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COMMON FIXED POINT RESULTS FOR GENERALIZED ORTHOGONAL F-SUZUKI CONTRACTION FOR FAMILY OF MULTIVALUED MAPPINGS IN ORTHOGONAL b-METRIC SPACES

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Abstract. In this paper, we introduce a new class of mappings called the generalized orthogonal F-Suzuki contraction for a family of multivalued mappings in the setup of orthogonal b-metric spaces. We established the existence of some common fixed point results without using any commutativity condition for this new class of mappings in orthogonal b-metric spaces. Moreover, we illustrate and support these common fixed point results with example. The results obtained in this work generalize and extend some recent and classical related results in the existing literature.

1. Introduction and preliminaries

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. In the last 50 years, fixed point theory has been a flourishing area of research for many mathematicians. The origins of the theory, which date to the later part of the nineteenth century, rest in the use of successive approximations to establish the existence and uniqueness of solutions, particularly to differential equations. Since the simplicity and usefulness of the Banach's fixed point theorem, many authors have extended, improved and generalized Banach's fixed point theorem from different perspectives. For more details, we cite the readers to (see [1,6,7, 9,12,14,17–20,31] and the references therein. The approximation of fixed points of nonlinear mappings and solutions of optimization problems is another area of research interest in fixed point theory (see [5, 23, 24, 30] and the references therein).

The concept of a b-metric space was introduced by Bakhtin [7] and Czerwik [12]. They also established the fixed point result in the setting of b-metric spaces

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which is a generalization of the Banach contraction principle. In 2015, Alsulami et al. [6] introduced the concepts of generalized F-Suzuki type contraction mappings and proved the fixed point theorems on complete b-metric spaces.

The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [26], and then by Nieto and Lopez [21, 22].

In [15], Gordji et al. introduced the new notion of the orthogonal sets and gave a real generalization of Banach's fixed point theorem. As an application, they studied the existence and uniqueness of solution for a first-order ordinary differential equation, while the Banach contraction mapping principle cannot be applied in this situation. Afterward Yang et al. [33] introduced the notion of an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping and proved some fixed point theorem for such contraction mappings in orthogonally metric spaces. Sawangsup et al. [28] introduced the new concept of an orthogonal F-contraction mappings and proved the fixed point theorems on orthogonal-complete metric spaces. Subsequently, many other researchers (see [13, 14, 33]) studied the orthogonal contractive type mappings and obtained significant results.

Recently, Abbas et al. [2] proved the existence of common fixed points of family of multivalued generalized F-contraction mappings without using any commutativity condition in the setup of partially ordered metric space.

The objective of this paper is to introduce a new concept of generalized orthogonal F-Suzuki contraction of a family of multivalued mappings in the setup of orthogonal b-metric space which generalizes some well-known results in the literature, especially [2]. In this paper, we present an improvement and generalization of the main results in the existing literature (see [3, 16, 27, 29]).

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of natural numbers, the set of nonnegative integer numbers, the set of real numbers and the set of positive real numbers, respectively.

Consistent with [2, 9, 12, 15, 28], the following definitions and results will be needed in the sequel.

Definition 1.1 ([7]). Let X be a nonempty set and $b \geq 1$. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, y) \leq b\,[d(x, z) + d(z, y)]$.

Then, the pair (X, d) is called a b-metric space with the coefficient b.

It is an obvious fact that a metric space is also a b-metric space with constant $b = 1$, but the converse is not generally true. To support this fact, we have the following example.

Example 1.1 ([12])**.** Consider the set $X = [0,1]$ endowed with the function $d : X \times X \to [0,\infty)$ defined by $d(x,y) = |x-y|^2$ for all $x,y \in X$. Clearly, (X,d) is a b-metric space with $b = 2$ but it is not a metric space.

Definition 1.2 ([10])**.** Let (X,d) be a b-metric space with constant b. The following notions are natural deductions from their metric counterparts.

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ in X converges if and only if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called a Cauchy sequence if and only if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $m,n \geq n_\varepsilon$. In this case, we write $\lim_{n,m\to\infty} d(x_n,x_m) = 0$.

(iii) A b-metric space (X,d) with constant b is said to be complete if and only if each Cauchy sequence in X converges to some $x \in X$.

Remark 1.2 ([10]). Notice that in a b-metric space (X,d) the following statements hold:

(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy;
(iii) in general, a b-metric is not continuous;
(iv) in general, a b-metric does not induce a topology on X.

Definition 1.3 ([6])**.** Let (X,d_X) and (Y,d_Y) be b-metric spaces; a mapping $f : X \to Y$ is called:

(i) continuous at a point $x \in X$, if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} d_X(x_n,x) = 0$, then $\lim_{n\to\infty} d_Y(f(x_n),f(x)) = 0$.

(ii) continuous on X, if it is continuous at each point $x \in X$.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.3 ([4])**.** *Let (X,d) be a b-metric space with $b \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x,y, respectively. Then we have*

$$\frac{1}{b^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq b^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{b}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq bd(x,z).$$

In 1922, Banach [8] proved Banach fixed point theorem as follows:

Theorem 1.4 ([8])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping, that is, there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Then, we have the following assertions hold*: (i) *T has a unique fixed point*; (ii) *for each $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ converges to the fixed point of T.*

In 2004, Ran and Reurings [26] extended the Banach's fixed point theorem to the setting of partially ordered sets as follows:

Theorem 1.5 ([26]). *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a continuous and monotone mapping such that the following conditions hold*: (i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \succeq y$*; (ii) *there exists $x_0 \preceq T(x_0)$ or $x_0 \succeq T(x_0)$. Then, T has a unique fixed point in X. Moreover, the Picard sequence $\{T^n x_0\}$ converges to the fixed point of T.*

After that, Nieto and Rodríguez [21] showed that Theorem 1.5 is still valid for T not necessarily continuous as follows:

Theorem 1.6 ([21]). *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a monotone nondecreasing mapping such that the following conditions hold*: (i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \succeq y$*; (ii) *there exists $x_0 \preceq T(x_0)$*. (iii) *if $\{x_n\} \subseteq X$ is a nondecreasing sequence such that $x_n \to x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then, T has a unique fixed point in X. Moreover, the Picard sequence $\{T^n x_0\}$ converges to the fixed point of T.*

Definition 1.4 ([31]). Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ such that:

- $(F1)$ F is strictly increasing, i.e., for all $a, b \in \mathbb{R}_+$ such that $F(a) < F(b)$ whenever $a < b$;
- $(F2)$ for each sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;
- $(F3)$ there exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Now let us review definitions of F-contraction mappings and some results on F-contraction mappings, related to the existing literature.

