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PROPERTIES OF FUNCTIONS WITH BOUNDED ROTATION ASSOCIATED WITH LIMAÇON CLASS

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Abstract. In this article, we initiate subclasses of functions with boundary and radius rotations that are related to limaçon domains and examine some of their geometric properties. Radius results associated with functions in these classes and their linear combination are studied. Furthermore, the growth rate of coefficients, arc length and coefficient estimates are derived for these novel classes. Overall, some useful consequences of our findings are also illustrated.

1. Introduction and preliminaries

Univalent function theory is an area of Mathematics characterized by a fascinating marriage between analysis and geometry, which had its history from the Reimann Mapping theory. One of the challenging problems of the subject is the Bieberbach conjecture, dating back to the year 1916, which asserts that the Taylor coefficients (a_n, $n \geq 2$) of every functions in S (the class of normalized univalent functions) satisfy the inequality $|a_n| \leq n$, $n \geq 2$ [3]. For many years, this problem challenged its people and inspired different methods and techniques that form the backbone of the whole subject. Among the early investigated of the subclass of S are the classes S^* and C of starlike and convex functions, respectively. These classes have received numerous attentions in different directions and perspectives (see [15, 22, 24, 27]).

A natural extension of the class C of convex functions is the class V_m ($m \geq 2$) of functions with bounded boundary rotation. The idea of this class originated from Loewner [6] in 1917 but Paatero [16, 17] systematically developed its properties and also made an intensive study of the class. In 1971, Pinchuk [18] introduced and investigated the classes P_m and R_m (the class of functions of

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bounded radius rotation bounded by $m\pi$), where R_m generalizes the class S^* in the same way V_m generalizes the class C. Pinchuk [18] gave the conditions for functions in V_m and R_m to be associated with the class P_m. These ideas of classes V_m and R_m have opened new door of research and attracted the audience of many researchers. Numerous articles have been written in this direction (see [1, 5, 8, 9, 11]). The article "A survey on functions of bounded boundary and bounded radius rotation" by Noor [10] is considered to be a masterpiece in this perspective.

Recently, Kanas and Masih [7] initiated a subfamily of analytic functions that are characterized by limaçon domains. The geometric properties of this function were examined and used to present convex and starlike limaçon classes of functions. Furthermore, Afis et al. [23] continued with the investigation of these classes and proved many interesting results associated with them.

Among the most fascinating geometric characterizations of univalent functions is the arc length properties. Let $l(r, f)$ be the length of the image curve of the line segment joining the points $-re^{i\theta}$ and $re^{i\theta}$ under the analytic function $f(z)$. Also, let $L(r, f)$ be the length of the image domain of the open unit disc under the analytic function $f(z)$. Cho et al. [2] obtained sharp $l(r, f)$ for the general classes of Ma and Minda starlike and convex functions. In particular, the case of Janowski starlike and convex functions was illustrated to validate their findings. In addition, they obtained $L(r, f)$ under the Janowski, lemniscate of Bernoulli and exponential functions. Considering some weaker assumptions on $f(z)$ with representation (1), Sokół and Nunokawa [25] presented arc length results for $f(z)$. Also, the relationship between $l(r, f)$ and $L(r, f)$ was established in an instance. In a recent time, Nunokawa and Sokół [13] proved some results for functions that are not necessarily univalent in U. For more information on arc length results, we refer to [12, 14, 20, 21, 28] and the references therein.

Motivated with these cited work, and the recent article by Afis and Noor [1], our objectives in this present investigation are to introduce subclasses of functions with boundary and radius rotations that are related to limaçon domains and study some of their geometric properties such as radius results, the growth rate of coefficients, arc length and coefficient estimates for these novel classes. Interestingly, a particular choice of $m = 2$ in our results (see Remark 2.2 and Remark 2.17) gives new results for limaçon, starlike and convex limaçon functions, which were not discussed in [7].

To put these findings in a clear direction, we begin with the following preliminaries.

