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# ON STRONG EXPONENTIAL LIMIT SHADOWING PROPERTY

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Abstract. In this study, we show that the strong exponential limit shadowing property (SELmSP, for short), which has been recently introduced, exists on a neighborhood of a hyperbolic set of a diffeomorphism. We also prove that $\Omega$-stable diffeomorphisms and $\mathcal{L}$-hyperbolic homeomorphisms have this type of shadowing property. By giving examples, it is shown that this type of shadowing is different from the other shadowings, and the chain transitivity and chain mixing are not necessary for it. Furthermore, we extend this type of shadowing property to positively expansive maps with the shadowing property.

## 1. Introduction

In the theory of shadowing, usually a Riemannian manifold $M$ with metric $d$ and a $C^1$-diffeomorphism $\phi : M \to M$ are considered. Given $\delta > 0$, a sequence $\xi = \{x_k\}_{k\in\mathbb{Z}} \subset M$ with the property

$$d(\phi(x_k), x_{k+1}) < \delta, \quad k \in \mathbb{Z},$$

is called a *$\delta$-pseudo-orbit*. Often, pseudo-orbits are obtained as a result of the numerical studies of dynamical systems. The dynamical system $\phi$ has the *shadowing property* (or *POTP*, for short) on a set $Y \subset M$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for a given $\delta$-pseudo-orbit $\xi = \{x_k\}_{k\in\mathbb{Z}} \subset Y$ there is some point $p \in M$ with the property that

$$d(\phi^k(p), x_k) < \epsilon, \quad k \in \mathbb{Z}.$$

We should recall that if $Y = M$, then it is said that $\phi$ has the POTP. It is well known that a diffeomorphism has the POTP on a neighborhood of its hyperbolic set. This property is thoroughly studied in [8] and [9].

We say that the dynamical system $\phi$ has the *Lipschitz shadowing property* (LpSP) on $Y$ if there exist constants $L, \delta_0 > 0$ such that for every $\delta$-pseudo-orbit $\xi = \{x_k\}_{k\in\mathbb{Z}} \subset Y$ with $0 < \delta < \delta_0$, there exists some point $p \in M$

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so that $d(\phi^k(p), x_k) < L\delta$, $k \in \mathbb{Z}$. Indeed, the LpSP is stronger than the POTP. Also, it is proved that the LpSP holds on a neighborhood of hyperbolic set [9]. Note that Pilyugin et al. in [11] interestingly, showed that the LpSP implies structural stability (and therefore, the LpSP is equivalent to structural stability). In addition, they proved that Anosov systems are equivalent to expansive systems that have the LpSP [11, Corollary 3].

Another kind of shadowing property is the *limit shadowing property* (*LmSP*) which was introduced by Eirola et al. in [4]. Precisely, we say that the dynamical system $\phi$ has the limit shadowing property on a set $Y$ if for any sequence $\xi = \{x_k\}_{k\geq 0} \subset Y$ with the property

$$d(\phi(x_k), x_{k+1}) \to 0, \quad k \to \infty$$

there is a point $p \in M$ such that

$$d(\phi^k(p), x_k) \to 0, \quad k \to \infty.$$

From the numerical point of view, this property means that if we apply a numerical method that approximates the orbits of $\phi$ with improving accuracy so that one-step errors go to zero as time goes to infinity, then the numerically obtained trajectories tend to the real ones.

If $b < \infty$ we say that a finite $\delta$-pseudo-orbit $\{x_i\}_{i=0}^{b}$ of $f$ is a $\delta$-*chain* from $x_0$ to $x_b$ of length $b$. A non-empty subset $A$ of $X$ is said to be *chain transitive* whenever for any $x, y \in A$ and any $\delta > 0$, there exists a $\delta$-chain from $x$ to $y$. A map $f$ is said *chain transitive* if $X$ itself is a chain transitive set.

Lee and Sakai in [7], proved that expansive systems with the shadowing property have limit shadowing property. More recently, Kulczycki et al. considered the converse case. In [6], they proved that in compact dynamical systems, chain transitivity together with limit shadowing property implies the shadowing property and transitivity. Therefore, in transitive expansive systems, the shadowing and the limit shadowing are equivalent. See [6, Corollary 7.5]. Later in [3], interestingly, it was proved that the shadowing, the limit shadowing, and the two-sided limit shadowing with a gap are equivalent.