Definition 1.5 ([31]). Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Wardowski [31] gave a generalization of Banach contraction principle as follows.

Theorem 1.7 ([31]). *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction mapping. Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.*

In 2014, Wardowski and Dung [32] introduced the notion of an F-weak contraction and proved a related fixed point theorem as follows.

Definition 1.6 ([32])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(T(x), T(y)) \leq F(M(x, y)),$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Theorem 1.8 ([32])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-weak contraction mapping. If T or F is continuous, then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.*

In 2014, Piri and Kumam [25] described a large class of functions by replacing the condition $(F3)$ in the definition of an F-contraction introduced by Wardowski [31] with the following one:

$(F3)'$ F is continuous on $\mathbb{R}_+$.

They denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions $(F1)$, $(F2)$, and $(F3')$. Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

Theorem 1.9 ([25])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

Theorem 1.10 ([25])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

Recently, Gordji et al. [15] introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with this new type of orthogonality.

Definition 1.7. Let $X \neq \emptyset$ and $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 \in X \left[(\forall x \in X, x \perp x_0) \text{ or } (\forall x \in X, x_0 \perp x)\right],$$

then it is called an orthogonal set (briefly O-set) and x_0 is called an orthogonal element. We denote this O-set by $(X, \perp)$.

Example 1.11 ([15])**.** (i) Let X be the set of all people in the world. Define the binary relation $\perp$ on X by $x \perp y$ if x can give blood to y. If x_0 is a person such that his (her) blood type is O^-, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an O-set. In this O-set, x_0 (in definition) is not unique. Note that x_0 may be a person with blood type AB+. In this case, we have $y \perp x_0$ for all $y \in X$.

(ii) In graph theory, a wheel graph W_n is a graph with n vertices for each $n \geq 4$, formed by connecting a single vertex to all vertices of an $(n-1)$-cycle. Let X be the set of all vertices of W_n for each $n \geq 4$. Define $a \perp b$ if there is a connection from a to b. Then $(X, \perp)$ is an O-set.

(iii) Let $X = Z$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.

Definition 1.8 ([15])**.** Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n)\,.$$

Definition 1.9 ([15])**.** The triplet $(X, \perp, d)$ is called an orthogonal metric space if $(X, \perp)$ is an O-set and (X, d) is a metric space.

Definition 1.10 ([15])**.** Let $(X, \perp, d)$ be an orthogonal metric space. Then a mapping $T : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $x \in X$ if for each O-sequence $\{x_n\} \subset X$ with $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$. Also, T is said to be $\perp$-continuous on X if T is $\perp$-continuous in each $x \in X$.

Note that every continuous mapping is $\perp$-continuous, but the converse is not true.

Definition 1.11 ([15])**.** Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $T(x) \perp T(y)$ if $x \perp y$. Also, $T : X \to X$ is said to be weakly $\perp$-preserving if $T(x) \perp T(y)$ or $T(y) \perp T(x)$ if $x \perp y$.

Definition 1.12 ([15])**.** Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

Also, note that every complete metric space is O-complete and the converse is not true.

In 2017, Gordji et al. [15] prove the following theorem:

Theorem 1.12. *Let $(X, \perp, d)$ be an O-complete metric space. Suppose that $T : X \to X$ is a $\perp$-continuous mapping such that the following conditions hold*: (i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \perp y$*; (ii) *T is $\perp$-preserving. Then T has a unique fixed point $u \in X$ such that for each $x \in X$, the sequence $\{T^n x\}$ converges to u.*

For a nonempty set X, let $P(X)$ and $CL(X)$ be the family of all nonempty and nonempty and closed subsets of X, respectively.

A point $u \in X$ is a fixed point of a multivalued mapping $T : X \to P(X)$ if and only if $u \in Tu$. The set of all fixed points of multivalued mapping T is denoted by $Fix(T)$. The idea of common fixed point theorems for a family of multivalued generalized F-contraction mappings without using any commutativity condition in the setup of partially ordered metric spaces is due to Abbas et al. [2].

Definition 1.13 ([2])**.** Let $(X, \preceq)$ be a partially ordered set. Define

$$\triangle_1 = \{(x, y) \in X \times X : x \preceq y\}$$

and

$$\triangle_2 = \{(x, y) \in X \times X : x \prec y \text{ or } y \prec x\}.$$

That is, $\triangle_2$ is the set of all comparable elements of X.

A subset $\mathcal{K}$ of a partially ordered set X is said to be well ordered if every two elements of $\mathcal{K}$ are comparable. Recently, Abbas et al. [2] introduced F_j-contraction family for $j = 1$ and $j = 2$ and then proved the existence of common fixed point of such contractions.

Definition 1.14 ([2])**.** Let $\{T_i\}_{i=1}^m$ be a family of mappings such that $T_i : X \to CL(X)$ for each $i \in \{1, 2, \dots, m\}$ and $T_{m+1} = T_1$. The set $\{T_i\}_{i=1}^m$ is said to be

1) an F_1-contraction family, whenever for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \dots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that the following condition holds

$$\tau(U(x, y; u_x, u_y)) + F(d(u_x, u_y)) \le F(U(x, y; u_x, u_y)), \tag{1.1}$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s\to t^+} \tau(s) \ge 0$ for all $t \ge 0$ and

$$U(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{d(x,u_y)+d(y,u_x)}{2}\right\}.$$

2) an F_2-contraction family, whenever for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \dots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that the following condition holds

$$\tau(V(x, y; u_x, u_y)) + F(d(u_x, u_y)) \le F(V(x, y; u_x, u_y)), \tag{1.2}$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s\to t^+} \tau(s) \ge 0$ for all $t \ge 0$ and

$$V(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$

for $\alpha, \beta, \gamma, \delta_1, \delta_2 \ge 0$, $\delta_1 \le \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \le 1$.