Let A be the class of analytic functions $f(z)$ having the series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U := \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Let $f, g \in A$. Then $f(z)$ is subordinate $g(z)$ (written as $f \prec g$ or $f(z) \prec g(g)$) if there exists a Schwarz function $w(z)$ such that $f(z) = g(w(z))$ for all $z \in U$. The class $P(\beta)$ denotes the class of all analytic functions $p(z)$ having the Taylor series of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in U$$

and satisfying the inequality

$$\left| p(z) - \frac{1 + (1 - 2\beta) r^2}{1 - r^2} \right| \leq \frac{2(1 - \beta) r}{1 - r^2}, \quad r \in (0,1),\ \beta \in [0,1) \text{ (see [4])}. \tag{2}$$

It is easy to notice that $P(\beta)$ reduces to the class P of functions $h(z)$ having positive real part for $\beta = 0$. Using the concept of the class $P(\beta)$, Robertson [19] introduce the classes $S^*(\beta)$ and $C(\beta)$ as follows:

$$S^*(\beta) = \left\{ f \in A : \frac{zf'}{f} \in P(\beta) \right\}$$

and

$$C(\beta) = \{ f \in A : zf' \in S^*(\beta) \}.$$

As a special case, we have $S^*(0) = S^*$ and $C(0) = C$ of starlike and convex functions, respectively.

Recently, Masih and Kanas [7] demonstrated the class of functions that mapped U onto a limaçon domain and presented the following:

Definition 1 ([7])**.** Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then $p \in P(\mathbb{L}_s)$ if and only if

$$p(z) \prec (1 + sz)^2, \quad 0 < s \leq \frac{1}{\sqrt{2}},\ z \in U,$$

or equivalently, if $p(z)$ satisfies the inequality

$$|p(z) - 1| < 1 - (1 - s)^2.$$

Demonstrated in [7], was the inclusion relation

$$\{ w \in \mathbb{C} \colon |w - 1| < 1 - (1 - s)^2 \} \subset \mathbb{L}_s(U) \subset \{ w \in \mathbb{C} \colon |w - 1| < (1 + s)^2 - 1 \}. \tag{3}$$

It is worthy of note that the function $\mathbb{L}_s(z) = (1 + sz)^2$ is the analytic characterization of $\mathbb{L}_s(U)$ of functions that map U onto a limaçon domain. Also, $\mathbb{L}_s(z)$ is starlike and convex univalent in U for $0 < s \leq \frac{1}{2}$. Furthermore, $\mathbb{L}_s(z) \in P(\beta)$, where $\beta = (1 - s)^2$, $0 < s \leq \frac{1}{2}$.

Definition 2. Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}}(s)$ if and only if

$$\frac{zf'(z)}{f(z)} \in P(\mathbb{L}_s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}}(s)$ if and only if

$$zf'(z) \in ST_{\mathcal{L}}(s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Spurred by the work of Masih and Kanas [7], Afis and Noor [1, 23], we initiated the following classes of functions.

Definition 3. Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then $p \in P_m(\mathbb{L}_s)$ if and only if there exist $p_1, p_2 \in P(\mathbb{L}_s)$ such that

$$p(z) = \left(\frac{m+2}{4}\right) p_1(z) - \left(\frac{m-2}{4}\right) p_2(z),\ z \in U,\ m \geq 2.$$

Definition 4. Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}_m}(s)$ if and only if

$$\frac{zf'(z)}{f(z)} \in P_m(\mathbb{L}_s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}_m}(s)$ if and only if

$$zf'(z) \in ST_{\mathcal{L}_m}(s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

For $m = 2$, we are back to Definitions 1 and 2.

To prove the main results of this article we need Corollaries 1 and 2 in [7] and the following results.

Lemma 1.1. *Let $f \in ST_{\mathcal{L}_m(s)}$. Then for $m \geq 2$,*

$$\begin{aligned} &\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\beta)r^2}{1-r^2}\right| \\ \leq\ &\frac{m(1-\beta)r}{1-r^2}, \quad r \in (0,1),\ \beta = (1-s)^2,\ 0 < s \leq \frac{1}{2}. \end{aligned} \tag{4}$$

