Commun. Korean Math. Soc. **37** (2022), No. 4, pp. 1055–1072
https://doi.org/10.4134/CKMS.c210344
pISSN: 1225-1763 / eISSN: 2234-3024

CERTAIN IMAGE FORMULAS OF (p,ν)–EXTENDED GAUSS' HYPERGEOMETRIC FUNCTION AND RELATED JACOBI TRANSFORMS

Purnima Chopra, Mamta Gupta, and Kanak Modi

Abstract. Our aim is to establish certain image formulas of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a,b;c;z)$ by using Saigo's hypergeometric fractional calculus (integral and differential) operators. Corresponding assertions for the classical Riemann-Liouville(R-L) and Erdélyi-Kober(E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the (p,ν)–extended Gauss's hypergeometric function $F_{p,\nu}(a,b;c;z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a,b;c;z)$.

1. Introduction and preliminaries

We recall Saigo's fractional integral and differential operators involving the Gauss's hypergeometric function ${}_2F_1$ as kernel. Let $\xi,\mu,\eta \in \mathbb{C}$, $\Re(\xi) > 0$ and $x > 0$. Then the Saigo's fractional integral and differential operators $\left(I_{0+}^{\xi,\mu,\eta}f\right)(x)$, $\left(I_{-}^{\xi,\mu,\eta}f\right)(x)$ and $\left(D_{0+}^{\xi,\mu,\eta}f\right)(x)$, $\left(D_{-}^{\xi,\mu,\eta}f\right)(x)$ are defined as (see, *e.g.*, [8, 9, 18, 19, 22]):

$$\left(I_{0+}^{\xi,\mu,\eta}f\right)(x) = \frac{x^{-\xi-\mu}}{\Gamma(\xi)}\int_0^x (x-t)^{\xi-1}\,{}_2F_1\left(\xi+\mu,-\eta;\xi;1-\frac{t}{x}\right)f(t)\,\mathrm{d}t, \tag{1.1}$$

$$\left(I_{-}^{\xi,\mu,\eta}f\right)(x) = \frac{1}{\Gamma(\xi)}\int_x^\infty (t-x)^{\xi-1}t^{-\xi-\mu}\,{}_2F_1\left(\xi+\mu,-\eta;\xi;1-\frac{x}{t}\right)f(t)\,\mathrm{d}t \tag{1.2}$$

and

$$\left(D_{0+}^{\xi,\mu,\eta}f\right)(x) = \left(I_{0+}^{-\xi,-\mu,\xi+\eta}f\right)(x)$$

Received October 18, 2021; Revised December 21, 2021; Accepted December 30, 2021.
2010 *Mathematics Subject Classification.* Primary 26A33, 33B20, 33C20; Secondary 26A09, 33B15, 33C05.
Key words and phrases. (p,ν)-extended Gauss hypergeometric function $F_{p,\nu}(a,b;c;z)$, extended beta function, fractional calculus operators.

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\xi+n,-\mu-n,\xi+\eta-n} f\right)(x) \quad (n = [\Re(\xi)] + 1), \tag{1.3}$$

$$\left(D_{-}^{\xi,\mu,\eta} f\right)(x) = \left(I_{-}^{-\xi,-\mu,\xi+\eta} f\right)(x)$$

$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{-\xi+n,-\mu-n,\xi+\eta} f\right)(x) \quad (n = [\Re(\xi)] + 1), \tag{1.4}$$

respectively. When $\mu = -\xi$, (1.1), (1.2), (1.3) and (1.4) coincide with the classical Riemann-Liouville fractional integrals and derivatives of order $\xi \in \mathbb{C}$ ($\Re(\xi) > 0$) and $x > 0$ (see, *e.g.*, [8,9,19]):

$$\left(I_{0+}^{\xi,-\xi,\eta} f\right)(x) = \left(I_{0+}^{\xi} f\right)(x) \equiv \frac{1}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} f(t)\,\mathrm{d}t,$$

$$\left(I_{-}^{\xi,-\xi,\eta} f\right)(x) = \left(I_{-}^{\xi} f\right)(x) = \frac{1}{\Gamma(\xi)} \int_x^\infty (t-x)^{\xi-1} f(t)\,\mathrm{d}t$$

and

$$\left(D_{0+}^{\xi,-\xi,\eta} f\right)(x) = \left(D_{0+}^{\xi} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\xi)} \int_0^x (x-t)^{n-\xi-1} f(t)\,\mathrm{d}t$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\xi} f\right)(x) \qquad (n = [\Re(\xi)] + 1),$$

$$\left(D_{-}^{\xi,-\xi,\eta} f\right)(x) = \left(D_{-}^{\xi} f\right)(x) = (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\xi)} \int_x^\infty (t-y)^{n-\xi-1} f(t)\,\mathrm{d}t$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{n-\xi} f\right)(x) \qquad (n = [\Re(\xi)] + 1),$$

respectively, where $[\Re(\xi)]$ is the integral part of $\Re(\xi)$.

If $\mu = 0$, (1.1), (1.2), (1.3) and (1.4) are the so-called Erdélyi-Kober fractional integrals and derivatives defined for $\xi \in \mathbb{C}$ ($\Re(\xi) > 0$) and $x > 0$ (see, *e.g.*, [8,9,19]):

$$\left(I_{0+}^{\xi,0,\eta} f\right)(x) = \left(I_{\eta,\xi}^{+} f\right)(x) = \frac{x^{-\xi-\eta}}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} t^{\eta} f(t)\,\mathrm{d}t,$$

$$\left(I_{-}^{\xi,0,\eta} f\right)(x) = \left(K_{\eta,\xi}^{-} f\right)(x) \equiv \frac{x^{\eta}}{\Gamma(\xi)} \int_x^\infty (t-x)^{\xi-1} t^{-\xi-\eta} f(t)\,\mathrm{d}t$$

and

$$\left(D_{0+}^{\xi,0,\eta} f\right)(x) = \left(D_{\eta,\xi}^{+} f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\xi+n,-\xi,\xi+\eta-n} f\right)(x) \quad (n = [\Re(\xi)] + 1),$$

$$\left(D_{-}^{\xi,0,\eta} f\right)(x) = \left(D_{\eta,\xi}^{-} f\right)(x)$$

$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_-^{-\xi+n,-\xi,\xi+\eta} f\right)(x) \quad (n = [\Re(\xi)] + 1),$$

$$\left(D_{\eta,\xi}^+ f\right)(x)$$
$$= x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\xi)} \int_0^x t^{\xi+\eta}(x-t)^{n-\xi-1} f(t)\,\mathrm{d}t \quad (n = [\Re(\xi)] + 1),$$

$$\left(D_{\eta,\xi}^- f\right)(x)$$
$$= x^{\eta+\xi} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\xi)} \int_x^\infty t^{-\eta}(t-x)^{n-\xi-1} f(t)\,\mathrm{d}t \quad (n = [\Re(\xi)] + 1),$$

respectively. In recent years, extensions of a number of well-known special functions have been investigated and studied the (p,q)–variant, and in turn, when $p = q$ the p–variant together with the set of related higher transcendental hypergeometric type special functions (see, for details, [1–6, 10, 11, 14–16]). In particular, Parmar *et al.* [13, p. 98, Eq. (40)] introduced and studied the (p,ν)-extended Gauss hypergeometric function $F_{p,\nu}(a,b;c;z)$ with $(p \geq 0; |z| < 1, \Re(c) > \Re(b) > 0)$ in the form:

$$F_{p,\nu}(a,b;c;z) = \sum_{n=0}^\infty (a)_n \frac{\mathrm{B}_\nu(b+n,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!}, \tag{1.5}$$

where $\mathrm{B}_\nu(x,y;p)$ is the extension of the extended Beta function introduced by Parmar *et al.* [13]

