \]

and

\[ S(X, Y) = \sum_{i=1}^n g(R(X, e_i)e_i, Y), \]

where \( \{e_i\}_{1 \leq i \leq n} \) is an orthonormal basis on \( M \).

**Proof.** The relation (16) follows from (18) and (15) with \( Z = \xi \). For the second relation, we have for all vector fields \( X, Y, Z \) on \( M \),

\[ g(R(X, \xi)Y, Z) = -g(R(Y, Z)\xi, X), \]

and using (16). Finally, knowing that

\[ S(X, \xi) = \sum_{i=1}^n g(R(X, e_i)e_i, \xi) \]

\[ = -\sum_{i=1}^n g(R(X, e_i)\xi, e_i), \]

and using (16), we obtain (17). This completes the proof of the proposition. □

**Definition 4.6.** Let \((M, \varphi, \xi, \eta, g)\) be an almost Golden almost para-contact metric manifold. If

\[ d\eta = 0 \quad \text{and} \quad 2\alpha\Omega = \mathcal{L}_\xi g, \]

then, \( M \) is called a Golden \( \alpha \)-para-Sasakian manifold. For \( \alpha = 1 \), we get a Golden para-Sasakian manifold.

**Theorem 4.7.** Let \((M, \varphi, \xi, \eta, g)\) be an almost Golden almost para-contact metric manifold. Then \( M \) is a Golden \( \alpha \)-para-Sasakian manifold if and only if

\[ \nabla_X \xi = \alpha \varphi X \]

for all vector field \( X \) on \( M \).

**Proof.** The proof is direct. □

**Theorem 4.8.** Every generalized \( \mathcal{G} \)-Golden manifold \((M, \Phi, \xi, \eta, g)\) induces an Golden \( \alpha \)-para-Sasaki manifold \((M, \varphi, \xi, \eta, g)\) and conversely.

**Proof.** From formula (8), we obtain

\[ (\nabla_X \Phi)Y = \frac{\sqrt{5}}{2} \left( \varepsilon(\nabla_X \varphi)Y + g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi \right). \]

Suppose that \((M, \Phi, \xi, \eta, g)\) is a generalized \( \mathcal{G} \)-Golden manifold. Using formulas (4) and (5), we get

\[ (\nabla_X \varphi)Y = -\varepsilon\sigma \left( -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \right), \]
i.e., $(M, \varphi, \xi, \eta, g)$ is a Golden $\alpha$-para-Sasaki manifold with $\alpha = -\varepsilon \sigma$.

Conversely, suppose that $(M, \varphi, \xi, \eta, g)$ is a Golden $\alpha$-para-Sasaki manifold. From formulas (20), (14) and (15), we obtain

$$ (\nabla_X \Phi) Y = -\varepsilon \alpha \sqrt{5} (g(X, Y) \xi + \eta(Y)X - 2\eta(X) \eta(Y) \xi), $$

i.e., $(M, \Phi, \xi, \eta, g)$ is a generalized $G$-Golden manifold with $\sigma = -\varepsilon \alpha$. □

5. A class of examples

Let $\mathbb{R}^{n+1}$ be an Euclidean space with Cartesian coordinates $\{x_1, \ldots, x_n, z\}$. We put

$$
\begin{align*}
\xi & = \frac{\partial}{\partial z}, \\
\eta & = dz - \tau dx_1, \\
\varphi & = \varepsilon (-I + \eta \otimes \xi), \\
g & = \eta \otimes \eta + \rho^2 \sum_{i=1}^{n} dx_i^2,
\end{align*}
$$

where $\rho$ and $\tau$ are two functions on $\mathbb{R}^{n+1}$.

It's clear that $(\varphi, \xi, \eta, g)$ is an almost Golden almost para-contact metric structure.

