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PASCAL'S HEXAGON THEOREM REPROVED BY ELEMENTARY TOOLS ONLY

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Abstract. In this paper, we prove Pascal's hexagon theorem by elementary tools only. We follow the well-known route to prove the theorem by Bezóut's theorem, explaining all the details in elementary argument. In particular, we prove a toy version of Study's lemma.

1. Introduction

Blaise Pascal's mystic hexagon theorem ([3], [4]) states that if six arbitrary points are chosen on a smooth conic curve and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon, extended if necessary, meet at three points which lie on a straight line, called the Pascal line of the hexagon. The most famous picture of the theorem is the ellipse version as given in Figure 1. The pascal line in this picture is the line through P, Q, and R.

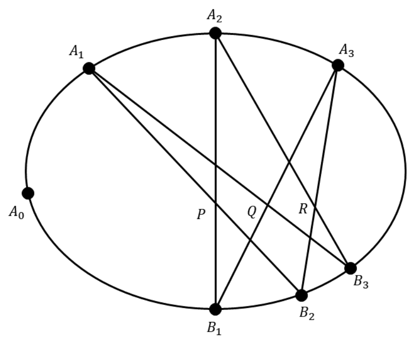

Figure 1.

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Among the various proofs, there is a way to prove by using Bezóut's theorem. For example, one can use M. Noether's theorem ([2, §5.6]) which is a variant of Bezóut's theorem.

A simpler version of proof suggested by Plücker goes as follows (see [5, p. 20]): Let $h_1(x,y)$ and $h_2(x,y)$ be the cubic polynomials defining the union of three lines $\overleftrightarrow{A_1B_2}$, $\overleftrightarrow{A_2B_3}$, $\overleftrightarrow{A_3B_1}$ and $\overleftrightarrow{A_1B_3}$, $\overleftrightarrow{A_2B_1}$, $\overleftrightarrow{A_3B_2}$, respectively. Symbolically, we can write:

$$\begin{aligned} h_1 &= \overleftrightarrow{A_1B_2} \cdot \overleftrightarrow{A_2B_3} \cdot \overleftrightarrow{A_3B_1}, \\ h_2 &= \overleftrightarrow{A_1B_3} \cdot \overleftrightarrow{A_2B_1} \cdot \overleftrightarrow{A_3B_2}. \end{aligned} \tag{1.1}$$

Let A_0 be any point on the conic other than the six points. Then $h := h_1 + \lambda h_2$ vanishes on A_0 for a suitable constant λ. Now the cubic polynomial h vanishes at the seven points $A_1, A_2, A_3, B_1, B_2, B_3$, and A_0. But by Bezóut's theorem, the number of intersection points of a conic and a cubic is bounded from above by six, provided that it is finite. $(*)$ "Therefore, the cubic polynomial $h(x,y)$ defines a reducible curve consisting of the conic and a line, say L. Note that $h(x,y)$ also vanishes at the three points P, Q, R in Figure 1. Hence they must lie on the line L."

This proof is simple and elegant, and provides a nice connection between projective geometry and algebraic geometry. But we note that some of the steps of the proof require techniques outside "elementary" level. In particular, to guarantee $(*)$, we need the following result:

Proposition 1.1 (Study's lemma). *Let k be an algebraically closed field, $f, g \in k[x,y]$ be polynomials with f irreducible. If all the zeros of f are contained in the zeros of g, then f divides g in $k[x,y]$.*

The requirement that k being algebraically closed is necessary, and logically speaking, we need to pass to the complexified polynomials defining the conic and cubic in order to use the Study's lemma. Also, to show this, one must either invoke to the Nullstelensatz (a stronger form of Study's lemma) or use the technique of ring theory, including resultants. In this sense, the standard proof via Study's lemma lies on a little higher level than elementary.

The goal of this short note is to give a completely elementary argument to prove Pascal's hexagon theorem (over $\mathbb{R}$). The idea is not different from the above line of proof, but we observe that in our context all the machinery boils down to the theory of polynomials in single variable. We expect this can be used to illustrate a seed idea of polynomial ring theory and algebraic geometry to the university students.

We remark that there are algebraic proofs which consider an intersection of two cubics ([1, II §6]) and Cayley-Bacharach's theorem ([6, pp. 241–242]). Also there is another simple proof based on the homogeneous coordinates and cross product ([7]).

2. Proof

First we show the following special case of Bezóut's theorem.

Lemma 2.1. *Let C be the smooth plane conic curve. Let H be any curve defined by a polynomial equation of degree d. If C is not contained in H, then the number $\sharp(C \cap H)$ of intersection points is finite and $\sharp(C \cap H) \leq 2d$.*

Proof. Suppose that C and H are defined by a qudratic $q(x, y)$ and an equation $h(x, y)$ of degree d, respectively.