$$\mathrm{B}_\nu(x,y;p) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\, K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \mathrm{d}t\,,$$

when $\Re(p) > 0$. They developed and studied its several properties such as integral representations, Mellin transforms, summation formulas, transformation formulas, and so on. The concept of the Hadamard product (or convolution) of two analytic functions is required in our current investigation. It can aid in the decomposition of a newly emerged function into two known functions. If one of the power series, in particular, describes an entire function, then the Hadamard product series also defines an entire function. If we assume

$$g(z) := \sum_{n=0}^\infty c_n z^n \ (|z| < R_f) \ \text{ and } \ h(z) := \sum_{n=0}^\infty d_n z^n \ (|z| < R_g)$$

two given power series and whose radii of convergence are given by R_f and R_g, respectively, then their Hadamard product (or convolution) is the power series defined by

$$(g * h)(z) := \sum_{n=0}^\infty c_n\, d_n z^n = (h * g)(z) \ (|z| < R), \tag{1.6}$$

where

$$R = \lim_{n\to\infty} \left| \frac{c_n\, d_n}{c_{n+1}\, d_{n+1}} \right| = \left(\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \right) \cdot \left(\lim_{n\to\infty} \left| \frac{d_n}{d_{n+1}} \right| \right) = R_f \cdot R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

The Fox-Wright function${}_r\Psi_s(z)$ $(r,\, s \in \mathbb{N}_0)$, which is a generalization of hypergeometric function, is defined as follows (see, for details, [8, 12]; see also [19, 21]):

$$(1.7)\quad {}_r\Psi_s \left[\begin{matrix} (a_1, A_1), \dots, (a_s, A_s); \\ (b_1, B_1), \dots, (b_s, B_s); \end{matrix}\; z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_r + A_r n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_s + B_s n)} \, \frac{z^n}{n!}$$

$$\left(A_\ell \in \mathbb{R}^+ \ (\ell = 1, \dots, r);\ B_\ell \in \mathbb{R}^+ \ (\ell = 1, \dots, s);\ 1 + \sum_{\ell=1}^{s} B_\ell - \sum_{\ell=1}^{r} A_\ell \geqq 0 \right),$$

where the equality in the convergence condition holds for

$$|z| < \nabla := \left(\prod_{\ell=1}^{r} A_\ell^{-A_\ell} \right) \cdot \left(\prod_{\ell=1}^{s} B_\jmath^{B_\ell} \right).$$

In this paper, we obtain certain image formulas of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a, b; c; z)$ by using Saigo's hypergeometric fractional calculus (integral and differential) operators (1.1), (1.2), (1.3) and (1.4). Corresponding assertions for the classical Riemann-Liouville(R-L) and Erdélyi-Kober(E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the (p,ν)–extended Gauss's hypergeometric function $F_{p,\nu}(a, b; c; z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a, b; c; z)$.

2. Fractional integration of the $F_{p,\nu}(a, b; c; z)$

We begin the main results exposition with presenting a composition formulas of generalized fractional integrals (1.1) and (1.2) involving (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a, b; c; z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.6) of (p,ν)–extended Gauss' hypergeometric function (1.5) and Fox-Wright function ${}_r\Psi_s(z)$ (1.7).

Lemma 1. *Let $\xi, \mu, \eta \in \mathbb{C}$. Then there exist the relations*

(a) *If $\Re(\xi) > 0$ and $\Re(\sigma) > \max[0, \Re(\mu - \eta)]$, then*

$$(2.1)\qquad (I_{0+}^{\xi,\mu,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma + \eta - \mu)}{\Gamma(\sigma - \mu)\Gamma(\sigma + \xi + \eta)} x^{\sigma-\mu-1}.$$

In particular, for $x > 0$ we have

$$(I_{0+}^{\xi} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma + \xi)} x^{\sigma+\xi-1} \quad (\Re(\xi) > 0,\ \Re(\sigma) > 0),$$

$$(I^{+}_{\eta,\xi}t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\xi+\eta)}x^{\sigma-1} \quad (\Re(\xi)>0,\ \Re(\sigma)>-\Re(\eta)).$$

(b) *If* $\Re(\xi)>0$ *and* $\Re(\sigma)<1+\min[\Re(\mu),\Re(\eta)]$, *then*

$$(I^{\xi,\mu,\eta}_{-}t^{\sigma-1})(x) = \frac{\Gamma(\mu-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\xi+\mu+\eta-\sigma+1)}x^{\sigma-\mu-1}. \tag{2.2}$$

In particular, for $x>0$ *we have*

$$(I^{\xi}_{-}t^{\sigma-1})(x) = \frac{\Gamma(1-\xi-\sigma)}{\Gamma(1-\sigma)}x^{\sigma+\xi-1} \quad (0<\Re(\xi)<1-\Re(\sigma)),$$

$$(K^{-}_{\eta,\xi}t^{\sigma-1})(x) = \frac{\Gamma(\eta-\sigma+1)}{\Gamma(\xi+\eta-\sigma+1)}x^{\sigma-1} \quad (\Re(\sigma)<1+\Re(\sigma)).$$

Theorem 1. *Let* $\rho,\xi,\mu,\eta,\sigma,\omega,\in\mathbb{C}$, $\rho>0$ *be such that* $\Re(p)>0$, $\Re(\xi)>0$ *and* $\Re(\sigma)>\max[0,\Re(\mu-\eta)]$. *Then the following Saigo hypergeometric fractional integral* $I^{\xi,\mu,\eta}_{0+}$ *of* $F_{p,\nu}(a,b;c;\omega t^{\rho})$ *holds*:

$$\left(I^{\xi,\mu,\eta}_{0+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x)$$
$$= x^{\sigma-\mu-1}F_{p,\nu}(a,b;c;\omega x^{\rho})] * {}_3\Psi_2\left[\begin{array}{c}(1,1),(\sigma,\rho),(\sigma+\eta-\mu,\rho);\\ (\sigma-\mu,\rho),(\sigma+\xi+\eta,\rho);\end{array}\ \omega x^{\rho}\right], \tag{2.3}$$

where it is assumed that the left-sided hypergeometric fractional integral in (2.3) *exists.*

Proof. Applying definition (1.5), using (1.1) and (2.1) and changing the orders of integration and summation, we find for $x>0$

$$\left(I^{\xi,\mu,\eta}_{0+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x)$$
$$= \sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_{\nu}(b+k,\ c-b\,;\,p)}{\mathrm{B}(b,\ c-b)}\,\frac{\omega^k}{k!}\ \left(I^{\xi,\mu,\eta}_{0+}t^{\sigma+\rho k-1}\right)(x)$$
$$= x^{\sigma-\mu-1}\ \sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_{\nu}(b+k,\ c-b\,;\,p)}{\mathrm{B}(b,\ c-b)\,k!}$$
$$\times\frac{\Gamma(1+k)\Gamma(\sigma+\rho k)\Gamma(\sigma+\eta-\mu+\rho k)}{\Gamma(\sigma+\xi+\eta+\rho k)\Gamma(\sigma-\mu+\rho k)\,k!}(\omega x^{\rho})^k. \tag{2.4}$$

By applying the Hadamard product (1.6) in (2.4), which in view of (1.5) and (1.7), yields the desired formula (2.3). □

Theorem 2. *Let* $\rho,\xi,\mu,\eta,\sigma,\omega\in\mathbb{C}$, $\rho>0$ *be such that* $\Re(p)>0$, $\Re(\xi)>0$ *and* $\Re(\sigma)<1+\min[\Re(\mu),\Re(\eta)]$. *Then the following Saigo hypergeometric fractional integral* $I^{\xi,\mu,\eta}_{-}$ *of* $F_{p,\nu}(a,b;c;\left(\frac{\omega}{t^{\rho}}\right))$ *holds*:

$$\left(I^{\xi,\mu,\eta}_{-}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)$$
$$= x^{\sigma-\mu-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{x^{\rho}}\right) * {}_3\Psi_2\left[\begin{array}{c}(1,1),(1+\mu-\sigma,\rho),(1+\eta-\sigma,\rho);\\ (1-\sigma,\rho),(1+\xi+\mu+\eta-\sigma,\rho);\end{array}\ \frac{\omega}{x^{\rho}}\right], \tag{2.5}$$

where it is assumed that the right-sided hypergeometric fractional integral in (2.5) *exists.*