Remember that $T : X \to CL(X)$ is said to be upper semi-continuous, if for $x_n \in X$ and $y_n \in Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, then we have $y_0 \in Tx_0$.

Theorem 1.13 ([2]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m$ an F_1-contraction family of multivalued mappings. Then, the following hold*:

(i) $Fix(T_i) \neq \emptyset$ *for any* $i \in \{1, 2, \ldots, m\}$ *if and only if* $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$.

(ii) $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ *provided that there exists some* $x_0 \in X$ *such that* $\{x_0\} \preceq_1 T_k(x_0)$ *for any* $k \in \{1, 2, \ldots, m\}$ *and any one of* T_i *is upper semi-continuous for* $i \in \{1, 2, \ldots, m\}$.

(iii) $\cap_{i=1}^m Fix(T_i)$ *is well ordered if and only if* $\cap_{i=1}^m Fix(T_i)$ *is a singleton set.*

Theorem 1.14 ([2]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m$ an F_2-contraction family of multivalued mappings. Then, the following hold*:

(i) $Fix(T_i) \neq \emptyset$ *for any* $i \in \{1, 2, \ldots, m\}$ *if and only if* $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$.

(ii) $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ *provided that there exists some* $x_0 \in X$ *such that* $\{x_0\} \preceq_1 T_k(x_0)$ *for any* $k \in \{1, 2, \ldots, m\}$ *and any one of* T_i *is upper semi-continuous for* $i \in \{1, 2, \ldots, m\}$.

(iii) $\cap_{i=1}^m Fix(T_i)$ *is well ordered if and only if* $\cap_{i=1}^m Fix(T_i)$ *is a singleton set.*

2. Existence of common fixed point results

In this section we state and prove our main results. Throughout the paper, we assume that b-metric is continuous and define $\frac{0}{0} := 0$.

We use $\mathcal{F}_T$ to denote the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions $(F1)$ and $(F3')$ and Φ to denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that φ is lower semi-continuous and $\varphi(t) = 0$ if and only if $t = 0$.

Inspired by the notion of F_j-contraction family of multivalued mappings for $j = 1$ and $j = 2$, defined by Abbas et al. [2], we introduce the notion of generalized orthogonal F-Suzuki type contraction family of multivalued mappings and prove the existence of some common fixed point theorems for a family of multivalued generalized orthogonal F-Suzuki type contraction mappings without using any commutativity condition in the framework of orthogonal b-metric spaces as follows.

Definition 2.1. Let $(X, \perp)$ be an orthogonal set (briefly, O-set), Y and Z two nonempty subsets of $(X, \perp)$. We say that $Y \perp_1 Z$, whenever for every $y \in Y$, there exists $z \in Z$ such that $y \perp z$.

Definition 2.2. Let $(X, \perp)$ be an orthogonal set. Define

$$^{\perp}\triangle_1 = \{(x, y) \in X \times X : x \perp y\}$$

and

$$^{\perp}\triangle_2 = \{(x, y) \in X \times X : x \perp y \text{ or } y \perp x\}.$$

That is, ${}^{\perp}\triangle_2$ is the set of all orthogonable elements of X.

We define that a subset A of an orthogonal set X is said to be well orthogonal if $(x,y) \in {}^{\perp}\triangle_2$ for every $x, y \in A$.

Definition 2.3. Let $(X, \perp)$ be an orthogonal set and $T_i : X \to P(X)$ multivalued mappings for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. The family of mappings $\{T_i\}_{i=1}^m$ is said to be orthogonal preserving (briefly, $\perp$-preserving) if for any $x, y \in X$ with $(x,y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$.

Definition 2.4. Let $(X, d, \perp)$ be an orthogonal b-metric space. Let $T_i : X \to CL(X)$ for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. The set $\{T_i\}_{i=1}^m$ is said to be a generalized orthogonal F-Suzuki contraction family if for any $x, y \in X$ with $x \neq y$, $(x,y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds

$$\begin{aligned} &\frac{1}{2b} d(x, u_x) < d(x,y) \\ &\Rightarrow F(b^5 d(u_x, u_y)) \leq F(M_T(x, y; u_x, u_y)) - \varphi(M_T(x, y; u_x, u_y)) \end{aligned} \tag{2.1}$$

in which $\varphi \in \Phi$ and

$$\begin{aligned} M_T(x, y; u_x, u_y) = \max\{ &d(x,y), d(x,u_x), d(y,u_y), \tfrac{1}{2b} d(x,u_y), \\ &d(y,u_x), (d(x,y))^{-1} d(x,u_x) d(y,u_y)\}. \end{aligned}$$

Remark 2.1. (i) If $\perp \equiv \preceq$ such that $(X, \preceq)$ is a partial ordered set and $\varphi = \tau$, then Equation (2.1) of Definition 2.4 reduced to Equation (1.1) of Definition 1.14 in [2] since $U(x, y; u_x, u_y) \leq M_T(x, y; u_x, u_y)$.

(ii) If $\perp \equiv \preceq$ such that $(X, \preceq)$ is a partial ordered set and $\varphi = \tau$, then Equation (2.1) of Definition 2.4 reduced to Equation (1.2) of Definition 1.14 in [2] since $V(x, y; u_x, u_y) \leq M_T(x, y; u_x, u_y)$ as $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

From this remark, we deduce that generalized orthogonal F-Suzuki contraction family of multivalued mappings comprise F_1-contraction family and F_2-contraction family of multivalued mappings.

A multivalued mapping $T : X \to CL(X)$ is said to be orthogonal upper semi-continuous, if for orthogonal sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, then we have $y_0 \in Tx_0$.

Now we state the main result of this section.

Theorem 2.2. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be a generalized orthogonal F-Suzuki contraction family and $\perp$-preserving. Then the following hold*:

(i) *$Fix(T_i) \neq \emptyset$ for any $i \in \{1, 2, \ldots, m\}$ if and only if $Fix(T_1) = Fix(T_2)$ $= \cdots = Fix(T_m) \neq \emptyset$.*

(ii) $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ *provided that any one of* T_i *is orthogonal upper semi-continuous for* $i \in \{1, 2, \ldots, m\}$.