In addition,

$$
d\eta = \sum_{k=2}^{n} \tau_k dx_1 \wedge dx_k \text{ with } \tau_k = \frac{\partial \tau}{\partial x_k}.
$$

So, if $\tau_k = 0$ for all $k \in \{2, \ldots, n\}$ (i.e., $\tau = \tau(x_1)$) then, $(\varphi, \xi, \eta, g)$ is a Golden almost para-contact metric structure.

Now, for the metric $g$, form the G-basis $\{e_1, e_i, \xi\}$ where $i \in \{2, \ldots, n\}$ as follows:

$$
e_1 = \frac{1}{\rho} \left( \frac{\partial}{\partial x_1} + \tau \xi \right), \quad e_i = \frac{1}{\rho} \frac{\partial}{\partial x_i},
$$

Taking into account the above condition, i.e., $\tau = \tau(x_1)$, one can find:

$$
\begin{align*}
[e_1, e_i] &= \frac{\rho_i}{\rho^2} e_1 - \frac{\rho_1 + \tau \xi (\rho)}{\rho^2} e_i, \quad [e_1, \xi] = \frac{\rho'}{\rho} e_1, \\
[e_i, e_j] &= \frac{\rho_j}{\rho^2} e_i - \frac{\rho_i}{\rho^2} e_j, \quad [e_i, \xi] = \frac{\rho'}{\rho} e_i,
\end{align*}
$$

where $\rho' = \xi (\rho) = \frac{\partial \rho}{\partial z}$ and $\rho_i = \frac{\partial \rho}{\partial x_i}$.

Using Koszul's formula for the metric $g$,

$$
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),
$$

we obtain:

$$
\begin{align*}
\nabla_{e_1} e_1 &= -\frac{\rho_1}{\rho^2} e_i - \frac{\rho'}{\rho} \xi, \quad \nabla_{e_1} e_i = \frac{\rho_i}{\rho^2} e_1, \quad \nabla_{e_1} \xi = \frac{\rho'}{\rho} e_i,
\end{align*}
$$
\( \nabla e_i e_j = -\frac{\rho_1 + \tau \xi(\rho)}{\rho^2} \delta_{ij} \psi_1 + \frac{1}{\rho^2} \sum_{k=2}^{n}(\rho_j \delta_{ik} - \rho_k \delta_{ij}) e_k - \frac{\rho'}{\rho} \delta_{ij} \xi, \)

\( \nabla e_i e_1 = \frac{\rho_1 + \tau \xi(\rho)}{\rho^2} e_i, \)

\( \nabla e_i \xi = \frac{\rho'}{\rho} e_i, \quad \nabla \xi e_1 = \nabla \xi e_i = \nabla \xi = 0. \)

On the other hand, with a simple calculation, we can find

\( \Omega = -\varepsilon \sum_{i=1}^{n} dx_i \otimes dx_i, \)

and

\( \mathcal{L}_{\xi} g = \frac{2\rho'}{\rho} \sum_{i=1}^{n} dx_i \otimes dx_i, \)

using formulas (19) or Theorem 4.7, we have immediately that, \((\mathbb{R}^{n+1}, \varphi, \xi, \eta, g)\) is a Golden \(\alpha\)-para-Sasakian manifold if and only if \(\tau = \tau(x_1)\) and \(\alpha = -\varepsilon \frac{\rho'}{\rho} \).

For \(\alpha = 1\), \((\mathbb{R}^{n+1}, \varphi, \xi, \eta, g)\) is a Golden para-Sasakian manifold if and only if \(\tau = \tau(x_1)\) and \(\rho = ce^{-\varepsilon z}\) where \(c \in \mathbb{R}\).

References


GHERICI BELDJILALI

LABORATORY OF QUANTUM PHYSICS AND MATHEMATICAL MODELING (LPQ3M)

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MASCARA

ALGERIA

Email address: gherici.beldjilali@univ-mascara.dz
Habib Bouzir
Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M)
Department of Mathematics
University of Mascara
Algeria
Email address: habib.bouzir@univ-mascara.dz