Proof. Applying definition (1.5), using (1.2) and (2.2) and changing the orders of integration and summation, we find for $x > 0$

$$\begin{aligned}
&\left(I_{-}^{\xi,\mu,\eta}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)\\
&= \sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_{\nu}(b+k,\,c-b\,;\,p)}{\mathrm{B}(b,\,c-b)\,k!}\,(\omega)^k\ \left(I_{-}^{\xi,\mu,\eta}t^{\sigma-\rho k-1}\right)(x)\\
&= x^{\sigma-\mu-1}\ \sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_{\nu}(b+k,\,c-b\,;\,p)}{\mathrm{B}(b,\,c-b)\,k!}\\
&\qquad\times\frac{\Gamma(1+k)\Gamma(1+\mu-\sigma+\rho k)\Gamma(1+\eta-\sigma+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1+\xi+\mu+\eta-\sigma+\rho k)\,k!}\left(\frac{\omega}{x^{\rho}}\right)^k. \qquad (2.6)
\end{aligned}$$

By applying the Hadamard product (1.6) in (2.6), which in view of (1.5) and (1.7), yields the desired formula (2.5). □

Next by putting $\mu = -\xi$ and $\mu = 0$ in Theorem 1 and Theorem 2, we obtain results for the classical Riemann-Liouville fractional integrals given in Corollary 2.1 and Corollary 2.3 and Erdélyi-Kober fractional integrals given in Corollary 2.2 and Corollary 2.4, respectively.

Corollary 2.1. *Let* $\rho,\xi,\sigma,\omega \in \mathbb{C}$, $\rho > 0$, *be such that* $\Re(p) > 0$, $\Re(\xi) > 0$, $\Re(\sigma) > 0$. *Then the following Riemann-Liouville fractional integral* I_{0+}^{ξ} *of* $F_{p,\nu}(a,b;c;\omega t^{\rho})$ *holds*:

$$\begin{aligned}
&\left(I_{0+}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x)\\
&= x^{\sigma+\xi-1}F_{p,\nu}(a,b;c;\omega x^{\rho})\,*\,{}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma,\rho);\\(\sigma+\xi,\rho);\end{array}\ \omega x^{\rho}\right], \qquad (2.7)
\end{aligned}$$

where it is assumed that the left-sided Riemann-Liouville fractional integral in (2.7) *exists.*

Corollary 2.2. *Let* $\rho,\xi,\eta,\sigma,\omega \in \mathbb{C}$, $\rho > 0$, *be such that* $\Re(p) > 0$, $\Re(\xi) > 0$ *and* $\Re(\sigma) > -\Re(\eta)$. *Then the following Erdélyi-Kober fractional integral* $I_{\eta,\xi}^{+}$ *of* $F_{p,\nu}(a,b;c;\omega t^{\rho})$ *holds*:

$$\begin{aligned}
&\left(I_{\eta,\xi}^{+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x)\\
&= x^{\sigma-1}F_{p,\nu}(a,b;c;\omega x^{\rho})\,*\,{}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma+\eta,\rho);\\(\sigma+\xi+\eta,\rho);\end{array}\ \omega x^{\rho}\right], \qquad (2.8)
\end{aligned}$$

where it is assumed that the left-sided Erdélyi-Kober fractional integral in (2.8) *exists.*

Corollary 2.3. *Let $\rho,\xi,\sigma,\omega\in\mathbb{C}$, $\rho>0$ be such that $\Re(p)>0$, $0<\Re(\xi)<1-\Re(\sigma)$. Then the following Riemann-Liouville fractional integral I_{-}^{ξ} of $F_{p,\nu}(a,b;c;\frac{\omega}{t^\rho})$ holds*:

$$\left(I_{-}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t^\rho}\right)\right\}\right)(x)$$

$$= x^{\sigma+\xi-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{x^\rho}\right) * {}_2\Psi_1\left[\begin{array}{cc}(1,1),(1-\xi-\sigma,\rho); & \frac{\omega}{x^\rho}\\ (1-\sigma,\rho); & \end{array}\right], \tag{2.9}$$

where it is assumed that the right-sided Riemann-Liouville fractional integral in (2.9) *exists.*

Corollary 2.4. *Let $\rho,\xi,\eta,\sigma,\omega\in\mathbb{C}$, $\rho>0$ be such that $\Re(p)>0$, $\Re(\xi)>0$ and $\Re(\sigma)<1+\Re(\eta)$. Then the following Erdélyi-Kober fractional integral $K_{\eta,\xi}^{-}$ of $F_{p,\nu}\left(a,b;c;\frac{\omega}{t^\rho}\right)$ holds*:

$$\left(K_{\eta,\xi}^{-}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{t^\rho})\right\}\right)(x)$$

$$= x^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{x^\rho}) * {}_2\Psi_1\left[\begin{array}{cc}(1,1),(1+\eta-\sigma,\rho); & \frac{\omega}{x^\rho}\\ (1+\xi+\eta-\sigma,\rho); & \end{array}\right], \tag{2.10}$$

where it is assumed that the right-sided Erdélyi-Kober fractional integral in (2.10) *exists.*

3. Fractional differentiation of the $F_{p,\nu}(a,b;c;z)$

In this section, we obtain a composition formulas of generalized fractional differentiation (1.3) and (1.4) involving (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a,b;c;z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.6) of (p,ν)–extended Gauss' hypergeometric function and Fox-Wright function ${}_p\Psi_q(z)$.

Lemma 2. *Let ξ, μ, $\eta\in\mathbb{C}$, $\rho>0$. Then there exist the relations*

(a) *If $\Re(\xi)>0$ and $\Re(\sigma)>-\min[0,\Re(\xi+\mu+\eta)]$, then*

$$(D_{0+}^{\xi,\mu,\eta}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)\Gamma(\sigma+\xi+\mu+\eta)}{\Gamma(\sigma+\mu)\Gamma(\sigma+\eta)}x^{\sigma+\mu-1}. \tag{3.1}$$

In particular, for $x>0$ we have

$$(D_{0+}^{\xi}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\xi)}x^{\sigma-\xi-1}\quad(\Re(\xi)>0,\ \Re(\sigma)>0),$$

$$(D_{\eta,\xi}^{+}t^{\sigma-1})(x)=\frac{\Gamma(\sigma+\xi+\eta)}{\Gamma(\sigma+\eta)}x^{\sigma-1}\quad(\Re(\xi)>0,\ \Re(\sigma)>-\Re(\xi+\eta)).$$

(b) *If $\Re(\xi)>0$, $\Re(\sigma)<1+\min[\Re(-\mu-n),\Re(\xi+\eta)]$ and $n=[\Re(\xi)]+1$, then*

$$(D_{-}^{\xi,\mu,\eta}t^{\sigma-1})(x)=\frac{\Gamma(1-\sigma-\mu)\Gamma(1-\sigma+\xi+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\mu)}x^{\sigma+\mu-1}. \tag{3.2}$$

In particular, for $x > 0$ we have

$$(D_{-}^{\xi} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\xi)}{\Gamma(1-\sigma)} x^{\sigma-\xi-1} \quad (\Re(\xi) > 0,\ \Re(\sigma) < 1 + \Re(\xi) - n),$$

$$(D_{\eta,\xi}^{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\xi+\eta)}{\Gamma(1-\sigma-\eta)} x^{\sigma-1} \quad (\Re(\xi) > 0,\ \Re(\sigma) < 1 + \Re(\xi+\eta) - n).$$