(iii) $\cap_{i=1}^{m} Fix(T_i)$ *is well orthogonal if and only if* $\cap_{i=1}^{m} Fix(T_i)$ *is a singleton set.*

Proof. At first, to prove (i): Suppose $Fix(T_k) \neq \emptyset$ for any $k \in \{1, 2, \ldots, m\}$, that is, there exists $w \in T_k(w)$ for any $k \in \{1, 2, \ldots, m\}$. If $w \notin T_{k+1}(w)$, then there exists a $z \in T_{k+1}(w)$ with $(w, z) \in {}^{\perp}\triangle_2$ and since $d(w, w) = 0$, then (2.1) holds such that

$$F(d(w, z)) \leq F(M_T(w, w; w, z)) - \varphi(M_T(w, w; w, z))$$

holds, where

$$\begin{aligned} M_T(w, w; w, z) = \ & \max\{d(w, w), d(w, w), d(w, z), \tfrac{1}{2b} d(w, z), d(w, w), \\ & (d(w, w))^{-1} d(w, w) d(w, z)\} = d(w, z). \end{aligned}$$

Thus, we have

$$F(d(w, z)) \leq F(d(w, z)) - \varphi(d(w, z)),$$

which is a contradiction as $\varphi(d(w, z)) > 0$. Thus $w = z$, that is, $w \in T_{k+1}(w)$ and $Fix(T_k) \subseteq Fix(T_{k+1})$. Similarly, one can show that $Fix(T_{k+1}) \subseteq Fix(T_{k+2})$. Arguing this way, we obtain that $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$. The converse is obvious.

To prove (ii): Since $(X, \perp)$ is an O-set,

$$\exists x_0 \in X : (x_0, x) \in {}^{\perp}\triangle_2, \ \forall x \in X.$$

Due to the fact that $T_k(x)$ is a nonempty set for all $x \in X$ and for any $k \in \{1, 2, \ldots, m\}$, it follows that $\{x_0\} \perp_1 T_{k_0}(x_0)$ or $T_{k_0}(x_0) \perp_1 \{x_0\}$ for any $k_0 \in \{1, 2, \ldots, m\}$. If $x_0 \in T_{k_0}(x_0)$, then we deduce that x_0 is a fixed point of T_{k_0} and so by (i), we conclude the proof is finished. Thus, we assume that $x_0 \notin T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$. Then $d(x_0, T_{k_0}(x_0)) > 0$ since $T_{k_0}(x_0)$ is closed. For $i \in \{1, 2, \ldots, m\}$, $x_1 \in T_i(x_0)$, there exists $x_2 \in T_{i+1}(x_1)$ with $(x_1, x_2) \in {}^{\perp}\triangle_2$. Due to $d(x_0, x_1) > 0$, we have

$$\frac{1}{2b} d(x_0, x_1) < d(x_0, x_1).$$

Thus, by assumption that $\{T_i\}_{i=1}^{m}$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$F(d(x_1, x_2)) \leq F(M_T(x_0, x_1; x_1, x_2)) - \varphi(M_T(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M_T(x_0, x_1; x_1, x_2) = \ & \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \tfrac{1}{2b} d(x_0, x_2), d(x_1, x_1), \\ & (d(x_0, x_1))^{-1} d(x_0, x_1) d(x_1, x_2)\} \\ \leq \ & \max\left\{d(x_0, x_1), d(x_1, x_2), \tfrac{d(x_0, x_1) + d(x_1, x_2)}{2}\right\} \\ = \ & \max\left\{d(x_0, x_1), d(x_1, x_2)\right\}. \end{aligned}$$

If $\max\{d(x_0,x_1),d(x_1,x_2)\} = d(x_1,x_2)$, then we have

$$F(d(x_1,x_2)) \le F(d(x_1,x_2)) - \varphi(d(x_1,x_2)),$$

which is a contradiction (from $d(x_0,x_1) > 0$ and property of φ, we have $\varphi(d(x_1,x_2)) > 0$). Thus, we conclude that

$$F(d(x_1,x_2)) \le F(d(x_0,x_1)) - \varphi(d(x_0,x_1)).$$

Similarly, for the point $x_2 \in T_{i+1}(x_1)$, there exists $x_3 \in T_{i+2}(x_2)$ for $i \in \{1,2,\dots,m\}$ with $(x_2,x_3) \in {}^{\perp}\triangle_2$. Due to $d(x_1,x_2) > 0$, we have

$$\frac{1}{2b}d(x_1,x_2) < d(x_1,x_2).$$

Thus, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$F(d(x_2,x_3)) \le F(M_T(x_1,x_2;x_2,x_3)) - \varphi(M_T(x_1,x_2;x_2,x_3)),$$

where

$$\begin{aligned} M_T(x_1,x_2;x_2,x_3) = {} & \max\{d(x_1,x_2), d(x_1,x_2), d(x_2,x_3), \tfrac{1}{2b}d(x_1,x_3), d(x_2,x_2), \\ & (d(x_1,x_2))^{-1}d(x_1,x_2)d(x_2,x_3)\} \\ \le {} & \max\left\{d(x_1,x_2), d(x_2,x_3), \tfrac{d(x_1,x_2)+d(x_2,x_3)}{2}\right\} \\ = {} & \max\{d(x_1,x_2), d(x_2,x_3)\}. \end{aligned}$$

If $\max\{d(x_1,x_2),d(x_2,x_3)\} = d(x_2,x_3)$, then we have

$$F(d(x_2,x_3)) \le F(d(x_2,x_3)) - \varphi(d(x_2,x_3)),$$

which is a contradiction (from $d(x_1,x_2) > 0$ and property of φ, we have $\varphi(d(x_2,x_3)) > 0$). Thus, we conclude that

$$F(d(x_2,x_3)) \le F(d(x_1,x_2)) - \varphi(d(x_1,x_2)).$$

Continuing this approach, for $x_{2n} \in T_i(x_{2n-1})$, there exists $x_{2n+1} \in T_{i+1}(x_{2n})$ for $i \in \{1,2,\dots,m\}$ with $(x_{2n},x_{2n+1}) \in {}^{\perp}\triangle_2$. Due to $d(x_{2n-1},x_{2n}) > 0$, we have

$$\frac{1}{2b}d(x_{2n-1},x_{2n}) < d(x_{2n-1},x_{2n}).$$

Thus, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$\begin{aligned} & F(d(x_{2n},x_{2n+1})) \\ & \le F(M_T(x_{2n-1},x_{2n};x_{2n},x_{2n+1})) - \varphi(M_T(x_{2n-1},x_{2n};x_{2n},x_{2n+1})), \end{aligned}$$

that is,

$$F(d(x_{2n},x_{2n+1})) \le F(d(x_{2n-1},x_{2n})) - \varphi(d(x_{2n-1},x_{2n})).$$