Ahmadi and Molaei in [1] introduced a new type of limit shadowing such that one-step errors tend to zero with exponential rate. Their definition is as follows:

**Definition** ([1])**.** The dynamical system $\phi$ has the *strong exponential limit shadowing* (*SELmSP, for short*) on $M$ if there exist constants $L > 0$ and $\lambda \in (0, 1)$ such that for every sequence $\xi = \{x_k\}_{k\geq 0}$ with

$$d(\phi(x_k), x_{k+1}) \leq \lambda^k, \quad k \geq k_1,\ k_1 \in \mathbb{N} \tag{1.1}$$

there exist a point $p \in M$ and $k_2 \in \mathbb{N}$ such that

$$d(\phi^k(p), x_k) \leq L\lambda^k, \quad k \geq k_2,\ k_2 \in \mathbb{N}. \tag{1.2}$$

In our opinion, the SELmSP has not been studied extensively yet. In [1] the authors studied a weaker form of the strong exponential limit shadowing, named the *exponential limit shadowing property* (ELmSP, for short). Indeed, their definition replaces the exponential term $\lambda^k$ by $\lambda^{\frac{k}{2}}$ in relation (1.2). They proved that the ELmSP holds on a neighborhood of a hyperbolic set. Also, they deduced that $\Omega$-stability implies the ELmSP.

In what follows, we prove the existence of the strong exponential limit shadowing on a neighborhood of a hyperbolic set and prove that positively expansive open maps have this type of limit shadowing. In order to show the above existence, we invoke a result in [4].

This paper is organized as follows. In Section 2, we prove that the SELmSP holds near a hyperbolic set. In Section 3, it is shown that $\Omega$-stability implies the SELmSP. Some examples represented in Section 4, show that the SELmSP is different from other shadowing properties.

## 2. The SELmSP property

Let us recall the following theorem, which is a stronger version of the shadowing lemma.

**Theorem 2.1** ([9])**.** *If $\Lambda$ is a hyperbolic set for a diffeomorphism $\phi$, then there exists a neighborhood $W$ of $\Lambda$ on which $\phi$ has the LpSP.*

We invoke the following lemma to prove Theorem 2.3.

**Lemma 2.2** ([4])**.** *Let $\Lambda$ be a hyperbolic set for a diffeomorphism $\phi$. Then there exist a neighborhood $U$ of $\Lambda$, and the constants $\delta > 0$ and $\nu \in (0, 1)$ such that if two points $x, y$ have the properties $\phi^k(x), \phi^k(y) \in U$ and $d(\phi^k(x), \phi^k(y)) \leq \delta$ for $k \geq 0$, then $d(\phi^k(x), \phi^k(y)) \leq 2\nu^k d(x, y)$, $k \geq 0$.*

**Theorem 2.3.** *Let $\Lambda$ be a hyperbolic set for a diffeomorphism $\phi$ of $M$. Then there exists a neighborhood $W$ of $\Lambda$ on which $\phi$ has the SELmSP on $W$.*

*Proof.* By Theorem 2.1 there exists a neighborhood $U_0$ of $\Lambda$ on which $\phi$ has the LpSP property with the constants $L_0$ and $d_0$. Take the neighborhood $U_1$ of $\Lambda$ and numbers $\nu \in (0, 1)$ and $\delta > 0$ given by Lemma 2.2.

If we put

$$U = U_0 \cap U_1,$$

then we can find a neighborhood $W$ of $\Lambda$ such that $N(W, \delta) \subset U$ (by decreasing $\delta$, if necessary), where $N(W, \delta)$ is the $\delta$-neighborhood of $W$. We claim that for $\lambda^2 = \nu \in (0, 1)$, $\phi$ has the $SELmSP$ on $W$ with constants $\lambda^2$ and $L = L_0 + 2\delta$. The rest of the proof is a mimic of the proof of [1, Theorem 2.1]. □

Note that it is easy to show that the strong exponential limit shadowing property is invariant of topological conjugacy. In fact, suppose $(X, f)$ and $(Y, g)$ are two conjugate systems, i.e., $h \circ f = g \circ h$, where $h$ is a conjugacy. Assume that $f$ has the SELmSP with constants $L$ and $\lambda$, and $\xi = \{y_k\}_{k \geq 0}$ is a sequence such that $d_Y(g(y_k), (y_{k+1})) \leq \lambda^{\frac{k}{2}}$ for $k \geq k_1$. Fix $i, k$ big enough ($i \geq k \geq k_1$). By uniform continuity of $h^{-1}$ choose $\delta > 0$ corresponding to