As well-known, C has a rational parameterization: This is given by fixing a point $P_0 = (x_0, y_0) \in C$ and considering a pencil of lines

$$L_t : y = t(x - x_0) + y_0$$

passing through P_0. By a linear change of coordinates, we may assume that the tangent line at P_0 is parallel to the y-axis. Then by associating to each t the intersection point $C \cap L_t$ other than P_0, we get a map $\alpha : U \to C$ for some nonempty open subset $U \subset \mathbb{R}$. This is an injective map which is surjective onto $C \setminus \{P_0\}$, since C is a conic curve. Algebraically, we first plug in the equation of L_t to $q(x, y)$:

$$q(x, t(x - x_0) + y_0) = 0$$

and then find a linear factor other than $x - x_0$. This gives a formula

$$\alpha(t) = \left(\frac{p_1(t)}{p_3(t)}, \frac{p_2(t)}{p_3(t)}\right)$$

for some polynomials p_1, p_2, p_3 of degree ≤ 2.

Assume that C is not contained in H and $\sharp(C \cap H) > 2d$. We choose $P_0 \in C \setminus H$. Note that every intersection point satisfies the equation $h(\alpha(t)) = 0$. Clearing its common denominators, we get a polynomial equation of degree $\leq 2d$. From the assumption that $\sharp(C \cap H) > 2d$, we have an identity $h(\alpha(t)) \equiv 0$. This implies that $C \setminus \{P_0\} \subset H$ and also $P_0 \in H$ by taking limit. This is a contradiction. $\square$

Next we show the following result which is a special case of Study's lemma.

Lemma 2.2. *Let C be a smooth conic curve defined by a quadratic equation $q(x, y) = 0$. Let H be a curve defined by a cubic equation $h(x, y) = 0$. If C is contained in H, then $q(x, y)$ divides $h(x, y)$ in $\mathbb{R}[x, y]$, in other words, $h(x, y) = q(x, y)\ell(x, y)$ for some linear polynomial $\ell(x, y)$.*

Proof. Under a suitable linear change of coordinates,[1] we may assume that

$$\begin{aligned} q(x, y) &= y^2 + a_1(x)y + a_2(x), \\ h(x, y) &= y^3 + b_1(x)y^2 + b_2(x)y + b_3(x), \end{aligned}$$

[1] This amounts to moving the curves C and H so that their projectivizations avoid the point $[0 : 1 : 0]$.

where $a_i(x)$ and $b_i(x)$ are polynomials of degree $\leq i$. Fix any $x_0 \in \mathbb{R}$ such that $(x_0, y_0) \in C$ for some y_0. Since $\widetilde{q}(y) := q(x_0, y)$ is quadratic in y having a real zero, it factorized into a product of linear polynomials. Since C is contained in H, all the zeros of $\widetilde{q}(y)$ are also zeros of $\widetilde{h}(y) := h(x_0, y)$. Therefore,

$$\widetilde{h}(y) = \widetilde{q}(y) \cdot (y + c_1)$$

for some $c_1 \in \mathbb{R}$. By comparing the coefficients of y^2, we have $c_1 = b_1(x_0) - a_1(x_0)$. Hence the following identity hold:

$$h(x_0, y) = q(x_0, y) \cdot (y + b_1(x_0) - a_1(x_0)).$$

Since the projection map $(x, y) \mapsto x$ of the curve C to the x-axis has an image containing an interval, the above polynomial identity holds for any $x_0 \in \mathbb{R}$, and we get an identity:

$$h(x, y) = q(x, y) \cdot (y + b_1(x) - a_1(x)). \tag{2.1}$$

This shows that the cubic equation $h(x, y) = 0$ defines a union of the conic C and a line. $\square$

Now we can give a proof of Pascal's theorem. Let $h_1(x, y)$ and $h_2(x, y)$ be the cubic polynomials as in (1.1). Let H be the cubic curve defined by $h = h_1 + \lambda h_2$ for some λ which passes through all the seven points $A_1, A_2, A_3, B_1, B_2, B_3$, and A_0 of C. By Lemma 2.1, $\sharp(C \cap H) \leq 6$ if it is finite, thus C must be contained in H. By Lemma 2.2, $h(x, y)$ factorizes into a product of the quadratic polynomial $q(x, y)$ defining C and a linear polynomial defining a line, say L. Since the points P, Q, R in Figure 1 lie on H, they must lie on the line L. This completes the proof of Pascal's theorem.

Remark 2.3. (1) The procedure shows how to reformulate the problem in $k[x, y]$ into a problem in $k[y]$, and then go back to the original one. This can be thought as a baby example of the elimination theory.

(2) The above algebraic proof still works when C is a degenerate conic given by a pair of two distinct lines. The point is that there are infinitely many x_0's such that $(x_0, y_0) \in C$ for some y_0. When C is a pair of distinct lines, Pascal's theorem reduces to the celebrated Pappus' theorem.

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