Proof. Since $f \in ST_{\mathcal{L}_m(s)}$, we have

$$\frac{zf'(z)}{f(z)} = \left(\frac{m+2}{4}\right) p_1(z) - \left(\frac{m-2}{4}\right) p_2(z)$$

for some $p_1, p_2 \in P(\mathbb{L}_s)$. But $P(\mathbb{L}_s) \subset P(\beta)$, where $\beta = (1-s)^2$ (see Definition 2 and Theorem 1 in [7]). Thus $p_1(z)$ and $p_2(z)$ satisfy (2). Therefore,

$$\begin{aligned} &\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\beta)r^2}{1-r^2}\right| \\ =&\left|\left(\frac{m+2}{4}\right)\left(p_1(z) - \frac{1+(1-2\beta)r^2}{1-r^2}\right) - \left(\frac{m-2}{4}\right)\left(p_2(z) - \frac{1+(1-2\beta)r^2}{1-r^2}\right)\right| \\ \leq&\left(\frac{m+2}{4}\right)\left|p_1(z) - \frac{1+(1-2\beta)r^2}{1-r^2}\right| + \left(\frac{m-2}{4}\right)\left|p_2(z) - \frac{1+(1-2\beta)r^2}{1-r^2}\right| \\ \leq&\frac{m(1-\beta)r}{1-r^2}. \end{aligned}$$

□

Lemma 1.2 ([26])**.** *If $|u-a| \leq d$ and $|v-a| \leq d$, where a and d are real with $a > d \geq 0$, and*

$$w = u\frac{1}{1+\lambda e^{i\rho}} + v\frac{1}{1+\lambda^{-1}e^{-i\rho}}, \tag{5}$$

where ρ is real, $\lambda > 0$ and $\rho \in [0, \pi)$, then

$$\operatorname{Re} w \geq a - d \sec \frac{\rho}{2}. \tag{6}$$

2. Main results

In this section, we establish our main findings.

Theorem 2.1. *Let $p \in P_m(\mathbb{L}_s)$. Then for $z = re^{i\theta}$,*

(i)

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \, d\theta \leq 1 + m^2 s^2 \frac{r^2}{1 - r^2} \,,$$

(ii)

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} |p(z)|^2 \, d\theta d\rho \leq r + \frac{m^2 s^2}{2} \left[\log \left(\frac{1 + r}{1 - r} \right) - 2r \right] \,,$$

(iii) *for $0 < s \leq \frac{1}{2}$, $p \in P$ in the disc*

$$|z| < \frac{2}{s\left(m + \sqrt{m^2 - 4}\right)} \,, \quad m > \frac{1 + s^2}{s}. \tag{7}$$

This result is sharp for the function given by

$$p(z) = \left(\frac{m + 2}{4} \right) \mathbb{L}_s(-z) - \left(\frac{m - 2}{4} \right) \mathbb{L}_s(z), \; z \in U, \, m > 2. \tag{8}$$

Proof. (i) Since $p \in P_m(\mathbb{L}_s)$, we have

$$p(z) = \left(\frac{m + 2}{4} \right) p_1(z) - \left(\frac{m - 2}{4} \right) p_2(z)$$

for some

$$p_i(z) = 1 + \sum_{n=1}^{\infty} c_{i,n} z^n \prec (1 + sz)^2, \quad i = 1, 2.$$

Using the Perseval's identity and the bounds for $p \in P(\mathbb{L}_s)$ given in [23], we have

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta &\leq 1 + \sum_{n=1}^{\infty} \left| \left(\frac{m + 2}{4} \right) c_{1,\,n} - \left(\frac{m - 2}{4} \right) c_{2,\,n} \right|^2 r^{2n} \\
&\leq 1 + \sum_{n=1}^{\infty} \left[\left(\frac{m + 2}{4} \right) |c_{1,\,n}| + \left(\frac{m - 2}{4} \right) |c_{2,\,n}| \right]^2 r^{2n} \\
&\leq 1 + \sum_{n=1}^{\infty} m^2 s^2 r^{2n} \\
&= 1 + m^2 s^2 \frac{r^2}{1 - r^2}. \qquad (9)
\end{aligned}$$

(ii) Applying (9), we obtain

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,d\theta d\rho \leq \int_0^r\left(1+m^2s^2\frac{\rho^2}{1-\rho^2}\right)d\rho$$
$$= r+\frac{m^2s^2}{2}\left[\log\left(\frac{1+r}{1-r}\right)-2r\right].$$