Theorem 3. *Let $\rho, \xi, \mu, \eta, \sigma, \omega \in \mathbb{C}$, $\rho > 0$ be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$ and $\Re(\sigma) > -\min[0, \Re(\xi+\mu+\eta)]$. Then the following Saigo hypergeometric fractional differentiation $D_{0+}^{\xi,\mu,\eta}$ of $F_{p,\nu}(a,b;c;\omega t^{\rho})$ holds*:

$$\begin{aligned} &\left(D_{0+}^{\xi,\mu,\eta}\left\{t^{\sigma-1} F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x) \\ &= x^{\sigma+\mu-1} F_{p,\nu}(a,b;c;\omega x^{\rho}) * {}_3\Psi_2\left[\begin{matrix} (1,1),(\sigma,\rho),(\xi+\sigma+\eta+\mu,\rho); \\ (\sigma+\mu,\rho),(\sigma+\eta,\rho); \end{matrix}\ \omega x^{\rho}\right], \end{aligned} \tag{3.3}$$

where it is assumed that the left-sided hypergeometric fractional derivative in (3.3) *exists.*

Proof. By virtue of the formulas (1.3) and (1.5), the term-by-term fractional differentiation and the application of the relation (3.1), yields for $x > 0$

$$\begin{aligned} &\left(D_{0+}^{\xi,\mu,\eta}\left\{t^{\sigma-1} F_{p,\nu}(a,b;c;\omega t^{\rho})\right\}\right)(x) \\ &= \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}_{\nu}(b+k,\ c-b\,;\ p)}{\mathrm{B}(b,\ c-b)} \frac{\omega^k}{k!} \left(D_{0+}^{\xi,\mu,\eta} t^{\sigma+\rho k-1}\right)(x) \\ &= x^{\sigma+\mu-1} \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}_{\nu}(b+k,\ c-b\,;\ p)}{\mathrm{B}(b,\ c-b)\, k!} \\ &\qquad \times \frac{\Gamma(1+k)\Gamma(\sigma+\rho k)\Gamma(\sigma+\xi+\eta+\mu+\rho k)}{\Gamma(\sigma+\mu+\rho k)\Gamma(\sigma+\eta+\rho k)\, k!} (\omega x^{\rho})^k. \end{aligned} \tag{3.4}$$

By applying the Hadamard product (1.6) in (3.4), which in view of (1.5) and (1.7), yields the desired formula (3.3). □

Theorem 4. *Let $\rho, \xi, \mu, \eta, \sigma, \omega \in \mathbb{C}$, $\rho > 0$ be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$ and $\Re(\sigma) < 1 + \min[\Re(-\mu-n), \Re(\xi+\eta)]$, $n = [\Re(\xi)] + 1$. Then the following Saigo hypergeometric fractional differentiation $D_{-}^{\xi,\mu,\eta}$ of $F_{p,\nu}\left(a,b;c;\frac{\omega}{t^{\rho}}\right)$ holds*:

$$\begin{aligned} &\left(D_{-}^{\xi,\mu,\eta}\left\{t^{\sigma-1} F_{p,\nu}\left(a,b;c;\frac{\omega}{t^{\rho}}\right)\right\}\right)(x) \\ &= x^{\sigma+\mu-1} F_{p,\nu}\left(a,b;c;\frac{\omega}{x^{\rho}}\right) * {}_3\Psi_2\left[\begin{matrix} (1,1),(1-\sigma-\mu,\rho),(1+\xi+\eta-\sigma,\rho); \\ (1-\sigma,\rho),(1-\mu+\eta-\sigma,\rho); \end{matrix}\ \frac{\omega}{x^{\rho}}\right], \end{aligned} \tag{3.5}$$

where it is assumed that the right-sided hypergeometric fractional derivative in (3.5) *exists.*

Proof. By virtue of the formulas (1.4) and (1.5), the term-by-term fractional differentiation and the application of the relation (3.2), yields for $x > 0$

$$\left(D_{-}^{\xi,\mu,\eta}\left\{t^{\sigma-1} F_{p,\nu}\left(a,b;c;\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)$$

$$= \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}_\nu(b+k,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)\, k!} (\omega)^k \; \left(D_{-}^{\xi,\mu,\eta} t^{\sigma-\rho k-1}\right)(x)$$

$$= x^{\sigma+\mu-1} \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}_\nu(b+k,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)\, k!}$$

$$\times \frac{\Gamma(1+k)\Gamma(1-\mu-\sigma+\rho k)\Gamma(1+\xi+\eta-\sigma+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1-\sigma+\eta-\mu+\rho k)\, k!} \left(\frac{\omega}{x^\rho}\right)^k. \tag{3.6}$$

By applying the Hadamard product (1.6) in (3.6), which in view of (1.5) and (1.7), yields the desired formula (3.5). □

Next by putting $\mu = -\xi$ and $\mu = 0$ in Theorem 3 and Theorem 4, we obtain results for the classical Riemann-Liouville fractional derivatives given in Corollary 4.1 and Corollary 4.3 and Erdélyi-Kober fractional derivatives given in Corollary 4.2 and Corollary 4.4, respectively.

Corollary 4.1. *Let $\rho, \xi, \sigma, \omega \in \mathbb{C}$, $\rho > 0$, be such that $\Re(p) > 0$, and $\Re(\xi) \geq 0$, $\Re(\sigma) > 0$. Then the following Riemann-Liouville fractional differentiation D_{0+}^{ξ} of $F_{p,\nu}(a,b;c;\omega t^\rho)$ holds*:

$$\left(D_{0+}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t^\rho)\right\}\right)(x)$$

$$= x^{\sigma-\xi-1}F_{p,\nu}(a,b;c;\omega x^\rho) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma,\rho);\\(\sigma-\xi,\rho);\end{array}\ \omega x^\rho\right], \tag{3.7}$$

where it is assumed that the left-sided Riemann-Liouville fractional derivative in (3.7) *exists.*

Corollary 4.2. *Let $\rho, \xi, \eta, \sigma, \omega \in \mathbb{C}$, $\rho > 0$, be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$ and $\Re(\sigma) > -\Re(\eta+\xi)$. Then the following Erdélyi-Kober fractional differentiation $D_{\eta,\xi}^{+}$ of $F_{p,\nu}(a,b;c;\omega t^\rho)$ holds*:

$$\left(D_{\eta,\xi}^{+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)$$

$$= x^{\sigma-1}F_{p,\nu}(a,b;c;\omega x^\rho) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma+\xi+\eta,\rho);\\(\sigma+\eta,\rho);\end{array}\ \omega x^\rho\right], \tag{3.8}$$

where it is assumed that the left-sided Erdélyi-Kober fractional derivative in (3.8) *exists.*

Corollary 4.3. *Let $\rho, \xi, \sigma, \omega \in \mathbb{C}$, $\rho > 0$ be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$, $\Re(\sigma) < \Re(\xi) - [\Re(\xi)]$. Then the following Riemann-Liouville fractional differentiation D_{-}^{ξ} of $F_{p,\nu}\left(a,b;c;\frac{\omega}{t^\rho}\right)$ holds*:

$$\left(D_{-}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{t^\rho})\right\}\right)(x)$$

$$= x^{\sigma-\xi-1}F_{p,\nu}(a,b;c;\frac{\omega}{x^\rho}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\xi-\sigma,\rho);\\(1-\sigma,\rho);\end{array}\ \frac{\omega}{x^\rho}\right], \tag{3.9}$$

where it is assumed that the right-sided Riemann-Liouville fractional derivative in (3.9) *exists.*