$\epsilon = \lambda^i$. Note that, if necessary, by increasing $k_1$ we can assume that $\lambda^{\frac{k}{2}} < \delta$. Now $d_X\big(fh^{-1}(y_k), h^{-1}(y_{k+1})\big) = d_X\big(h^{-1}g(y_k), h^{-1}(y_{k+1})\big) < \lambda^i < \lambda^k$, $k \geq k_1$. So, $h^{-1}(\xi)$ satisfies relation (1.1) for $f$. Therefore, there exist $z \in X$ and $k_2 \in \mathbb{N}$ so that $d_X(f^k(z), h^{-1}(y_k)) < L\lambda^k$ for $k \geq k_2$. Again, fix $i, k$ big enough ($i \geq k \geq k_2$). By uniform continuity of $h$ choose $\eta > 0$ corresponding to $\epsilon = L\lambda^i$. Note that, if necessary, by increasing $k_2$ we can assume that $L\lambda^k < \eta$. By choosing $\eta$ and $k_2$ we have $d_Y(g^k h(z), y_k) = d_Y(hf^k(z), hh^{-1}(y_k)) < L\lambda^i < L\lambda^{\frac{k}{2}}$ for $k \geq k_2$, hence $h(z)$ is the required point and $g$ has the SELmSP with constants $L$ and $\lambda^{\frac{1}{2}}$. The remaining part is similar.

**Proposition 2.4.** *If $f$ is a surjection that has the SELmSP, then so does $f^n$ for all $n > 0$.*

*Proof.* Fix $n > 0$ and suppose $\lambda \in (0,1)$ and $L > 0$ are the constants in definition of the SELmSP for $f$. Let $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence which satisfies the relation (1.1) with constant $\lambda^n$ in place of $\lambda$ for the map $f^n$, define the sequence $\eta = \{y_i\}_{i=0}^{\infty}$ with

$$y_k = \begin{cases} x_0 & k = 0, \\ f^{k-nq-1}(x_{q+1}) & nq < k \leq n(q+1). \end{cases}$$

Indeed,

$$\begin{aligned} \eta = &\{x_0, x_1, f(x_1), f^2(x_1), \dots, f^{n-1}(x_1), x_2, f(x_2), f^2(x_2), \dots, \\ & f^{n-1}(x_2), x_3, f(x_3), f^2(x_3), \dots, f^{n-1}(x_3), \dots\} \end{aligned}$$

then for $k = nq$ we have

$$d(f(y_k), y_{k+1}) = d(f^n(x_q), x_{q+1}) < (\lambda^n)^q = \lambda^k.$$

So, $\eta = \{y_i\}_{i=0}^{\infty}$ satisfies the relation (1.2) for the map $f$. Hence there exists $z \in X$ such that $d(f^k(z), y_k) < L\lambda^k$. If we put $k = nq - (n-1)$ in the last inequality we get $y_k = x_q$, and $d\big((f^n)^q(p), x_q\big) < L\lambda^{nq-(n-1)} = L\lambda^{-(n-1)}(\lambda^n)^q$, which $p = f^{-(n-1)}(z)$. Therefore $f^n$ has the SELmSP with constants $\lambda^n$ and $L_0 = L\lambda^{-(n-1)}$ $\square$

**Definition.** A diffeomorphism $\phi$ satisfies Axiom $A$ if $\overline{Per(\phi)} = \Omega(\phi)$ is a hyperbolic set, which $\Omega(\phi)$ is the set of non-wandering points of $\phi$.

Let us remind the reader of the Smale's spectral decomposition Theorem [9]. If $\phi$ satisfies Axiom $A$ property, then there is a unique representation of $\Omega(\phi)$,

$$\Omega(\phi) = \Omega_1 \cup \cdots \cup \Omega_k$$

as a disjoint union of closed $\phi$-invariant sets (called basic set) such that

(1) each $\Omega_i$ is a locally maximal hyperbolic set of $\phi$.
(2) $\phi$ is topologically transitive on each $\Omega_i$.
(3) each $\Omega_i$ is a disjoint union of closed sets $\Omega_i^j$, $1 \leq j \leq m_i$, the diffeomorphism $\phi$ cyclically permutes the sets $\Omega_i^j$, and $\phi^{m_i}$ is topologically mixing on each $\Omega_i^j$.

