Commun. Korean Math. Soc. **37** (2022), No. 4, pp. 1259–1267
https://doi.org/10.4134/CKMS.c210333
pISSN: 1225-1763 / eISSN: 2234-3024

CIRCLE APPROXIMATION USING PARAMETRIC POLYNOMIAL CURVES OF HIGH DEGREE IN EXPLICIT FORM

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Abstract. In this paper we present a full circle approximation method using parametric polynomial curves with algebraic coefficients which are curvature continuous at both endpoints. Our method yields the n-th degree parametric polynomial curves which have a total number of $2n$ contacts with the full circle at both endpoints and the midpoint. The parametric polynomial approximants have algebraic coefficients involving rational numbers and radicals for degree higher than four. We obtain the exact Hausdorff distances between the circle and the approximation curves.

1. Introduction

Circle approximation using polynomial curve is a challenging task in CAGD (Computer Aided Geometric Design) or Geometric Modeling. Many works for circle approximation with different foci, e.g., best approximation, geometric continuity, explicit form, full circle approximation, have been developed in recent forty years.

de Boor et al. [3] presented the approximation method of planar curves using curvature continuous cubic Bézier curves. The approximation methods of circular arcs using quadratic and cubic Bézier curves having approximation order four and six, respectively, were proposed [4,7,16]. A lot of papers concerned the quartic circle approximation in explicit form have been published [2,8,12,13,15]. The approximation methods of circular arc using quintic Bézier curve having approximation order ten were developed [2, 5, 19]. Floater [6] found the polynomial curves of odd degree n which have $2n$ contacts with the conic and are G^{n-1} continuous. Jaklič et al. [9–11] presented the full circle approximation using polynomial curves of degree n having very high precision and $2n$ contacts with the full circle. The best circle approximation scheme using G^k polynomial

Received October 10, 2021; Accepted December 30, 2021.
2020 *Mathematics Subject Classification.* Primary 65D17.
Key words and phrases. Full circle approximation, parametric polynomial approximant, curvature continuous, Hausdorff distance, algebraic coefficients.
This study was supported by research funds from Chosun University, 2021.

curves having the constrained Chebyshev polynomial error functions was proposed [18]. The optimal G^2 circle approximation of degree less than or equal to four can be obtained in explicit form [1]. However, there is no full circle approximation using parametric polynomial curves of degree lager than four in explicit form with G^2 continuity at both endpoints.

This study aims to obtain the full circle approximation using parametric polynomial curves of high degree with algebraic coefficients and at least G^2 continuity at both endpoints. It can be useful in Geometric Modeling. We are inspired largely by Floater's method [6] which yields a conic approximation curve having $2n$ contacts with conic at both endpoints and the shoulder point. We present the approximation method using the parametric polynomial curve of degree n which has $2n$ contacts with the full circle at both endpoints and the midpoint. This method is suitable to find the parametric polynomial approximation curve with algebraic coefficients. Using our approximation method we obtain the circle approximation by parametric polynomial curves of high degree with G^2 continuity at both endpoints and with algebraic coefficients involving radicals and rational numbers.

The rest of our paper is constructed as follows. In Section 2, preliminaries for circle approximation are given, and in Section 3, our approximation method is presented. In Section 4, the curvature continuous circle approximation curves of high degree with algebraic coefficients are obtained, and in Section 5, we summarize our works.

2. Preliminaries

Let $\mathbf{c}$ be a full circle parameterized by $\mathbf{c}(\theta) = (\cos\theta, \sin\theta)^T$, $\theta \in [-\pi, \pi]$, and let $\mathbf{b}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{b}_i$ be the Bézier curve of degree n, where $\mathbf{b}_0, \ldots, \mathbf{b}_n$ are control points and $B_i^n = \binom{n}{i} t^i (1-t)^{n-i}$ is the Bernstein polynomial of degree n. The Hausdorff distance between two curves $\mathbf{b}$ and $\mathbf{c}$ is

$$d_H(\mathbf{b}, \mathbf{c}) = \max_{t \in [0,1]} |\phi(t)|,$$

where $\phi(t) = ||\mathbf{b}(t)|| - 1$. We use another error function

$$\psi(t) = ||\mathbf{b}(t)||^2 - 1$$

for convenience, which is a polynomial of degree $2n$.