Similarly, for $x_{2n+1} \in T_{i+1}(x_{2n})$, there exists $x_{2n+2} \in T_{i+2}(x_{2n+1})$ for $i \in \{1, 2, \dots, m\}$ with $(x_{2n}, x_{2n+1}) \in {}^{\perp}\triangle_2$. Due to $d(x_{2n}, x_{2n+1}) > 0$, we have

$$\frac{1}{2b} d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1}).$$

Thus, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})).$$

Hence, we get a sequence $\{x_n\}$ in X such that $x_n \in T_i(x_{n-1})$ and $x_{n+1} \in T_{i+1}(x_n)$ with $(x_n, x_{n+1}) \in {}^{\perp}\triangle_2$ which implies that $\{x_n\}$ is an orthogonal sequence for all $n \in \mathbb{N}_0$.

If $x_k \in T_i(x_k)$ for some $k \in \mathbb{N}_0$ and all $i \in \{1, 2, \dots, m\}$, then $Fix(T_i) \neq \emptyset$, which completes the proof. So, we may assume that $x_n \notin T_i(x_n)$ for all $n \in \mathbb{N}_0$. Then due to $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$, we have

$$\frac{1}{2b} d(x_{n-1}, x_n) < d(x_{n-1}, x_n).$$

Thus, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$\begin{aligned} &F(d(x_n, x_{n+1})) \\ &\leq F(M_T(x_{n-1}, x_n; x_n, x_{n+1})) - \varphi(M_T(x_{n-1}, x_n; x_n, x_{n+1})). \end{aligned}$$

Since

$$\begin{aligned} &M_T(x_{n-1}, x_n; x_n, x_{n+1}) \\ =\ &\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{1}{2b} d(x_{n-1}, x_{n+1}), \\ &\qquad d(x_n, x_n), (d(x_{n-1}, x_n))^{-1} d(x_{n-1}, x_n) d(x_n, x_{n+1})\} \\ \leq\ &\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\ =\ &\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}, \end{aligned}$$

we obtain

$$\begin{aligned} F(d(x_n, x_{n+1})) \leq\ &F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\ &- \varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned} \tag{2.2}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, so (2.2) becomes

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})),$$

which is a contradiction (from $d(x_n, x_{n+1}) > 0$ and the property of φ, we have $\varphi(d(x_n, x_{n+1})) > 0$). Thus, we conclude that

$$\begin{aligned} F(d(x_n, x_{n+1})) \leq\ &F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\ <\ &F(d(x_{n-1}, x_n)). \end{aligned} \tag{2.3}$$

It follows from (2.3) and $(F1)$ that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Therefore, $\{d(x_n, x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers which is bounded from below. Thus, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta.$$

Letting $n \to \infty$ in (2.3) and by applying $(F3')$ and property of φ, we obtain

$$F(\delta) \leq F(\delta) - \varphi(\delta).$$

This implies that $\varphi(\delta) = 0$ and thus $\delta = 0$. Consequently, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

Now, we claim that $\{x_n\}$ is a Cauchy orthogonal sequence, that is,

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Suppose on the contrary, we assume that there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$, where $n(k)$ is the smallest integer with $n(k) > m(k) \geq k$ for all $k \in \mathbb{N}$ satisfying

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \text{ and } d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.5}$$

Using (2.5) and the triangle inequality in b-metric space, we have that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq bd(x_{n(k)}, x_{n(k)-1}) + bd(x_{n(k)-1}, x_{m(k)})\\ &< bd(x_{n(k)}, x_{n(k)-1}) + b\varepsilon.\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.4), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq b\varepsilon. \tag{2.6}$$

From the triangle inequality, we have

$$\begin{aligned}\varepsilon &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq bd(x_{m(k)}, x_{n(k)+1}) + bd(x_{n(k)+1}, x_{n(k)})\\ &\leq b^2 d(x_{m(k)}, x_{n(k)}) + b^2 d(x_{n(k)}, x_{n(k)+1}) + bd(x_{n(k)+1}, x_{n(k)})\\ &= b^2 d(x_{m(k)}, x_{n(k)}) + (b^2 + b) d(x_{n(k)}, x_{n(k)+1}).\end{aligned}$$

Thus, from (2.4) and (2.6), we obtain

$$\frac{\varepsilon}{b} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq b^2\varepsilon. \tag{2.7}$$

Again, from the triangle inequality, we have

$$\begin{aligned}\varepsilon &\leq d(x_{n(k)}, x_{m(k)})\\ &\leq bd(x_{n(k)}, x_{m(k)+1}) + bd(x_{m(k)+1}, x_{m(k)})\\ &\leq b^2 d(x_{n(k)}, x_{m(k)}) + b^2 d(x_{m(k)}, x_{m(k)+1}) + bd(x_{m(k)+1}, x_{m(k)})\\ &= b^2 d(x_{n(k)}, x_{m(k)}) + (b^2 + b) d(x_{m(k)}, x_{m(k)+1}).\end{aligned}$$

Thus, from (2.4) and (2.6), we obtain

$$\frac{\varepsilon}{b} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq b^2\varepsilon. \tag{2.8}$$

From (2.8) and the inequality

$$d(x_{m(k)+1}, x_{n(k)}) \leq bd(x_{m(k)+1}, x_{n(k)+1}) + bd(x_{n(k)+1}, x_{n(k)}),$$

we have

$$\frac{\varepsilon}{b^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{2.9}$$

From (2.6) and the inequality

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) &\leq b[d(x_{m(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})] \\ &\leq b^2[d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)})] \\ &\quad + bd(x_{n(k)}, x_{n(k)+1}), \end{aligned}$$

we get

$$\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq b^3\varepsilon. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\frac{\varepsilon}{b^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq b^3\varepsilon. \tag{2.11}$$