Corollary 4.4. *Let* $\rho,\xi,\eta,\sigma,\omega\in\mathbb{C}$, $\rho>0$ *be such that* $\Re(p)>0$, $\Re(\xi)\geq 0$ *and* $\Re(\sigma)<\Re(\xi+\eta)-[\Re(\xi)]$. *Then the following Erdélyi-Kober fractional differentiation* $D^{-}_{\eta,\xi}$ *of* $F_{p,\nu}\left(a,b;c;\frac{\omega}{t^\rho}\right)$ *holds*:

$$\left(D^{-}_{\eta,\xi}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t^\rho}\right)\right\}\right)(x)$$

$$= x^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{x^\rho}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\xi-\sigma+\eta,\rho);\\(1-\sigma-\eta,\rho);\end{array}\ \frac{\omega}{x^\rho}\right], \tag{3.10}$$

where it is assumed that the right-sided Erdélyi-Kober fractional derivative in (3.10) *exists.*

4. Special cases to generalized hypergeometric function ${}_rF_s$

The Fox–Wright function ${}_r\Psi_s(z)$ extends the generalized hypergeometric function ${}_rF_s[z]$ which power series form reads

$${}_rF_s\left[\begin{array}{c}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{array}\Big|z\right]=\sum_{k\geq 0}\frac{\prod_{\ell=1}^{r}(a_\ell)_k}{\prod_{\ell=1}^{s}(b_\ell)_k}\frac{z^k}{k!},$$

where, as usual, we make use of the Pochhammer symbol (or raising factorial)

$$(\tau)_0=1;\quad (\tau)_k=\tau(\tau+1)\cdots(\tau+k-1)=\frac{\Gamma(\tau+k)}{\Gamma(\tau)},\quad k\in\mathbb{N}.$$

In the special case $A_r=B_s=1;\ell=1,\ldots,r;\ell=1,\ldots,s$, the Fox–Wright function ${}_r\Psi_s[z]$ reduces to the generalized hypergeometric function

$${}_r\Psi_s\left[\begin{array}{c}(a_1,1),\ldots,(a_r,1)\\(b_1,1),\ldots,(b_s,1)\end{array}\Big|z\right]=\frac{\Gamma(a_1)\cdots\Gamma(a_r)}{\Gamma(b_1)\cdots\Gamma(b_s)}\,{}_rF_s\left[\begin{array}{c}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{array}\Big|z\right].$$

As particular cases, if we put $\rho=1$ in all the results derived in Section 2 and Section 3, then they can be represented in terms of the Hadamard product of the (p,ν)–extended Gauss's hypergeometric function $F_{p,\nu}(a,b;c;z)$ and Gauss's hypergeometric function ${}_rF_s(z)$.

Theorem 5. *Let* $\xi,\mu,\eta,\sigma,\omega,\in\mathbb{C}$ *be such that* $\Re(p)>0$, $\Re(\xi)>0$ *and* $\Re(\sigma)>\max[0,\Re(\mu-\eta)]$. *Then the following Saigo hypergeometric fractional integral* $I^{\xi,\mu,\eta}_{0+}$ *of* $F_{p,\nu}(a,b;c;\omega t)$ *holds*:

$$\left(I^{\xi,\mu,\eta}_{0+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)$$

$$= x^{\sigma-\mu-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\mu)}{\Gamma(\sigma-\mu)\Gamma(\sigma+\xi+\eta)}$$

$$F_{p,\nu}(a,b;c;\omega x) * {}_3F_2\left[\begin{array}{c}1,\sigma,\sigma+\eta-\mu;\\\sigma-\mu,\sigma+\xi+\eta;\end{array}\ \omega x\right], \tag{4.1}$$

where it is assumed that the left-sided hypergeometric fractional integral in (4.1) *exists.*

Theorem 6. *Let* $\xi, \mu, \eta, \sigma, \omega \in \mathbb{C}$ *be such that* $\Re(p) > 0$, $\Re(\xi) > 0$ *and* $\Re(\sigma) < 1+\min[\Re(\mu), \Re(\eta)]$. *Then the following Saigo hypergeometric fractional integral* $I_{-}^{\xi,\mu,\eta}$ *of* $F_{p,\nu}(a, b; c; \left(\frac{\omega}{t}\right))$ *holds*:

$$\begin{aligned}&\left(I_{-}^{\xi,\mu,\eta}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)\right\}\right)(x)\\ &= x^{\sigma-\mu-1}\frac{\Gamma(1+\mu-\sigma)\Gamma(1+\eta-\sigma)}{\Gamma(1-\sigma)\Gamma(1+\xi+\mu+\eta-\sigma)}F_{p,\nu}\left(a,b;c;\frac{\omega}{x}\right)\\ &* {}_3F_2\left[\begin{array}{c}1, 1+\mu-\sigma, 1+\eta-\sigma;\\ 1-\sigma, 1+\xi+\mu+\eta-\sigma;\end{array}\ \frac{\omega}{x}\right],\end{aligned} \tag{4.2}$$

where it is assumed that the right-sided hypergeometric fractional integral in (4.2) *exists.*

Corollary 6.1. *Let* $\xi, \sigma, \omega \in \mathbb{C}$, *be such that* $\Re(p) > 0$, $\Re(\xi) > 0$, $\Re(\sigma) > 0$. *Then the following Riemann-Liouville fractional integral* I_{0+}^{ξ} *of* $F_{p,\nu}(a, b; c; \omega t)$ *holds*:

$$\begin{aligned}&\left(I_{0+}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)\\ &= x^{\sigma+\xi-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma+\xi)}F_{p,\nu}(a,b;c;\omega x) * {}_2F_1\left[\begin{array}{c}1, \sigma;\\ \sigma+\xi;\end{array}\ \omega x\right],\end{aligned} \tag{4.3}$$

where it is assumed that the left-sided Riemann-Liouville fractional integral in (4.3) *exists.*

Corollary 6.2. *Let* $\xi, \eta, \sigma, \omega \in \mathbb{C}$, *be such that* $\Re(p) > 0$, $\Re(\xi) > 0$ *and* $\Re(\sigma) > -\Re(\eta)$. *Then the following Erdélyi-Kober fractional integral* $I_{\eta,\xi}^{+}$ *of* $F_{p,\nu}(a, b; c; \omega t)$ *holds*:

$$\begin{aligned}&\left(I_{\eta,\xi}^{+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)\\ &= x^{\sigma-1}\frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\xi+\eta)}F_{p,\nu}(a,b;c;\omega x) * {}_2F_1\left[\begin{array}{c}1, \sigma+\eta;\\ \sigma+\xi+\eta;\end{array}\ \omega x\right],\end{aligned} \tag{4.4}$$

where it is assumed that the left-sided Erdélyi-Kober fractional integral in (4.4) *exists.*

Corollary 6.3. *Let* $\xi, \sigma, \omega \in \mathbb{C}$ *be such that* $\Re(p) > 0$, $0 < \Re(\xi) < 1 - \Re(\sigma)$. *Then the following Riemann-Liouville fractional integral* I_{-}^{ξ} *of* $F_{p,\nu}\left(a, b; c; \frac{\omega}{t}\right)$ *holds*:

$$\begin{aligned}&\left(I_{-}^{\xi}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)\right\}\right)(x)\\ &= x^{\sigma+\xi-1}\frac{\Gamma(1-\xi-\sigma)}{\Gamma(1-\sigma)}F_{p,\nu}\left(a,b;c;\frac{\omega}{x}\right) * {}_2F_1\left[\begin{array}{c}1, 1-\xi-\sigma;\\ 1-\sigma;\end{array}\ \frac{\omega}{x}\right],\end{aligned} \tag{4.5}$$

where it is assumed that the right-sided Riemann-Liouville fractional integral in (4.5) *exists.*