Lemma 2.1. *The error function ψ has zeros of multiplicity k at $t = 0, 1$ with $\mathbf{c}'(-\pi) \cdot \mathbf{b}'(0) > 0$ and $\mathbf{c}'(\pi) \cdot \mathbf{b}'(1) > 0$ if and only if the Bézier curve $\mathbf{b}$ is a G^{k-1} endpoint interpolation of the full circle $\mathbf{c}$.*

The proof of this lemma can be found in [2].

3. Circle approximation using Bézier curves

Let m be the greatest integer less than or equal to $n/2$. Since $\mathbf{c}$ is symmetric with respect to x-axis, we assume the same symmetry for the approximation curve, i.e.,

$$\mathbf{b}_{n-i} = R_x \mathbf{b}_i, \qquad i = 0, \ldots, m, \tag{1}$$

where R_x is the reflection with respect to x-axis. Let $\mathbf{b}_i = (x_i, y_i)^T$ for $i = 0, 1, \ldots, n$. By the symmetry, $y_{\frac{n}{2}} = 0$ for even n. ψ has double zeros at $t = 0, 1$ if and only if

$$\mathbf{b}_0 = \mathbf{b}_n = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ and } x_1 = x_{n-1} = -1. \tag{2}$$

Moreover, if $y_1 < 0$, then $\mathbf{b}$ is a G^1 endpoint interpolation of $\mathbf{c}$. By the symmetry, if $\mathbf{b}$ is G^1 endpoint interpolation of $\mathbf{c}$, then $\mathbf{b}$ has the same curvature at both endpoints so that $\mathbf{b}$ becomes to be curvature continuous at both endpoints.

Proposition 3.1. *The circle approximation* $\mathbf{b}$ *has two contacts with* $\mathbf{c}$ *at* $t = 1/2$ *if*

$$x_m = \begin{cases} \frac{(1+n+2^{n-1})}{\binom{n}{m}} - \sum_{i=2}^{m-1} \frac{\binom{n}{i}}{\binom{n}{m}} x_i & \textit{for odd } n, \\ 2\left(\frac{(1+n+2^{n-1})}{\binom{n}{m}} - \sum_{i=2}^{m-1} \frac{\binom{n}{i}}{\binom{n}{m}} x_i\right) & \textit{for even } n. \end{cases} \tag{3}$$

Proof. If x_m satisfies Eq. (3), then $\mathbf{b}(1/2) = (1, 0)^T$ and so $\psi(1/2) = 0$. Since ψ is symmetric with respect to $t = 1/2$, ψ has double zeros at $t = 1/2$. Hence $\mathbf{b}$ has two contacts at $t = 1/2$ with $\mathbf{c}$. $\square$

If $\mathbf{b}$ satisfies Eqs. (1)-(3), then ψ has double zeros at $t = 0, 1/2, 1$. Let $\psi_1(t) = \psi(t)/(t^2(1-t)^2(t-1/2)^2)$. ψ_1 is a polynomial of degree $2n - 6$. $\mathbf{b}$ has $n - 3$ unknowns $x_2, \ldots, x_{m-1}$ and $y_1, \ldots, y_m$ for odd n, and $x_2, \ldots, x_{m-1}$ and $y_1, \ldots, y_{m-1}$ for even n. Now we consider the approximation $\mathbf{b}$ satisfying Eqs. (1)-(3).

Proposition 3.2. *The circle approximation* $\mathbf{b}$ *has contact orders* $k, 2(n-k), k$ *with* $\mathbf{c}$ *at* $t = 0, 1/2, 1$*, respectively, if*

$$\begin{aligned} \psi_1^{(i)}(0) &= 0, \quad i = 0, \ldots, k-3, \\ \psi_1^{(2j)}(1/2) &= 0, \quad j = 0, \ldots, n-k-2, \end{aligned} \tag{4}$$

for $k = 2, 3, \ldots, n-1$.

Proof. If $\mathbf{b}$ satisfies the first equation in Eq. (4), then ψ_1 has zeros of order $k-2$ at $t = 0, 1$ and so ψ has zeros of order k at $t = 0, 1$. Thus $\mathbf{b}$ has k contacts with $\mathbf{c}$ at $t = 0, 1$. If $\mathbf{b}$ satisfies the second equation in Eq. (4), by symmetry of ψ_1 with respect to $t = 1/2$, ψ_1 has zeros of order $2(n-k) - 2$ at $t = 1/2$. Hence $\mathbf{b}$ has $2n - 2k$ contacts with $\mathbf{c}$ at $t = 1/2$. Therefore $\mathbf{b}$ has totally $2n$ contacts with $\mathbf{c}$ in the interval $[0, 1]$. $\square$

4. Full circle approximation using parametric polynomial curves in explicit form

In this section we obtain full circle approximations by parametric polynomial curves with algebraic coefficients solving the equations in Proposition 3.2. They are at least G^2 continuous at both endpoints. The following propositions represent the full circle approximation using parametric polynomial curves with algebraic coefficients involving rational numbers and radicals for degree $n = 5, 6, 7, 9$.