We use the following lemma, which is proved in [10, Lemma 1].

**Lemma 2.5.** *If a diffeomorphism $\phi$ is $\Omega$-stable and the sequence $\{x_k\}$ satisfies $\lim_{k\to 0} d(\phi(x_k), x_{k+1}) = 0$, then there exists a basic set $\Omega_i$ such that*

$$d(\Omega_i, x_k) \to 0, \quad k \to \infty.$$

## 3. Ω-stable implies SELmSP

Now, we are going to prove $\Omega S \subset SELmSP$.

**Theorem 3.1.** *If a diffeomorphism $\phi$ is $\Omega$-stable, then it has the SELmSP.*

*Proof.* Let $\Omega(\phi)$ be the non-wandering set of $\phi$, since $\phi$ is $\Omega$-stable it satisfies in Axiom $A$ property, so $\Omega(\phi)$ is a hyperbolic set. Therefore, by Theorem 2.3 there exists a neighborhood $W$ of $\Omega(\phi)$ on which $\phi$ has the $SELmSP$ for some $\lambda \in (0, 1)$ and $L > 0$. Now, if a sequence $\{x_k\}$ in $W$ satisfies the relation (1.1), then by Lemma 2.5, there exists a basic set $\Omega_i$ such that

$$d(\Omega_i, x_k) \to 0, \quad k \to \infty.$$

So, there exists $k_0 \geq k_1$ such that $\{x_k\}_{k \geq k_0} \subset W$. Hence there exist $p \in M$ and $k_2 \in \mathbb{N}$ so that

$$d(\phi^k(p), x_k) \leq L\lambda^k, \quad k \geq k_2.$$ □

**Definition.** If $(X, d)$ is a compact metric space and $f : X \to X$ is a homeomorphism, then $f$ is called *$\mathcal{L}$-hyperbolic* if

(1) $f$ is a Lipschitz homeomorphism;
(2) There is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there exists $\delta > 0$ such that for any two points $x, y$ with $d(x, y) < \delta$, $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point $\alpha(x, y)$;
(3) There is a constant $\kappa > 0$ such that

$$d(\alpha(x, y), x) \leq \kappa d(x, y), \quad d(\alpha(x, y), y) \leq \kappa d(x, y);$$

(4) There are $\Delta, \nu \in (0, 1)$ such that for all $x \in X$,

$$d(f^n(x), f^n(y)) \leq \nu^n d(x, y) \quad \text{for } \ y \in W^s_\Delta(x),\ n \geq 0,$$
$$d(f^{-n}(x), f^{-n}(y)) \leq \nu^n d(x, y) \quad \text{for } \ y \in W^u_\Delta(x),\ n \geq 0.$$

In analogous to the proof of Theorem 2.3 in [1], it is easy to prove the following result.

**Theorem 3.2.** *Let $f : X \to X$ be an $\mathcal{L}$-hyperbolic homeomorphism on a compact metric space $(X, d)$. Then $f$ has the SELmSP.*

*Proof.* In [13], it was shown that an $\mathcal{L}$-hyperbolic homeomorphism has the LpSP. Take $\lambda^2 = \nu \in (0, 1)$ and repeat the steps of the proof [1, Theorem 2.3]. Note that here $\nu$ is the constant in the definition of $\mathcal{L}$-hyperbolicity. □

**Theorem 3.3** ([13])**.** *Let $f$ be a homeomorphism on a compact metric space $X$. Then the following conditions are equivalent*:

(1) *f is expansive and has the POTP.*
(2) *There is a compatible metric $D$ for $X$ such that $f$ is $\mathcal{L}$-hyperbolic.*
(3) *$(X, f)$ is a Smale space.*

By using the above theorem, we have the following corollaries.