On the other hand, from (2.4) and (2.5), we can choose a positive integer $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2b} d(x_{n(k)}, x_{n(k)+1}) < \frac{1}{2b}\varepsilon < d(x_{n(k)}, x_{m(k)}),\ \forall n \geq n_1.$$

Since $\{x_n\}$ is an orthogonal sequence, hence for all $\forall n \geq n_1$ $(x_{n(k)}, x_{m(k)}) \in {}^{\perp}\triangle_2$. Thus, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we have

$$\begin{aligned} F(d(x_{n(k)+1}, x_{m(k)+1})) &\leq F(M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1})) \\ &\quad - \varphi(M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1})). \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\leq M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1}) \\ &= \max\Big\{ d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}), \\ &\qquad \tfrac{1}{2b} d(x_{n(k)}, x_{m(k)+1}), d(x_{m(k)}, x_{n(k)+1}), \\ &\qquad (d(x_{n(k)}, x_{m(k)}))^{-1} d(x_{n(k)}, x_{n(k)+1}) d(x_{m(k)}, x_{m(k)+1}) \Big\}, \end{aligned}$$

taking the limit supremum as $k \to \infty$ on each side of the above inequality and using (2.6), (2.7), (2.8) and (2.11), we have

$$\varepsilon \leq \limsup_{k\to\infty} M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1}) < b^3\varepsilon. \tag{2.13}$$

Similarly, we can obtain

$$\varepsilon \leq \liminf_{k\to\infty} M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1}) < b^3\varepsilon. \tag{2.14}$$

Taking the limit supremum as $k \to \infty$ in (2.12) and using $(F3')$, property of φ, (2.11), (2.13) and (2.14), we get

$$\begin{aligned} F(b^3\varepsilon) = F(b^5\frac{\varepsilon}{b^2}) &\leq F(b^5 \limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1})) \\ &\leq F(\limsup_{k\to\infty} M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1})) \\ &\quad - \varphi(\limsup_{k\to\infty} M_T(x_{n(k)}, x_{m(k)}; x_{n(k)+1}, x_{m(k)+1})) \\ &< F(b^3\varepsilon) - \varphi(\varepsilon), \end{aligned}$$

which is a contradiction with $\varepsilon > 0$, and it follows that $\{x_n\}$ is a Cauchy orthogonal sequence in X. On the account of O-completeness of $(X, d, \perp)$, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

Now, if T_i is orthogonal upper semi-continuous for any of $i \in \{1, 2, \ldots, m\}$, then $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $\lim_{n\to\infty} d(x_{2n}, x^*) = 0$ and $\lim_{n\to\infty} d(x_{2n+1}, x^*) = 0$ imply that $x^* \in T_i(x^*)$. Using (i), we obtain $x^* \in T_1(x^*) = T_2(x^*) = \cdots = T_m(x^*)$.

Finally, to prove (iii): Assume $\cap_{i=1}^m Fix(T_i)$ is well orthogonal. Suppose that there exist w and z such that $w, z \in \cap_{i=1}^m Fix(T_i)$ but $w \neq z$. Since $(w, z) \in {}^\perp\triangle_2$ and $\frac{1}{2b}d(w, w) < d(w, z)$, by assumption that $\{T_i\}_{i=1}^m$ is a generalized orthogonal F-Suzuki contraction family of multivalued mappings, we obtain that

$$\begin{aligned} &F(d(w, z)) \\ &\leq F(M_T(w, z; w, z)) - \varphi(M_T(w, z; w, z)) \\ &= F\big(\max\big\{d(w, z), d(w, w), d(z, z), \tfrac{1}{2b}d(w, z), d(z, w), (d(w, z))^{-1}d(w, w)d(z, z)\big\}\big) \\ &\quad - \varphi\big(\max\big\{d(w, z), d(w, w), d(z, z), \tfrac{1}{2b}d(w, z), d(z, w), (d(w, z))^{-1}d(w, w)d(z, z)\big\}\big) \\ &= F(d(w, z)) - \varphi(d(w, z)), \end{aligned}$$

and that implies $F(d(w, z)) \leq F(d(w, z)) - \varphi(d(w, z))$, which is a contradiction as $\varphi(d(w, z)) > 0$. Hence, $w = z$, that is, $\cap_{i=1}^m Fix(T_i)$ is a singleton set. The converse is straightforward. □

Corollary 2.3. *Let $(X, d, \perp)$ be an O-complete b-metric space. Let $T_1, T_2 : X \to CL(X)$. Suppose that for any $x, y \in X$ with $x \neq y$, $(x, y) \in {}^\perp\triangle_1$ and*

$u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i \neq j$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds

$$\begin{aligned} &\frac{1}{2b} d(x, u_x) < d(x, y) \\ &\Rightarrow F(b^5 d(u_x, u_y)) \leq F(M_T(x, y; u_x, u_y)) - \varphi(M_T(x, y; u_x, u_y)) \end{aligned} \tag{2.15}$$

in which $\varphi \in \Phi$, where $i, j \in \{1, 2\}$ and

$$\begin{aligned} M_T(x, y; u_x, u_y) = \ &\max\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{1}{2b} d(x, u_y), \\ &d(y, u_x), (d(x, y))^{-1} d(x, u_x) d(y, u_y)\}. \end{aligned}$$

Then the following hold:

(i) *$Fix(T_i) \neq \emptyset$ for any $i \in \{1, 2\}$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.*
(ii) *$Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that either T_1 or T_2 is orthogonal upper semi-continuous.*
(iii) *$Fix(T_1) \cap Fix(T_2)$ is well orthogonal if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton set.*

Example 2.4. Let $X = \{\frac{1}{2^n}\}_{n=0}^{\infty}$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Define the binary relation $\perp$ on X by $x \perp y$ if and only if $\frac{y}{x} \in \mathbb{N}$. Then $(X, d, \perp)$ is an O-complete b-metric space with $b = 2$. Define the mappings $T_1, T_2 : X \to CL(X)$ by

$$T_1(x) = \begin{cases} \{\frac{1}{4^n}\}, & \text{if } x = \frac{1}{2^n},\ n \in \mathbb{N}, \\ \{1, \frac{1}{2}\}, & \text{if } x = 1. \end{cases}$$

$$T_2(x) = \begin{cases} \{\frac{1}{8^n}\}, & \text{if } x = \frac{1}{2^n},\ n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 1. \end{cases}$$