Proposition 4.1. *The quintic Bézier curve* $\mathbf{b}$ *with the control points satisfying*

$$\mathbf{b}_1 = \begin{pmatrix} -1 \\ -\frac{6\sqrt{2}}{5} \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} \frac{11}{5} \\ -\frac{7\sqrt{2}}{5} \end{pmatrix} \tag{5}$$

and Eqs. (1)-(3) *is curvature continuous at both endpoint, and*

$$d_H(\mathbf{c}, \rho\mathbf{b}) = 1 - \rho \approx 4.28 \times 10^{-3}, \tag{6}$$

where $\rho = 250/(125 + 17\sqrt{55})$.

Proof. The quintic Bézier curve $\mathbf{b}$ with the control points in Eq. (5) has the error function ψ factorized as

$$\psi(t) = 8t^2(2t-1)^6(t-1)^2.$$

Since ψ has double zeros at $t = 0, 1$, $\mathbf{b}$ is a G^1 endpoint interpolation of the full circle $\mathbf{c}$. Moreover, $\mathbf{b}$ is symmetric with respect to the x-axis, $\mathbf{b}$ has the same curvature at both endpoints, and so is curvature continuous at the both endpoints. Since ψ has the maximum at $t_0 = \frac{1}{2} \pm \frac{\sqrt{15}}{10}$, we have

$$0 \leq \psi(t) \leq \psi(t_0) = \frac{54}{5^5}$$

and $1 \leq ||\mathbf{b}(t)|| \leq ||\mathbf{b}(t_0)|| = 17\sqrt{55}/5^3$. We choose a scaling factor $\rho = 2/(1 + ||\mathbf{b}(t_0)||)$, so that

$$\rho - 1 \leq ||\rho\mathbf{b}(t)|| - 1 \leq 1 - \rho.$$

Thus

$$d_H(\mathbf{c}, \rho\mathbf{b}) = \max_{0 \leq t \leq 1} |||\rho\mathbf{b}(t)|| - 1| = ||\rho\mathbf{b}(t_0)|| - 1 = 1 - \rho$$

and so the assertion follows. $\square$

Proposition 4.2. *The hexic Bézier curve* $\mathbf{b}$ *with the control points satisfying*

$$\mathbf{b}_1 = \begin{pmatrix} -1 \\ -\frac{2\sqrt{2+\sqrt{2}}}{3} \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} \frac{8\sqrt{2}+1}{15} \\ \frac{(16\sqrt{2}-40)\sqrt{2+\sqrt{2}}}{15} \end{pmatrix}, \ \mathbf{b}_3 = \begin{pmatrix} \frac{19-4\sqrt{2}}{5} \\ 0 \end{pmatrix}, \tag{7}$$

and Eqs. (1)-(3) *is a* G^3 *endpoint interpolation of the full circle, and*

$$d_H(\mathbf{c}, \rho\mathbf{b}) = 1 - \rho \approx 4.70 \times 10^{-4}, \tag{8}$$

where $\rho = 54/(27 + \sqrt{753 - 16\sqrt{2}})$.

Proof. The hexic Bézier curve $\mathbf{b}$ with the control points in Eq. (7) has the error function ψ factorized as

$$\psi(t) = 128(3 - 2\sqrt{2})t^4(2t - 1)^4(t - 1)^4.$$

Thus ψ has four zeros at $t = 0, 1$ and so $\mathbf{b}$ is a G^3 endpoint interpolation of the full circle. ψ has the maximum at $t = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$, and

$$\max_{0\le t\le 1} \psi(t) = \frac{8(3 - 2\sqrt{2})}{3^6},$$
$$\max_{0\le t\le 1} ||\mathbf{b}(t)|| = \frac{\sqrt{753 - 16\sqrt{2}}}{27}.$$