(iii) In view of Theorem 1 in [7] and the representation for $p\in P_m(\mathbb{L}_s)$, we arrive at

$$\begin{aligned}\operatorname{Re}p(z) &= \left(\frac{m+2}{4}\right)\operatorname{Re}p_1(z)-\left(\frac{m-2}{4}\right)\operatorname{Re}p_2(z)\\ &\geq \left(\frac{m+2}{4}\right)(1-sr)^2-\left(\frac{m-2}{4}\right)(1+sr)^2\\ &= 1-srm+s^2r^2. \end{aligned} \tag{10}$$

Let $T(r)=1-srm+s^2r^2$. Then $T(0)=1>0$ and $T(1)=s^2-sm+1<0$ for $m>\frac{1+s^2}{s}$. Thus, there exists $r\in(0,1)$ such that

$$1-srm+s^2r^2=0.$$

Hence, the right side of (10) is positive for all z in the disc given by (7). $\square$

Remark 2.2. For $m=2$, we obtain the integral inequalities (i) and (ii) for $p\in P(\mathbb{L}_s)$.

From (iii), we have:

Corollary 2.3. *Let $0<s\leq\frac{1}{\sqrt{2}}$ and suppose $m>\frac{1+s^2}{s}$. Then $ST_{\mathcal{L}_m}(s)\subset S^*$ and $CV_{\mathcal{L}_m}(s)\subset C$ in the disc given by (7).*

Theorem 2.4. *Let $f\in A$. Then $f\in ST_{\mathcal{L}_m}(s)$ has the representation*

$$f(z)=\frac{(f_1(z))^{\frac{m+2}{4}}}{(f_2(z))^{\frac{m-2}{4}}} \tag{11}$$

for some $f_1,f_2\in ST_{\mathcal{L}}(s)$.

Proof. The proof is direct from the definition of $ST_{\mathcal{L}_m}(s)$. $\square$

The next two theorems are the immediate consequences of the representation (11) and Corollaries 1 and 2 in [7].

Theorem 2.5. *If $f\in ST_{\mathcal{L}}(s)$, then*

(i)

$$\left|\arg\left(\frac{f(z)}{z}\right)\right|\leq\frac{m}{2}\left(2sr+\frac{(sr)^2}{2}\right),$$

(ii)

$$r\exp\left(\frac{(sr)^2}{2}-msr\right)\leq|f(z)|\leq r\exp\left(\frac{(sr)^2}{2}+msr\right),$$

(iii) *The range of U under the function $f \in ST_{\mathcal{L}_m}(s)$ contains the disc*

$$|z| \leq \exp\left(\frac{s^2}{2} - ms\right).$$

Theorem 2.6. *If $f \in CV_{\mathcal{L}_m}(s)$, then*

(i)

$$|\arg f'(z)| \leq \frac{m}{2}\left(2sr + \frac{(sr)^2}{2}\right),$$

(ii)

$$\exp\left(\frac{(sr)^2}{2} - msr\right) \leq |f'(z)| \leq \exp\left(\frac{(sr)^2}{2} + msr\right),$$

(iii) *The range of U under the function $f \in CV_{\mathcal{L}_m}(s)$ contains the disc*

$$|z| \leq \lim_{r \to 1^-} \int_0^r \exp\left(\frac{(s\rho)^2}{2} - ms\rho\right) d\rho.$$

Theorem 2.7. *Let $f \in ST_{\mathcal{L}_m}(s)$. Then the length $L(r, f)$ of the image domain of $U_r = \{z \in \mathbb{C} : |z| < 1\}$ under $f(z)$ satisfies the inequality*

$$L(r, f) \leq C(m, s)\left(\frac{1}{1-r}\right)^{\frac{1}{2}} \quad (r \to 1^-),$$

where $C(m, s)$ is a constant depending only on m and s.