**Corollary 3.4.** *Let $f : X \to X$ be an expansive homeomorphism on a compact metric space having the POTP. Then there exists a compatible metric $D$ on $X$ such that $f$ has the SELmSP with respect to $D$.*

**Corollary 3.5.** *Let $(X, f)$ be a Smale space. Then there exists a compatible metric on $X$ such that $f$ has SELmSP with respect to $D$.*

In the following, we are going to show that the SELmSP holds for a class of non-homeomorphisms. Precisely, we show that the SELmSP holds for positively expansive maps having the POTP. In [1, Corollary 3.1], the author deduced that expansive homeomorphisms on a compact metric space with the POTP also have the exponential limit shadowing property. Here, we are going to extend this result for positively expansive maps.

**Theorem 3.6.** *Let $f : X \to X$ be a positively expansive map on a compact metric space $X$ having the LpSP. Then $f$ has also the SELmSP.*

*Proof.* Let $\delta > 0$ be an expansive constant for $f$, without losing the generality assume that $0 < \delta < 1$. Take $\lambda = \delta$, and suppose that $\{x_i\}_{i\geq 0} \subset X$ is a sequence with

$$d(f(x_k), x_{k+1}) \leq \lambda^k, \quad k \geq k_1,\ k_1 \in \mathbb{N}.$$

Take $L > 0$, and $0 < \delta_0 < \delta$ by the LpSP, and let $0 < \Delta < \delta_0$ be given. Choose $k_2 \in \mathbb{N}$ large enough to have $\lambda^k < \min\{\delta_0, \frac{\Delta}{2L}\}$ for all $k \geq k_2$. Then by the LpSP for each fixed $k \geq k_2$ there exists $y_k$ such that

$$d(f^i(y_k), x_i) \leq L\lambda^k < \frac{\Delta}{2},\ i \geq k \geq k_2.$$

If we take $z = y_{k_2}$, then we have

$$\begin{aligned} d(f^i(z), f^i(y_k)) &\leq d(f^i(y_k), x_i) + d(f^i(z), x_i) \\ &< \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta,\ i \geq k \geq k_2. \end{aligned}$$

So, we get

$$d(f^{i+k}(y_k), f^{i+k}(z)) < \Delta,\ i \geq 0.$$

Hence, by expansivity, it follows $f^k(y_k) = f^k(z)$. Therefore, by the above inequality we obtain

$$d(f^k(z), x_k) = d(f^k(y_k), x_k) \leq L\lambda^k,\ k \geq k_2.$$

□

It is well-known that for positively expansive maps on a compact metric space, being an open map, the standard shadowing property and the Lipschitz shadowing are all equivalent [14, Theorem 1]. So, readily we get the following result.

**Corollary 3.7.** *Suppose that $f : X \to X$ is a positively expansive map on a compact metric space. If $f$ has the POTP (or equivalently being an open map), then it has the SELmSP.*

## 4. Examples

The following example shows that not every system has the SELmSP property.

**Example 4.1.** Let $X = \{0, 1, \frac{1}{2}, \dots\}$ be a metric space with the usual metric on $\mathbb{R}$, and define $f(0) = 1, f(\frac{1}{n}) = \frac{1}{n+1}$. Now, take $(x_i)_{i\in\mathbb{N}} = \{1, \frac{1}{2}, \dots\}$ so we have $d(f(x_i), x_{i+1}) = 0$. It can be easily shown that $f^i(z) = \frac{z}{iz+1}$ for every $i > 0$. Hence for every $L > 0$ and $\lambda \in (0, 1)$, there are no points $z \in X$ such that $d(f^i(z), x_i) \le L\lambda^i$.

In the following, we prove that the one-sided shift dynamical system has the SELmSP.

**Example 4.2.** The shift $(\Sigma_2^+, \sigma)$ has the SELmSP.

*Proof.* Although, the proof follows by Corollary 3.7, but here we give a direct proof. We use the metric $d(x, y) = \Sigma_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$, which generates the product topology on $\Sigma_2^+$. We show that $\sigma$ has the SELmSP with constants $\lambda = \frac{1}{2}, L = 1$. Let $\{x^{(i)}\}_{i=0}^{\infty} \subset \Sigma_2^+$ be a sequence with

$$d(\sigma(x^{(i)}), x^{(i+1)}) < \lambda^i,\ i \ge 0.$$

Therefore, it is easy to see that for all $i \ge 0$ we have $x_j^{(i)} = x_{j-1}^{(i+1)},\ 1 \le j \le i+1$. Put $z = (x_0^{(0)} x_0^{(1)} \cdots)$, that is for all $i \ge 0$, set $z_i = x_0^{(i)}$. Now it can be shown that

$$\begin{aligned} d(\sigma^i(z), x^{(i)}) &= d(x_0^{(i)} x_0^{(i+1)} \cdots x_0^{(2i+1)} \cdots, x_0^{(i)} x_1^{(i)} \cdots x_{i+1}^{(i)} \cdots) \\ &< \lambda^{i+1} < \lambda^i,\ i \ge 0. \end{aligned}$$