Hence Eq. (8) follows. □

Proposition 4.3. *The septic Bézier curve* $\mathbf{b}$ *with the control points satisfying*

$$\mathbf{b}_1 = \begin{pmatrix} -1 \\ \frac{4(1-\sqrt{5})}{7} \end{pmatrix}, \; \mathbf{b}_2 = \begin{pmatrix} \frac{27-16\sqrt{5}}{21} \\ \frac{4-20\sqrt{5}}{21} \end{pmatrix}, \; \mathbf{b}_3 = \begin{pmatrix} \frac{45+16\sqrt{5}}{35} \\ -\frac{96-16\sqrt{5}}{35} \end{pmatrix}, \tag{9}$$

and Eqs. (1)-(3) *is a* G^3 *endpoint interpolation of the full circle, and*

$$d_H(\mathbf{c}, \rho\mathbf{b}) = 1 - \rho \approx 3.06 \times 10^{-4} \tag{10}$$

for $\rho = 2 \cdot 7^4/(7^4 + \sqrt{2^7 \cdot 3^3 \cdot 7(7 - 3\sqrt{5}) + 7^8})$.

Proof. The sepic Bézier curve $\mathbf{b}$ with the control points in Eq. (9) has the error function ψ factorized as

$$\psi(t) = 128(7 - 3\sqrt{5})t^4(2t - 1)^6(t - 1)^4.$$

Thus ψ has four zeros at $t = 0, 1$ and so $\mathbf{b}$ is a G^3 endpoint interpolation of the full circle. ψ has the maximum at $t = \frac{1}{2} \pm \frac{\sqrt{21}}{14}$, and

$$\max_{0\le t\le 1} \psi(t) = \frac{2^7 \cdot 3^3(7 - 3\sqrt{5})}{7^7},$$
$$\max_{0\le t\le 1} ||\mathbf{b}(t)|| = \frac{\sqrt{2^7 \cdot 3^3 \cdot 7(7 - 3\sqrt{5}) + 7^8}}{7^4}.$$

Hence Eq. (10) follows. □

The approximation method by Proposition 3.2 cannot yield the octic circle approximation with algebraic coefficients involving radicals and rational numbers. The septic Bézier approximant in Proposition 4.3 can be used by degree elevation [14, 17] when the octic Bézier curve approximating the full circle is needed.

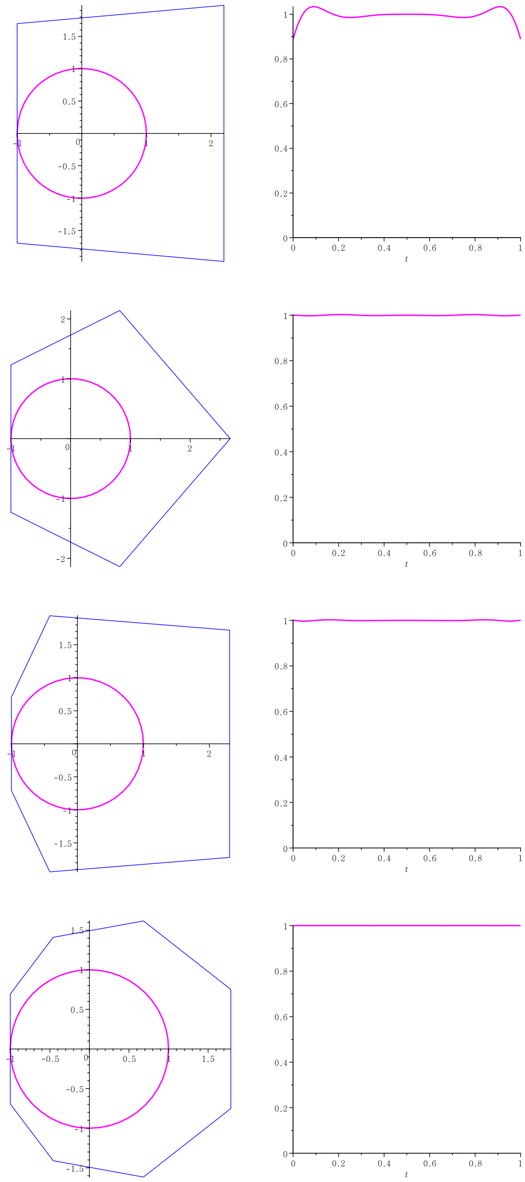

Figure 1. Left: Quintic, hexic, septic, and nonic (from top to bottom) circle approximations (magenta color) with control polygons (blue). Right: Curvatures (magenta) of the quintic, hexic, Septic, and nonic (from top to bottom) approximation curves.