Now we can easily show that T is $\perp$-preserving, for $\frac{1}{2^n}, \frac{1}{2^m} \in X$ with $\left(\frac{1}{2^n}, \frac{1}{2^m}\right) \in {}^{\perp}\triangle_1$ for $n \geq m$ and $n, m \in \mathbb{N}_0$, then for $\frac{1}{4^n} \in T_1(\frac{1}{2^n})$, there exists $\frac{1}{8^m} \in T_2(\frac{1}{2^m})$ with $(\frac{1}{4^n}, \frac{1}{8^m}) \in {}^{\perp}\triangle_2$ and also for $\frac{1}{2} \in T_1(1)$, there exists $\frac{1}{8^m} \in T_2(\frac{1}{2^m})$ with $(\frac{1}{2}, \frac{1}{8^m}) \in {}^{\perp}\triangle_2$ such that condition (2.15) holds for all $F \in \mathcal{F}_T$ and $\varphi \in \Phi$.

Thus, all the assumptions of Corollary 2.3 are satisfied. Hence, all the results of Corollary 2.3 hold. Moreover, $Fix(T_1) = Fix(T_2) = \{1\}$.

The following theorems can be obtained easily by repeating the steps in the proof of Theorem 2.2.

Theorem 2.5. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\frac{1}{2b} d(x, u_x) < d(x, y) \Rightarrow \tau + F(b^5 d(u_x, u_y)) \leq F(M_T(x, y; u_x, u_y)),$$

where $\tau > 0$. Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Taking $\varphi(t) = \tau$ for all $t > 0$ in (2.1), so the proof immediately follows from Theorem 2.2. $\square$

Theorem 2.6. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\begin{aligned}(2.16) \qquad & \frac{1}{2b} d(x, u_x) < d(x, y) \\ & \Rightarrow F(b^5 d(u_x, u_y)) \leq F(W(x, y; u_x, u_y)) - \varphi(W(x, y; u_x, u_y)),\end{aligned}$$

where $\varphi \in \Phi$ and

$$W(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{d(x,u_y)+d(y,u_x)}{2b}\right\}.$$

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since $W(x, y; u_x, u_y) \leq M_T(x, y; u_x, u_y)$ and from $(F1)$, so the proof immediately follows from Theorem 2.2. $\square$

Theorem 2.7. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\begin{aligned}(2.17) \qquad & \frac{1}{2b} d(x, u_x) < d(x, y) \\ & \Rightarrow F(b^5 d(u_x, u_y)) \leq F(V(x, y; u_x, u_y)) - \varphi(V(x, y; u_x, u_y)),\end{aligned}$$

where $\varphi \in \Phi$ and

$$V(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$

for $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since $V(x, y; u_x, u_y) \leq M_T(x, y; u_x, u_y)$ and from $(F1)$, so the proof immediately follows from Theorem 2.2. $\square$

Corollary 2.8. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for*

$i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds

$$\tau(W(x, y; u_x, u_y)) + F(d(u_x, u_y)) \leq F(W(x, y; u_x, u_y)),$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and

$$W(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{d(x,u_y)+d(y,u_x)}{2b}\right\}.$$

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Taking $\varphi(t) = \tau(t)$ for all $t > 0$ in (2.16), so the proof immediately follows from Theorem 2.6. $\square$

Corollary 2.9. *Let $(X, d, \perp)$ be an O-complete b-metric space with constant $b \geq 1$ and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\tau(V(x, y; u_x, u_y)) + F(d(u_x, u_y)) \leq F(V(x, y; u_x, u_y)),$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and

$$V(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$

for $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Taking $\varphi(t) = \tau(t)$ for all $t > 0$ in (2.17), so the proof immediately follows from Theorem 2.7. $\square$

Since a b-metric space is a metric space when $b = 1$, so we obtain the following theorems.

Theorem 2.10. *Let $(X, d, \perp)$ be an O-complete metric space and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be a generalized orthogonal F-Suzuki contraction family and $\perp$-preserving. Then the following hold:*

- (i) *$Fix(T_i) \neq \emptyset$ for any $i \in \{1, 2, \ldots, m\}$ if and only if $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$.*
- (ii) *$Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ provided that any one of T_i is orthogonal upper semi-continuous for $i \in \{1, 2, \ldots, m\}$.*
- (iii) *$\cap_{i=1}^{m} Fix(T_i)$ is well orthogonal if and only if $\cap_{i=1}^{m} Fix(T_i)$ is a singleton set.*

Proof. Since any metric space is a b-metric space with constant $b = 1$, so from Theorem 2.2 the proof is complete. $\square$

Corollary 2.11. *Let $(X, d, \perp)$ be an O-complete metric space and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\begin{aligned}&\frac{1}{2}d(x, u_x) < d(x, y)\\ &\Rightarrow F(d(u_x, u_y)) \leq F(W(x, y; u_x, u_y)) - \varphi(W(x, y; u_x, u_y)),\end{aligned}$$

where $\varphi \in \Phi$ and

$$W(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{d(x,u_y)+d(y,u_x)}{2}\right\}.$$

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since any metric space is a b-metric space with constant $b = 1$, so from Theorem 2.6 the proof is complete. □

Corollary 2.12. *Let $(X, d, \perp)$ be an O-complete metric space and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\begin{aligned}&\frac{1}{2}d(x, u_x) < d(x, y)\\ &\Rightarrow F(d(u_x, u_y)) \leq F(V(x, y; u_x, u_y)) - \varphi(V(x, y; u_x, u_y)),\end{aligned}$$

where $\varphi \in \Phi$ and

$$V(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$

for $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since any metric space is a b-metric space with constant $b = 1$, so from Theorem 2.7 the proof is complete. □