Proof. From the definition of $L(r, f)$, we have

$$\begin{aligned}
L(r, f) &= \int_0^{2\pi} |zf'(z)|\, d\theta, \quad z = re^{i\theta} \\
&= \int_0^{2\pi} |f(z)|\, |p(z)|\, d\theta, \quad p \in P_m(\mathbb{L}_s) \\
&\leq 2\pi \left(\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\, d\theta\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2\, d\theta\right)^{\frac{1}{2}} \\
&\leq 2\pi r \exp\left(\frac{(sr)^2}{2} + msr\right)\left(1 + \frac{m^2 s^2 r^2}{1 - r^2}\right)^{\frac{1}{2}} \\
&< 2\pi \exp\left(\frac{s^2}{2} + ms\right)\left(1 + \frac{m^2 s^2}{1-r}\right)^{\frac{1}{2}} \\
&= C(m, s)\left(\frac{1}{1-r}\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used Cauchy Schwarz inequality and Theorem 2.5(ii). □

Corollary 2.8. *Let $f \in ST_{\mathcal{L}_m}(s)$ be of the form (1). Then*

$$a_n = \mathrm{O}(1)\sqrt{\frac{1}{n}} \quad (n \to \infty),$$

where O(1) *is constant depending only on* m *and* s.

Proof. By Cauchy theorem, we have for $z = re^{i\theta}$,

$$\begin{aligned} n|a_n| &\leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)|\, d\theta \\ &= \frac{1}{2\pi r^n} L(r, f). \end{aligned}$$

Applying Theorem 2.7 and setting $r = 1 - \frac{1}{n}$ $(n \to \infty)$, we obtain the required result. □

Corollary 2.9. *Let* $f \in CV_{\mathcal{L}_m}(s)$ *be of the form* (1). *Then*

$$a_n = \mathrm{O}(1) \sqrt[3]{\frac{1}{n}} \quad (n \to \infty),$$

where O(1) *is constant depending only on* m *and* s.

Theorem 2.10. *Let* $A(r, f)$ *be the area bounded by the image curve* $|z| = r$, $r \in (0, 1)$ *under the function* $f \in CV_{\mathcal{L}_m}(s)$. *Then*

$$L(r, f) \leq 2 \left\{ \frac{\pi A(r, f)}{r} \left[r + \frac{m^2 s^2}{2} \left(\log \left(\frac{1 + r}{1 - r} - 2r \right) \right) \right] \right\}^{\frac{1}{2}} \quad (r \to 1^-).$$

Proof. We have

$$\begin{aligned} L(r, f) &= \int_0^{2\pi} |zf'(z)|\,, \quad z = ri\theta \\ &= \int_0^r \int_0^{2\pi} |(zf'(z))'|\, d\theta d\rho \\ &= \int_0^r \int_0^{2\pi} |f'(z)p(z)|\, d\theta d\rho\,, \quad p \in P_m(\mathbb{L}_s) \\ &\leq \left(\int_0^r \int_0^{2\pi} |f(z)|^2\, d\theta d\rho \right)^{\frac{1}{2}} \left(\int_0^r \int_0^{2\pi} |p(z)|^2\, d\theta d\rho \right)^{\frac{1}{2}} \\ &\leq \left(2\pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n-1} \right)^{\frac{1}{2}} \left(\int_0^r \int_0^{2\pi} |p(z)|^2\, d\theta d\rho \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used Cauchy Schwarz inequality and Perseval's identity. Applying Theorem 2.1(ii) with the fact that

$$A(r, f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \quad a_1 = 1,$$

we arrive at the required result. □

Theorem 2.11. *Let* $f, g \in ST_{\mathcal{L}_m}(s)$ *and*

$$F(z) = \alpha f(z) + (1 - \alpha) g(z), \tag{12}$$

where $0 \leq \arg\left(\frac{\alpha}{1-\alpha}\right) = \gamma < \pi$. *Suppose* $2 \leq m < \frac{2\sqrt{4s-2s^2-1}}{s(2-s)}$, $\frac{2-\sqrt{2}}{2} < s \leq \frac{1}{2}$. *Then* $F \in S^*$ *in the disc* $|z| < r(m,s) = r_*$, *where* $r(m,s)$ *is the smallest positive value of* r *satisfying the equation*

$$(13)\qquad \left(1-(2s^2-4s+1)r^2\right)\cos\left(\frac{\gamma}{2}+m\left(sr+\frac{(sr)^2}{4}\right)\right) - ms(2-s)r = 0.$$