Proposition 4.4. *The nonic Bézier curve* $\mathbf{b}$ *with the control points satisfying*

$$\begin{aligned} \mathbf{b}_1 &= \begin{pmatrix} -1 \\ \frac{2-2\sqrt{17}}{9} \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} -\frac{\sqrt{17}}{9} \\ -\frac{(13+3\sqrt{17})}{18} \end{pmatrix}, \\ \mathbf{b}_3 &= \begin{pmatrix} -\frac{(64-19\sqrt{17})}{21} \\ -\frac{(35-3\sqrt{17})}{14} \end{pmatrix}, \ \mathbf{b}_4 = \begin{pmatrix} \frac{(29-4\sqrt{17})}{7} \\ \frac{(501-133\sqrt{17})}{63} \end{pmatrix}, \end{aligned} \tag{11}$$

and Eqs. (1)-(3) *is a* G^5 *endpoint interpolation of the full circle, and*

$$d_H(\mathbf{c}, \rho\mathbf{b}) = 1 - \rho \approx 1.48 \times 10^{-7} \tag{12}$$

for $\rho = 486/(243 + \sqrt{67287 - 1998\sqrt{17}})$.

Proof. The nonic Bézier curve $\mathbf{b}$ with the control points in Eq. (11) has the error function ψ factorized as

$$\psi(t) = 128(1373 - 333\sqrt{17})t^6(2t-1)^6(t-1)^6.$$

Thus ψ has six zeros at $t = 0, 1$ and so $\mathbf{b}$ is a G^5 endpoint interpolation of the full circle. ψ has the maximum at $t_0 = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$, and

$$\max_{0 \le t \le 1} \psi(t) = \frac{2(1373 - 333\sqrt{17})}{3^9},$$

$$\max_{0 \le t \le 1} ||\mathbf{b}(t)|| = \frac{\sqrt{6(1373 - 333\sqrt{17}) + 3^{10}}}{3^5}.$$

Hence Eq. (12) follows. □

The graphes of full circle approximations proposed in Propositions 4.1-4.4 and their curvatures are plotted in Figure 1. Using change of variable $t = \frac{s+1}{2}$ for $s \in [-1, 1]$, the approximation curve $\mathbf{b}(\frac{s+1}{2})$ can be expanded in power basis, as shown in Table 1. The errors of the parametric polynomial curves approximating the full circle are listed in Table 1, and they are larger than those of the previous full circle approximation methods [9,10]. The advantages of our method is that the approximation curves are curvature continuous at both endpoints and have coefficients in explicit form.

5. Conclusion

In this paper we presented the full circle approximation method using parametric polynomial curves of degree n with algebraic coefficients which are curvature continuous at both endpoints and have a total number of $2n$ contacts with the full circle at both endpoints and the midpoint. The exact Hausdorff distances between the full circle and the approximation curves are obtained. Although our approximation method has larger error than those of previous full circle approximation methods, our method yields G^2 parametric polynomial curves at both endpoints with algebraic coefficients.

TABLE 1. Full circle approximation using parametric polynomial curves of degree n with algebraic coefficients involving rational numbers and radicals.

degree	k	$\mathbf{b}(\frac{s+1}{2})$, $s \in [-1,1]$	$d_H(\mathbf{c}, \rho\mathbf{b})$
5	2	$\begin{pmatrix} 1-4s^2+2s^4 \\ \frac{1}{\sqrt{2}}s(1-s)(1+s)(4-s^2) \end{pmatrix}$	4.28×10^{-3}
6	4	$\begin{pmatrix} 1-5s^2+4s^4-s^6+\frac{s^2}{\sqrt{2}}(1-2s^2+s^4) \\ \sqrt{2+\sqrt{2}}s(s+1)(1-s)(3-2s^2-\sqrt{2}(1-s^2)) \end{pmatrix}$	4.70×10^{-4}
7	4	$\begin{pmatrix} 1-3s^2+s^6-\sqrt{5}s^2(1-2s^2+s^4) \\ \frac{1}{2}s(1-s)(1+s)(-2+7s^2-3s^4+\sqrt{5}(-2-s^2+s^4)) \end{pmatrix}$	3.06×10^{-4}
9	6	$\begin{pmatrix} \frac{1}{4}(4-36s^2+69s^4-62s^6+21s^8+\sqrt{17}s^2(4-13s^2+14s^4-5s^6) \\ \frac{1}{8}s(1-s)(1+s)(-8+63s^2-96s^4+37s^6+\sqrt{17}(8-19s^2+24s^4-9s^6)) \end{pmatrix}$	1.48×10^{-7}

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