Corollary 2.13. *Let $(X, d, \perp)$ be an O-complete metric space and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\tau(W(x, y; u_x, u_y)) + F(d(u_x, u_y)) \leq F(W(x, y; u_x, u_y)),$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and

$$W(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \tfrac{d(x,u_y)+d(y,u_x)}{2}\right\}.$$

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since any metric space is a b-metric space with constant $b = 1$, so from Corollary 2.8 the proof is complete. □

Corollary 2.14. *Let $(X, d, \perp)$ be an O-complete metric space and an orthogonal element x_0. Let $T_i : X \to CL(X)$ be $\perp$-preserving for each $i \in \{1, 2, \dots, m\}$ and $T_{m+1} = T_1$. Suppose that for every $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \dots, m\}$ with $(u_x, u_y) \in {}^{\perp}\triangle_2$ and there exists $F \in \mathcal{F}_T$ such that the following condition holds*

$$\tau(V(x, y; u_x, u_y)) + F(d(u_x, u_y)) \leq F(V(x, y; u_x, u_y)),$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s\to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and

$$V(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$

for $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Then, the conclusions obtained in Theorem 2.2 remain true.

Proof. Since any metric space is a b-metric space with constant $b = 1$, so from Corollary 2.9 the proof is complete. □

Remark 2.15.

1) Theorem 2.2, Theorem 2.5, Theorem 2.6, Theorem 2.7 and Theorem 2.10 are new results in the existing literature.
2) Corollary 2.13 extends and generalizes the main results: Theorem 1 of Abbas et al. [2], Theorem 4.1 of Latif et al. [16], Theorem 3.4 of Rus et al. [27], Theorem 2.1 of Piri et al. [25], and Theorem 3.1 of Sgroi et al. [29].
3) Corollary 2.9 extends and generalizes main result of Theorem 14 of the Beg et al. [9], Theorem 3.3 of Sawangsup et al. [28].
4) Corollary 2.14 extends and generalizes the main results: Theorem 2 of Abbas et al. [2], Theorem 3.4 and Theorem 4.1 of Cosentino et al. [11], Theorem 3.4 of Rus et al. [27] and Theorem 3.4 of Sgroi et al. [29].
5) If we take $T_1 = T_2 = \cdots = T_m$ in generalized orthogonal F-Suzuki contraction family of multivalued mappings, then we obtain the fixed point results for generalized orthogonal F-Suzuki contraction of multivalued mappings which is also new in the literature.

All results in this paper may be stated with respect to single valued mappings in the setting of O-complete b-metric space. These results extend, unify and generalize the related results in the literature. For instance, we consider the following single valued results.

Corollary 2.16. *Let $(X, d, \perp)$ be O-complete b-metric space. Let $f_i : X \to X$ for $i \in \{1, 2\}$. Suppose that for any $x, y \in X$ with $x \neq y$, $(x, y) \in {}^{\perp}\triangle_1$ implies $(f_i(x), f_j(y)) \in {}^{\perp}\triangle_2$ for $i \neq j$ and there exists $F \in \mathcal{F}_T$ such that the following* *condition holds*

$$\begin{aligned}&\frac{1}{2b}d(x,f_i(x))<d(x,y)\\&\Rightarrow F(b^5d(f_i(x),f_j(y)))\leq F(M(x,y;f_i(x),f_j(y)))\\&\qquad\qquad\qquad\qquad\qquad-\varphi(M(x,y;f_i(x),f_j(y)))\end{aligned}$$

in which $\varphi\in\Phi$*, where* $i,j\in\{1,2\}$ *and*

$$\begin{aligned}M(x,y;f_i(x),f_j(y))=\ &\max\{d(x,y),d(x,f_i(x)),d(y,f_j(y)),\tfrac{1}{2b}d(x,f_j(y)),\\&d(y,f_i(x)),(d(x,y))^{-1}d(x,f_i(x))d(y,f_j(y))\}.\end{aligned}$$

Then the following hold:

(i) $Fix(f_i)\neq\emptyset$ *for any* $i\in\{1,2\}$ *if and only if* $Fix(f_1)=Fix(f_2)\neq\emptyset$.

(ii) $Fix(f_1)=Fix(f_2)\neq\emptyset$ *provided that either* f_1 *or* f_2 *is orthogonal upper semi-continuous.*

(iii) $Fix(f_1)\cap Fix(f_2)$ *is well orthogonal if and only if* $Fix(f_1)\cap Fix(f_2)$ *is a singleton set.*

Proof. Define $T_i:X\to CL(X)$ as $T_i(x)=\{f_i(x)\}$ for all $i\in\{1,2\}$. Note that T_i satisfy all conditions of Corollary 2.3. So the proof immediately follows from Corollary 2.3. □

Corollary 2.17. *Let* $(X,d,\perp)$ *be an O-complete metric space and an orthogonal element* x_0*. Let* $f:X\to X$ *be* $\perp$*-continuous. Suppose that for every* $x,y\in X$ *with* $x\neq y$*,* $(x,y)\in{}^{\perp}\triangle_1$*, then* $(fx,fy)\in{}^{\perp}\triangle_2$*, there exists* $F\in\mathcal{F}_T$ *and* $\tau>0$ *such that the following condition holds*

$$\frac{1}{2}d(x,fx)<d(x,y)\Rightarrow\tau+F(d(fx,fy))\leq F(W_f(x,y)),$$

where

$$W_f(x,y)=\max\left\{d(x,y),d(x,fx),d(y,fy),\tfrac{d(x,fy)+d(y,fx)}{2}\right\}.$$

Then f *has a fixed point.*

Proof. For $i\in\{1,2,\dots,m\}$ define $T_i:X\to CL(X)$ as $T_1(x)=T_2(x)=\cdots=T_m(x)=\{f(x)\}$. Note that T_i satisfy all conditions of Corollary 2.13. So the proof immediately follows from Corollary 2.13. □

3. Conclusion

A new notion of generalized orthogonal F-Suzuki contraction for a family of multivalued mappings in orthogonal b-metric space was introduced. This new notion generalizes some well-known results in the literature, especially [2]. We proved the existence of some common fixed point results for generalized orthogonal F-Suzuki contraction for a family of multivalued mappings in orthogonal b-metric space. We provided an example to illustrate and support these results. Our results are improvement and generalization of some related results in the existing literature (see [3], [16], [27] and [29]).

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