Corollary 6.4. *Let $\xi, \eta, \sigma, \omega \in \mathbb{C}$ be such that $\Re(p) > 0$, $\Re(\xi) > 0$ and $\Re(\sigma) < 1 + \Re(\eta)$. Then the following Erdélyi-Kober fractional integral $K^{-}_{\eta,\xi}$ of $F_{p,\nu}\left(a, b; c; \frac{\omega}{t}\right)$ holds:*

$$\left(K^{-}_{\eta,\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{t})\right\}\right)(x)$$

$$= x^{\sigma-1}\frac{\Gamma(1+\eta-\sigma)}{\Gamma(1+\xi+\eta-\sigma)}F_{p,\nu}(a,b;c;\frac{\omega}{x}) * {}_2F_1\left[\begin{array}{c} 1, 1+\eta-\sigma; \\ 1+\xi+\eta-\sigma; \end{array} \frac{\omega}{x}\right], \tag{4.6}$$

where it is assumed that the right-sided Erdélyi-Kober fractional integral in (4.6) *exists.*

Theorem 7. *Let $\xi, \mu, \eta, \sigma, \omega \in \mathbb{C}$ be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$ and $\Re(\sigma) > -\min[0, \Re(\xi + \mu + \eta)]$. Then the following Saigo hypergeometric fractional differentiation $D^{\xi,\mu,\eta}_{0+}$ of $F_{p,\nu}(a, b; c; \omega t)$ holds:*

$$\left(D^{\xi,\mu,\eta}_{0+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)$$

$$= x^{\sigma+\mu-1}\frac{\Gamma(\sigma)\Gamma(\xi+\sigma+\eta+\mu)}{\Gamma(\sigma+\mu)\Gamma(\sigma+\eta)}F_{p,\nu}(a,b;c;\omega x^{\rho})$$

$$* \ {}_3F_2\left[\begin{array}{c} 1, \sigma, \xi+\sigma+\eta+\mu; \\ \sigma+\mu, \sigma+\eta; \end{array} \omega x\right], \tag{4.7}$$

where it is assumed that the left-sided hypergeometric fractional derivative in (4.7) *exists.*

Theorem 8. *Let $\xi, \mu, \eta, \sigma, \omega \in \mathbb{C}$ be such that $\Re(p) > 0$, $\Re(\xi) \geq 0$ and $\Re(\sigma) < 1 + \min[\Re(-\mu - n), \Re(\xi + \eta)], n = [\Re(\xi)] + 1$. Then the following Saigo hypergeometric fractional differentiation $D^{\xi,\mu,\eta}_{-}$ of $F_{p,\nu}\left(a, b; c; \frac{\omega}{t}\right)$ holds:*

$$\left(D^{\xi,\mu,\eta}_{-}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)\right\}\right)(x)$$

$$= x^{\sigma+\mu-1}\frac{\Gamma(1-\sigma-\mu)\Gamma(1+\xi+\eta-\sigma)}{\Gamma(1-\sigma)\Gamma(1-\mu+\eta-\sigma)}F_{p,\nu}\left(a,b;c;\frac{\omega}{x}\right)$$

$$* \ {}_3F_2\left[\begin{array}{c} 1, 1-\sigma-\mu, 1+\xi+\eta-\sigma; \\ 1-\sigma, 1-\mu+\eta-\sigma; \end{array} \frac{\omega}{x}\right], \tag{4.8}$$

where it is assumed that the right-sided hypergeometric fractional derivative in (4.8) *exists.*

Corollary 8.1. *Let $\xi, \sigma, \omega \in \mathbb{C}$, be such that $\Re(p) > 0$, and $\Re(\xi) \geq 0$, $\Re(\sigma) > 0$. Then the following Riemann-Liouville fractional differentiation D^{ξ}_{0+} of $F_{p,\nu}(a, b; c; \omega t)$ holds:*

$$\left(D^{\xi}_{0+}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)$$

$$= x^{\sigma-\xi-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma-\xi)}F_{p,\nu}(a,b;c;\omega x) * {}_2F_1\left[\begin{array}{c} 1, \sigma; \\ \sigma-\xi; \end{array} \omega x\right], \tag{4.9}$$

where it is assumed that the left-sided Riemann-Liouville fractional derivative in (4.9) *exists.*

Corollary 8.2. *Let* $\xi,\eta,\sigma,\omega\in\mathbb{C}$, $\rho>0$, *be such that* $\Re(p)>0$, $\Re(\xi)\geq 0$ *and* $\Re(\sigma)>-\Re(\eta+\xi)$. *Then the following Erdélyi-Kober fractional differentiation* $D^{+}_{\eta,\xi}$ *of* $F_{p,\nu}(a,b;c;\omega t)$ *holds*:

$$\begin{aligned}&\left(D^{+}_{\eta,\xi}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\omega t)\right\}\right)(x)\\ &= x^{\sigma-1}\frac{\Gamma(\sigma+\xi+\eta)}{\Gamma(\sigma+\eta)}F_{p,\nu}(a,b;c;\omega x) * {}_2F_1\left[\begin{array}{c}1,\sigma+\xi+\eta;\\ \sigma+\eta;\end{array}\ \omega x\right],\end{aligned} \tag{4.10}$$

where it is assumed that the left-sided Erdélyi-Kober fractional derivative in (4.10) *exists.*

Corollary 8.3. *Let* $\xi,\sigma,\omega\in\mathbb{C}$ *be such that* $\Re(p)>0$, $\Re(\xi)\geq 0$, $\Re(\sigma)<\Re(\xi)-[\Re(\xi)]$. *Then the following Riemann-Liouville fractional differentiation* D^{ξ}_{-} *of* $F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)$ *holds*:

$$\begin{aligned}&\left(D^{\xi}_{-}\left\{t^{\sigma-1}F_{p,\nu}(a,b;c;\frac{\omega}{t})\right\}\right)(x)\\ &= x^{\sigma-\xi-1}\frac{\Gamma(1+\xi-\sigma)}{\Gamma(1-\sigma)}F_{p,\nu}(a,b;c;\frac{\omega}{x}) * {}_2F_1\left[\begin{array}{c}1,1+\xi-\sigma;\\ 1-\sigma;\end{array}\ \frac{\omega}{x}\right],\end{aligned} \tag{4.11}$$

where it is assumed that the right-sided Riemann-Liouville fractional derivative in (4.11) *exists.*

Corollary 8.4. *Let* $\xi,\eta,\sigma,\omega\in\mathbb{C}$ *be such that* $\Re(p)>0$, $\Re(\xi)\geq 0$ *and* $\Re(\sigma)<\Re(\xi+\eta)-[\Re(\xi)]$. *Then the following Erdélyi-Kober fractional differentiation* $D^{-}_{\eta,\xi}$ *of* $F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)$ *holds*:

$$\begin{aligned}&\left(D^{-}_{\eta,\xi}\left\{t^{\sigma-1}F_{p,\nu}\left(a,b;c;\frac{\omega}{t}\right)\right\}\right)(x)\\ &= x^{\sigma-1}\frac{\Gamma(1+\xi-\sigma+\eta)}{\Gamma(1-\sigma-\eta)}F_{p,\nu}(a,b;c;\frac{\omega}{x}) * {}_2F_1\left[\begin{array}{c}1,1+\xi-\sigma+\eta;\\ 1-\sigma-\eta;\end{array}\ \frac{\omega}{x}\right],\end{aligned} \tag{4.12}$$

where it is assumed that the right-sided Erdélyi-Kober fractional derivative in (4.12) *exists.*

5. Jacobi and related integral transforms

In this section, we obtain Jacobi and related integral transforms of the (p,ν)-extended Gauss' hypergeometric functions (1.5). The classical orthogonal Jacobi polynomials $P_n^{(\varpi,\theta)}(t)$ is defined by (see, for details, [17,20,21]):

$$P_n^{(\varpi,\theta)}(t)=(-1)^n(-t)=\binom{\varpi+n}{n}{}_2F_1\left[\begin{array}{c}-n,\varpi+\theta+n+1\\ \varpi+1\end{array}\Big|\frac{1-t}{2}\right],$$

where ${}_2F_1$ denotes the Gauss hypergeometric function [17].