Proof. Logarithmic differentiation of (12) gives

$$\begin{aligned}\frac{zF'(z)}{F(z)} &= \frac{\alpha zf'(z)+(1-\alpha)g'(z)}{\alpha zf(z)+(1-\alpha)g(z)}\\ &= \frac{zg'(z)}{g(z)}\left(\frac{1}{1+\left(\frac{\alpha}{1-\alpha}\right)\frac{f(z)}{g(z)}}\right)+\frac{zf'(z)}{f(z)}\left(\frac{1}{1+\left(\left(\frac{\alpha}{1-\alpha}\right)\frac{f(z)}{g(z)}\right)^{-1}}\right)\\ &:= u\frac{1}{1+\lambda e^{i\rho}}+v\frac{1}{1+\lambda^{-1}e^{-i\rho}},\end{aligned}$$

where

$$\begin{aligned}&u=\frac{zg'(z)}{g(z)},\ v=\frac{zf'(z)}{f(z)},\ \lambda=\left|\frac{\alpha}{1-\alpha}\frac{f(z)}{g(z)}\right|,\\ &\rho=\arg\left(\frac{\alpha}{1-\alpha}\frac{f(z)}{g(z)}\right)=\arg\left(\frac{\alpha}{1-\alpha}\right)+\arg\left(\frac{f(z)}{z}\right)-\arg\left(\frac{g(z)}{z}\right).\end{aligned}$$

Therefore,

$$(14)\qquad |\rho| \leq \gamma + m\left(2sr+\frac{(sr)^2}{2}\right),$$

where we have used Theorem 2.5(i). Now by Lemmas 1.1 and 1.2, we obtain

$$(15)\qquad \operatorname{Re}\frac{zF'(z)}{F(z)} \geq a - b\sec\frac{\rho}{2},$$

where

$$a=\frac{1-(2s^2-4s+1)r^2}{1-r^2},\quad b=\frac{ms(2-s)r}{1-r^2}.$$

We note that the condition $a > b$ in Lemma 1.2 holds here if $1-(2s^2-4s+1)r^2 > ms(2-s)r$, i.e., $(2s^2-4s+1)r^2+ms(2-s)r-1<0$. Let $h(r)=Ar^2+Br+C$ with $A=2s^2-4s+1$, $B=ms(2-s)$, $C=-1$. Thus, $h(r)<0$ if $A<0$ and $4AC-B^2>0 \iff B^2+4A<0$. But $A<0$ holds if $\frac{2-\sqrt{2}}{2}<s\leq\frac{1}{2}$ and the condition $B^2+4A<0$ holds if $m<\frac{2\sqrt{4s-2s^2-1}}{s(2-s)}$ for all $\frac{2-\sqrt{2}}{2}<s\leq\frac{1}{2}$.

From (14) and (15), we have

$$(16)\qquad \left(1-(2s^2-4s+1)r^2\right)\cos\left(\frac{\gamma}{2}+m\left(sr+\frac{(sr)^2}{4}\right)\right) - ms(2-s)r > 0.$$

Let $T(r)$ represent the left side of (16). Then $T(0) = \cos\frac{\gamma}{2} > 0$ and

$$T\left(\frac{2(\pi-\gamma)}{s\left[2m+\sqrt{2m(2m+\pi-\gamma)}\right]}\right) = \frac{-2m(2-s)(\pi-\gamma)}{s\left[2m+\sqrt{2m(2m+\pi-\gamma)}\right]} < 0.$$

Therefore, there exists $r \in \left(0, \frac{2(\pi-\gamma)}{s\left[2m+\sqrt{2m(2m+\pi-\gamma)}\right]}\right)$ such that $T(r) = 0$. Consequently, the right side of (15) is positive in the disc $|z| < r(m,s)$, where $r(m,s)$ is defined in the theorem. Hence, $F \in S^*$ in the disc $|z| < r(m,s)$. $\square$

Theorem 2.12. *Let $f, g \in CV_{\mathcal{L}_m}(s)$ and $F(z) = \alpha f(z) + (1-\alpha)g(z)$, where α, m and s satisfy the same conditions of Theorem 2.11. Then $F \in C$ in the disc $|z| < r(m,s)$, where $r(m,s)$ is the least positive root of* (13).