Note that we have used $x_j^{(i)} = x_0^{(i+j)},\ 1 \le j \le i+1$, in the above inequality. □

The next example, which is called a permutation of two points, is clearly an open positively expansive map on a compact metric space that does not have the two-sided limit shadowing. However, we observe that by Corollary 3.7, it has the strong exponential limit-shadowing. Therefore, the SELmSP is different from the two-sided limit shadowing.

**Example 4.3.** Let $X = \{a, b\}$ and define $f(a) = b$ and $f(b) = a$, so $f$ is a homeomorphism on $X$. Fix $0 < \lambda < 1$, for each exponentially-limit pseudo-orbit $\{x_n\}_{n\geq 0}$ (relation (1.1)) there exists $N \in \mathbb{N}$ such that $x_{N+k} = a$, if $k$ is even or $x_{N+k} = b$, if $k$ is odd. In other words, each exponentially limit pseudo-orbit is, by neglecting a finite beginning terms, a periodic sequence of the form $\overline{ab}$ or $\overline{ba}$. Then either $z = a$ or $b$, exponentially limit shadows the sequence $\{x_n\}_{n\geq 0}$, i.e., the relation (1.2) holds with constants $\lambda \in (0, 1)$ and $L = 1$.

*Remark* 4.4. It can be easily shown that the above example is transitive, but not topologically mixing (and since it has the POTP, so equivalently not chain mixing). Therefore, the chain mixing (and topologically mixing) is not necessary for the SELmSP.

*Remark* 4.5. Sakai in [13, Example 4.2], shows that the subshifts of finite type or briefly SFT are $\mathcal{L}$-hyperbolic. So, by Theorem 3.2, they have the SELmSP. On the other hand, there are examples of SFT in [2], which are not chain transitive. Therefore, chain transitivity (and topological transitivity) is not necessary for the SELmSP.

We can see that a dynamical system with the POTP need not have the SELmSP, but for expansive homeomorphisms on compact metric spaces, it has by Corollary 3.4. Note that in [1, Example 4.1], it is proved that the following example does not have the ELmSP, so it does not have the SELmSP. However, by using [9, Theorem 3.1.1], it has the POTP. In addition, by [12], there is an $\Omega$-stable diffeomorphism that does not have the weak shadowing (and so does not the POTP). That is, the SELmSP does not imply the POTP and vice versa.

**Example 4.6.** Consider the unit circle with the coordinate $x \in [0, 1)$. Let $\phi$ be a dynamical system on $S^1$ generated by the map $f : S^1 \to S^1$ defined by $f(x) = x - x^2(x - \frac{1}{2})(x - 1)^2$. Easily, $\phi$ has two fixed points $\{0, \frac{1}{2}\}$.

*Remark* 4.7. Note that by [9, Theorem 3.1.3], the above example also has the LmSP. Hence, the LmSP does not imply the SELmSP.

If $X$ is a totally disconnected compact metric space having at least two points, then it is easy to show that the identity map has the SELmSP. However, by [5, Example 5.1], the identity map does not have the asymptotic average shadowing property. Therefore, the following corollary is obvious.

**Corollary 4.8.** *The SELmSP does not imply the asymptotic average shadowing property* (*and so the specification property*).

## 5. Conclusion

In this paper, we studied the strong exponential limit shadowing (SELmSP) as a new kind of limit shadowing. We show that the SELmSP exists near a hyperbolic set of a diffeomorphism. Also, the $\Omega$-stable dynamical systems have this kind of limit shadowing. In addition, in a compact metric space any $\mathcal{L}$-hyperbolic dynamical system has the SELmSP. Moreover, Examples 4.3, 4.6 and Corollary 3.7 show that this limit shadowing is different from the usual shadowing, the limit shadowing, the asymptotic average shadowing and the two-sided limit shadowing property. Besides, we prove that open positively expansive maps have the strong exponential limit shadowing.

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