Definition 1 (see, for example, [7, p. 501])**.** The Jacobi transform of a function $f(t)$ is defined as follows:

$$\mathbb{J}^{(\varpi,\theta)}[f(t);n]=\int_{-1}^{1}(1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\,f(t)\mathrm{d}t \tag{5.1}$$

$$(\min\{\Re(\varpi),\,\Re(\theta),\}>-1;\;n\in\mathbb{N}_0),$$

provided that the function $f(t)$ is so constrained that the integral in (5.1) exists.

The Jacobi transform of the power function $t^{\rho-1}$ (see, for example, [7]) is given by

$$\begin{aligned}&\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1};\,n]\\&=\int_{-1}^{1}(1-t)^{\xi-1}\,(1+t)^{\eta-1}\,P_n^{(\varpi,\theta)}(t)\,t^{\rho-1}\,\mathrm{d}t\\&=2^{\xi+\eta-1}\binom{\varpi+n}{n}\mathrm{B}(\xi,\eta)F_{1:1;0}^{1:2;1}\left[\begin{array}{lr}\xi:-n,\varpi+\theta+n+1;1-\rho; & \\ & 1,2\\ \xi+\eta:\qquad\varpi+1;\quad —; & \end{array}\right]\end{aligned} \tag{5.2}$$

$$(\min\{\Re(\xi),\,\Re(\eta),\}>0;\;\rho\in\mathbb{C};\;n\in\mathbb{N}_0),$$

where $F_{p:l;\mu}^{q:m;\nu}$ denotes the Kampé de Fériet's function in two variables (see, *e.g.*, [21, p. 22, Eq. 1.3(2)] and [21, p. 37, Eq. 1.4(21)]). In particular, upon setting $\xi=\varpi+1$ and $\eta=\theta+1$, this last integral formula (5.2) would reduce immediately to the following form:

$$\begin{aligned}&\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1};\,n]\\&=\int_{-1}^{1}(1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\,t^{\rho-1}\,\mathrm{d}t\\&=2^{\varpi+\theta+1}\binom{\varpi+n}{n}\mathrm{B}(\varpi+1,\theta+1)\\&\quad F_{1:1;0}^{1:2;1}\left[\begin{array}{lr}\varpi+1:-n,\varpi+\theta+n+1;1-\rho; & \\ & 1,2\\ \varpi+\theta+2:\qquad\varpi+1;\quad —; & \end{array}\right]\end{aligned} \tag{5.3}$$

$$(\min\{\Re(\varpi),\,\Re(\theta),\}>-1;\;\rho\in\mathbb{C};\;n\in\mathbb{N}_0).$$

Indeed, in its further special case when $\rho=m+1\,(m\in\mathbb{N}_0)$, (5.3) yields the following well-known result for the Jacobi transform of $t^m\,(m\in\mathbb{N}_0)$, which is given by (see, for example, [17, p. 261, Eq. (14) and (15)])

$$\begin{aligned}&\mathbb{J}^{(\varpi,\theta)}[t^{m};\,n]\\&=\int_{-1}^{1}(1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\,t^{m}\,\mathrm{d}t\end{aligned}$$

(5.4)

$$= \begin{cases} 0 & (m=0,1,2,\ldots,n-1) \\ 2^{\varpi+\theta+n+1}\,\mathrm{B}(\varpi+n+1,\theta+n+1) & (m=n) \\ 2^{\varpi+\theta+n+1}\binom{m}{n}\mathrm{B}(\varpi+n+1,\theta+n+1) & \\ \cdot\ {}_2F_1\left[\begin{matrix} n-m,\varpi+n+1 \\ \varpi+\theta+2n+2 \end{matrix}\,\Big|\,2\right], & (m=n+1,n+2,n+3,\ldots) \end{cases}$$

$$(\min\{\Re(\varpi),\ \Re(\theta),\} > -1;\ m,n\in\mathbb{N}_0).$$

For various choices of the parameters ϖ and θ, the Jacobi polynomials $P_n^{(\varpi,\theta)}(t)$ contain, as their special cases, such other classical orthogonal polynomials as (for example) the Gegenbauer (or Ultraspherical) polynomials $C_n^\nu(t)$, the Legendre (or spherical) polynomials $P_n(t)$, and the Tchebycheff polynomials $T_n(t)$ and $U_n(t)$ of the first and second kind (see, for details, [21]). In fact, we have the following relationships with the Gegenbauer polynomials $C_n^\nu(z)$ and the Legendre polynomials $P_n(z)$:

$$C_n^\nu(t) = \binom{\nu+n-\frac{1}{2}}{n}^{-1}\binom{2\nu+n-1}{n}P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(t) \tag{5.5}$$

and

$$P_n(t) = C_n^{\frac{1}{2}}(t) = P_n^{(0,0)}(t),$$

respectively, which, in conjunction with (5.1), yields the corresponding Gegenbauer transform $\mathbb{G}^{(\nu)}[f(t);n]$ given by

$$\begin{aligned} \mathbb{G}^{(\nu)}[f(t);n] &= \binom{\nu+n-\frac{1}{2}}{n}^{-1}\binom{2\nu+n-1}{n}\mathbb{J}^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}[f(t);n] \\ &= \int_{-1}^{1}(1-t^2)^{\nu-\frac{1}{2}}\,C_n^\nu(z)\,f(t)\mathrm{d}t \quad (\Re(\nu) > -\frac{1}{2};\ n\in\mathbb{N}_0), \end{aligned}$$

and the corresponding Legendre transform $\mathbb{L}[f(t);n]$ defined by

$$\mathbb{L}[f(t);n] = \mathbb{G}^{(\frac{1}{2})}[f(t);n] = \int_{-1}^{1}P_n(t)\,f(z)\mathrm{d}t \quad (n\in\mathbb{N}_0). \tag{5.6}$$

Now, we prove three results which exhibit the connections between the Jacobi, Gegenbauer and Legendre transforms with the following (p,ν)-extended Gauss' hypergeometric function (1.5).

Theorem 9. *Under the condition stated in* (1.5), *the following Jacobi transform formula holds*:

$$\begin{aligned} &\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1}\,F_{p,\nu}(a,b;c;\omega t);\ n] \\ &= 2^{\varpi+\theta+1}\binom{\varpi+n}{n}\mathrm{B}(\varpi+1,\theta+1)\sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_\nu(b+k,\ c-b\,;\ p)}{\mathrm{B}(b,\ c-b)} \end{aligned}$$

$$\cdot\, F^{1:2;1}_{1:1;0}\left[\begin{array}{lll} \varpi+1: -n, \varpi+\theta+n+1; 1-\rho-k; & & \\ & & 1,2 \\ \varpi+\theta+2: \quad \varpi+1; \quad —; & & \end{array}\right] \frac{\omega^k}{k!}, \tag{5.7}$$

$$(\Re(p)>0;\, m,\, n\in\mathbb{N}_0; \min\{\Re(\varpi),\, \Re(\theta),\}>-1;\, \rho\in\mathbb{C}),$$

where it is assumed that the Jacobi transforms in (5.3) *exists.*

Proof. By applying the definition (5.1) in conjunction with (1.5), we have

$$\begin{aligned} &\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1}\ F_{p,\nu}(a,b;c;\omega t);\, n] \\ &= \int_{-1}^{1} t^{\rho-1}(1-t)^{\varpi}\,(1+t)^{\theta}\, P_n^{(\varpi,\theta)}(t)\ F_{p,\nu}(a,b;c;\omega t)\,\mathrm{d}t \\ &= \int_{-1}^{1} t^{\rho-1}(1-t)^{\varpi}\,(1+t)^{\theta}\, P_n^{(\varpi,\theta)}(t) \sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_\nu(b+k,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)}\,\frac{(\omega t)^k}{k!}\,\mathrm{d}t. \end{aligned}$$

Now, upon changing the order of integration and summation (which can be justified easily by absolute convergence), we make use of the Jacobi transform formula (5.3) with the parameter ρ replaced by $\rho+k$ ($\rho\in\mathbb{C}; k\in\mathbb{N}_0$). □

By applying the Jacobi transform formula (5.4), we can simplify the assertion (5.7) of Theorem 9 in their special case when $\rho=m+1$ ($m\in\mathbb{N}_0$). Moreover, in view of the relationship (5.5), Theorem 9 yields the following corollary by setting $\varpi=\theta=\nu-\frac{1}{2}$.