Proof. The proof follows the same techniques as in Theorem 2.11, where we used Theorem 2.6(i), Lemmas 1.1 and 1.2. Hence, we obtain the desired result. $\square$

Theorem 2.13. *If $f \in ST_{\mathcal{L}_m}(s)$, then*

$$\sum_{n=2}^{\infty}\left(4n^2 - m^2(1+s)^4\right)|a_n|^2 \le m^2(1+s)^4 - 4. \tag{17}$$

Proof. From the representation (11), we have for $f_1,\ f_2 \in ST_{\mathcal{L}}(s)$,

$$\frac{zf'(z)}{f(z)} = \frac{m+2}{4}\cdot\frac{zf_1'(z)}{f_1(z)} - \frac{m-2}{4}\cdot\frac{zf_2'(z)}{f_2(z)}. \tag{18}$$

But

$$f_i(z) \in ST_{\mathcal{L}}(s) \Rightarrow \frac{zf_i'(z)}{f_i(z)} \prec (1+sz)^2,\ z \in U,\ 0 < s \le \frac{1}{\sqrt{2}},\ i = 1,2.$$

Therefore, there exists a Schwartz function $w_i(z)$, $i = 1,2$, such that

$$\frac{zf_i'(z)}{f_i(z)} = (1+sw_i(z))^2.$$

Thus, (18) can be written as

$$f(z) = \frac{4zf'(z)}{(m+2)(1+sw_1(z))^2 - (m-2)(1+sw_2(z))^2}. \tag{19}$$

By Parseval's identity, it follows that

$$\begin{aligned}
2\pi\sum_{n=1}^{\infty}|a_n|^2 r^{2n} &= \int_0^{2\pi}|f(z)|^2\,d\theta,\ \ z = re^{i\theta}\\
&= 16\int_0^{2\pi}\left|\frac{zf'(z)}{(m+2)(1+sw_1(z))^2 - (m-2)(1+sw_2(z))^2}\right|^2 d\theta\\
&\ge 16\int_0^{2\pi}\frac{|zf'(z)|^2}{\left[(m+2)(1+s)^2 + (m-2)(1+s)^2\right]^2}d\theta
\end{aligned}$$

$$= 4\int_0^{2\pi} \frac{|zf'(z)|^2\, d\theta}{m^2(1+s)^4}$$
$$= \frac{8}{m^2(1+s)^4}\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}.$$

This implies that

$$\sum_{n=1}^{\infty} \Big(4n^2 - m^2(1+s)^4\Big)|a_n|^2 r^{2n} \le 0, \quad a_1 = 1.$$

As $r \to 1^-$, we obtain the required result. □

Corollary 2.14. *If $f \in CV_{\mathcal{L}_m}(s)$, then*

$$\sum_{n=2}^{\infty} n^2\Big(4n^2 - m^2(1+s)^4\Big)|a_n|^2 \le m^2(1+s)^4 - 4. \tag{20}$$

For any positive integer $n > \frac{m(1+s)^2}{2}$, we have the following results.

Corollary 2.15. *If $f \in ST_{\mathcal{L}_m}(s)$, then*

$$|a_n| \le \sqrt{\frac{m^2(1+s)^4 - 4}{4n^2 - m^2(1+s)^4}}.$$

Corollary 2.16. *If $f \in CV_{\mathcal{L}_m}(s)$, then*

$$|a_n| \le \frac{1}{n}\sqrt{\frac{m^2(1+s)^4 - 4}{4n^2 - m^2(1+s)^4}}.$$

Remark 2.17. For $m = 2$, we obtain the corresponding results for the classes $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$.

3. Conclusion

We inaugurated subclasses of functions with boundary and radius rotations that are related to limaçon domains and examined some of their geometric properties. Radius results associated with functions in these classes and their linear combination have been successfully investigated. Furthermore, the growth rate of coefficients, arc length and coefficient estimates were derived for these novel classes. Conclusively, we gave some consequences of our findings, which were entirely new results on their own.

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