Corollary 9.1. *Under the condition stated in* (1.5)*, the following Gegenbauer transform formula holds*:

$$\begin{aligned} &\mathbb{G}^{(\nu)}[t^{\rho-1}\ F_{p,\nu}(a,b;c;\omega t);\, n] \\ &= 2^{2\nu}\binom{2\nu+n-1}{n}\mathrm{B}(\nu+\frac{1}{2},\nu+\frac{1}{2})\sum_{k=0}^{\infty}(a)_k\,\frac{\mathrm{B}_\nu(b+k,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)} \end{aligned}$$

$$\cdot\, F^{1:2;1}_{1:1;0}\left[\begin{array}{lll} \nu+\frac{1}{2}: -n, 2\nu+n; 1-\rho-k; & & \\ & & 1,2 \\ 2\nu+1: \quad \nu+\frac{1}{2}; \quad —; & & \end{array}\right] \frac{\omega^k}{k!}, \tag{5.8}$$

$$(\Re(p)>0;\, m,\, n\in\mathbb{N}_0;\, \rho\in\mathbb{C}),$$

where it is assumed that the Gegenbauer transforms in (5.8) exists.

For the Legendre transform defined by (5.6), a special case of Theorem 9 when $\varpi=\theta=0$ (or, alternatively, Corollary 9.1 with $\nu=\frac{1}{2}$) yields the following result.

Corollary 9.2. *Under the condition stated in* (1.5)*, the following Legendre transform formula holds*:

$$\mathbb{L}[t^{\rho-1}\ F_{p,\nu}(a,b;c;\omega t);\, n]$$

$$(5.9)\quad = 2\sum_{k=0}^{\infty}(a)_k \frac{\mathrm{B}_\nu(b+k,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)}\, F^{1:2;1}_{1:1;0}\left[\begin{array}{c} 1:-n, n+1; 1-\rho-k; \\ 2: \qquad 1; \quad -; \end{array} 1, 2\right]\frac{\omega^k}{k!},$$

$$(\Re(p)>0;\ m,\, n\in\mathbb{N}_0;\ \rho\in\mathbb{C}p),$$

where it is assumed that the Legendre transform in (5.9) *exists.*

6. Concluding remarks

In this paper, we obtain certain image formulas of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a,b;c;z)$ by using Saigo's hypergeometric fractional calculus (integral and differential) operators (1.1), (1.2), (1.3) and (1.4). Corresponding assertions for the classical Riemann-Liouville(R-L) and Erdélyi-Kober(E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the (p,ν)–extended Gauss's hypergeometric function $F_{p,\nu}(a,b;c;z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the (p,ν)–extended Gauss' hypergeometric function $F_{p,\nu}(a,b;c;z)$.

Acknowledgments. The authors are grateful for comments and suggestions to anonymous referee to improve the manuscript in present form.

References

[1] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78** (1997), no. 1, 19–32. https://doi.org/10.1016/S0377-0427(96)00102-1

[2] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159** (2004), no. 2, 589–602. https://doi.org/10.1016/j.amc.2003.09.017

[3] J. Choi and R. K. Parmar, *Fractional integration and differentiation of the (p,q)-extended Bessel function*, Bull. Korean Math. Soc. **55** (2018), no. 2, 599–610. https://doi.org/10.4134/BKMS.b170193

[4] J. Choi and R. K. Parmar, *Fractional calculus of the (p,q)–extended Struve function*, Far East J. Math. Sci. **103** (2018), no. 2, 541–559 .

[5] J. Choi, R. K. Parmar, and T. K. Pogány, *Mathieu-type series built by (p,q)-extended Gaussian hypergeometric function*, Bull. Korean Math. Soc. **54** (2017), no. 3, 789–797. https://doi.org/10.4134/BKMS.b160313

[6] J. Choi, A. K. Rathie, and R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **36** (2014), no. 2, 357–385. https://doi.org/10.5831/HMJ.2014.36.2.357

[7] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, third edition, CRC Press, Boca Raton, FL, 2015.

[8] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[9] V. Kiryakova, *Generalized fractional calculus and applications*, Pitman Research Notes in Mathematics Series, 301, Longman Scientific & Technical, Harlow, 1994.

[10] M.-J. Luo, R. K. Parmar, and R. K. Raina, *On extended Hurwitz-Lerch zeta function*, J. Math. Anal. Appl. **448** (2017), no. 2, 1281–1304. https://doi.org/10.1016/j.jmaa.2016.11.046

[11] D. J. Maširević, R. K. Parmar, and T. K. Pogány, (p, q)-*extended Bessel and modified Bessel functions of the first kind*, Results Math. **72** (2017), no. 1-2, 617–632. https://doi.org/10.1007/s00025-016-0649-1

[12] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function*, Springer, New York, 2010. https://doi.org/10.1007/978-1-4419-0916-9

[13] R. K. Parmar, P. Chopra, and R. B. Paris, *On an extension of extended beta and hypergeometric functions*, J. Class. Anal. **11** (2017), no. 2, 91–106. https://doi.org/10.7153/jca-2017-11-07

[14] R. K. Parmar and T. K. Pogány, *Extended Srivastava's triple hypergeometric* $H_{A,p,q}$ *function and related bounding inequalities*, J. Contemp. Math. Anal. **52** (2017), no. 6, 276–287; translated from Izv. Nats. Akad. Nauk Armenii Mat. **52** (2017), no. 6, 47–61.

[15] R. K. Parmar, T. K. Pogány, and R. K. Saxena, *On properties and applications of* (p, q)-*extended* τ-*hypergeometric functions*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 3, 278–282. https://doi.org/10.1016/j.crma.2017.12.014

[16] T. K. Pogány and R. K. Parmar, *On* p-*extended Mathieu series*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **22(534)** (2018), 107–117.

[17] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[18] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ. **11** (1977/78), no. 2, 135–143.

[19] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*: *Theory and Applications*, Translated from the Russian: *Integrals and derivatives of fractional order and some of their applications* ("Nauka i Tekhnika", Minsk, 1987); Gordon and Breach Science Publishers: Reading, UK, 1993.

[20] I. N. Sneddon, *The Use of the Integral Transforms*, Tata McGraw-Hill, New Delhi, 1979.

[21] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.

[22] H. M. Srivastava and R. K. Saxena, *Operators of fractional integration and their applications*, Appl. Math. Comput. **118** (2001), no. 1, 1–52. https://doi.org/10.1016/S0096-3003(99)00208-8

Purnima Chopra
Marudhar Engineering College
Bikaner-334001, Rajasthan, India
Email address: purnimachopra05@gmail.com

Mamta Gupta
Amity School of Applied Sciences
Amity University
Jaipur-303002, Rajasthan, India
Email address: mamtagupta8384@gmail.com

Kanak Modi
Amity School of Applied Sciences
Amity University
Jaipur-303002, Rajasthan, India
Email address: kmodi@jpr.amity.edu