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CLASSIFICATION OF SOLVABLE LIE GROUPS WHOSE NON-TRIVIAL COADJOINT ORBITS ARE OF CODIMENSION 1

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Abstract. We give a complete classification of simply connected and solvable real Lie groups whose nontrivial coadjoint orbits are of codimension 1. This classification of the Lie groups is one to one corresponding to the classification of their Lie algebras. Such a Lie group belongs to a class, called the class of MD-groups. The Lie algebra of an MD-group is called an MD-algebra. Some interest properties of MD-algebras will be investigated as well.

1. Introduction

Throughout this paper, unless otherwise specified, the underlying field is always the field $\mathbb{R}$ of real numbers. As we know, an n-dimensional Lie algebra can be defined via the commutators of its basis vectors. More precisely, in order to define an n-dimensional Lie structure P on an n-dimensional vector space $\mathcal{G}$ spanned by $\{x_1, x_2, \ldots, x_n\}$, we need to define the Lie brackets

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k \quad (1 \le i, j \le n)$$

which satisfy two properties:

(skew-symmetrix) $\quad [x_i, x_j] = -[x_j, x_i] \quad \forall i, j,$

(Jacobi identity) $\quad [x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0 \quad \forall i, j, k.$

In other words, we need to consider the family $\{\mathrm{ad}_{x_i} : i = 1, 2, \ldots, n\}$ of n adjoint operators on $\mathcal{G}$ which satisfies the following requirements:

(skew-symmetrix) $\quad \mathrm{ad}_{x_i}(x_j) = -\mathrm{ad}_{x_j}(x_i) \quad \forall i, j,$

(Jacobi identity) $\quad \mathrm{ad}_{x_i} \circ \mathrm{ad}_{x_j} - \mathrm{ad}_{x_j} \circ \mathrm{ad}_{x_i} = \mathrm{ad}_{[x_i, x_j]} \quad \forall i, j.$

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Let's denote by A_i the $n \times n$ matrix of the adjoint operator ad_{x_i} with respect to the basis $\{x_1, \ldots, x_n\}$, $i = 1, \ldots, n$. Note that the family of matrices $\mathcal{A} := (A_1, \ldots, A_n)$ is determined by the n-dimensional Lie structure P.

According to Kirillov et al. [14], the set $\mathcal{P}$ of all n-dimensional Lie structures P on $\mathcal{G}$ determines the set $\mathcal{A}_n := \{\mathcal{A} : P \in \mathcal{P}\} \subset \mathcal{G} \otimes \mathcal{G}^* \otimes \mathcal{G}^*$. Due to the skew-symmetry and Jacobi identity, the set $\mathcal{A}_n$ forms an algebraic variety in n^3-dimensional space $\mathcal{G} \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ and it is called the variety of n-dimensional Lie algebra structures in term of A. A. Kirillov et al. [14].

Note that, by Kirillov et al. [14], the natural action of the general linear group $GL(\mathcal{G}) \equiv GL(n, \mathbb{R})$ on $\mathcal{G} \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ takes the variety $\mathcal{A}_n$ into itself and the orbits of $GL(n, \mathbb{R})$ in $\mathcal{A}_n$ correspond to the isomorphic classes of n-dimensional Lie algebras. In other words, the orbit space $\mathcal{A}_n / GL(n, \mathbb{R})$ coincides with the set of isomorphic classes of n-dimensional real Lie algebra. Hence, the number of parameters increases drastically and the volume of calculations, therefore, will become enormous when n not small [4]. In particular, while using the techniques of Gröbner bases and the computer algebra system Magma to solve the problem of classification of small-dimensional solvable Lie algebras, W. A. de Graaf [12] noted in 2005 that "the Gröbner basis computations (to classify solvable Lie algebras of dimension 5) can be rather time consuming, and there are even instances where it did not terminate in reasonable time". Recently, while trying to give an algorithm to test the isomorphism of two given Lie algebras (over real and complex fields) by using the triangular decomposition, L. A. Vu (the second author of this article) et al. found out the difficulty of using computer-based approach for the problem of classification of Lie algebras of dimension greater than 6 [19]. That explains why the problem of classification of general Lie algebras (as well as Lie groups) of dimension n strictly greater than 6 is still open until now (a complete classification of solvable Lie algebras of dimension 6 is given in [25], and of nilpotent Lie algebras of dimension 7 in [10]). In some certain cases, the problem of the classification is proven to be "wild" and "hopeless" [2, 3, 5, 9].

However, it is still possible to classify some interest subclass of solvable Lie algebras (of general finite dimensions) added one or some special properties in which we can use techniques in matrix theory to make the matrices in the family $\mathcal{A} = (A_1, \ldots, A_n)$ to be as simple as possible. An illustration for this idea is the existence of the Vergne bases of filiform Lie algebras [21]. In this paper, we will use geometric properties of coadjoint orbits (as known as K-orbits) to investigate Lie groups (as well as Lie algebras) whose non-trivial coadjoint orbits are all of the same dimension. This idea comes from the Kirillov's method [13] and proposed by D. N. Diep while searching for the class of Lie groups whose C^*-algebra can be characterized by BDF K-functions in 1980 [7]. More precisely, Diep recommended to consider finite-dimensional solvable and simply connected Lie groups whose coadjoint orbits are either of dimension 0 or of a constant positive dimension (which is called the maximal dimension). This Lie group is said to have the property MD and is called an

MD-group. The Lie algebra of an MD-group is called a Lie algebra having the property MD or an MD-algebra for brevity. It is worth pointing out that the family of K-orbits of the maximal dimension of an MD-group forms a measured foliation in terms of A. Connes [6, 26].

The problem of classifying MD-algebras has received much attention in recent decades, but the results are not much.

- In 1984, V. M. Son and H. H. Viet gave a complete classification of MD-groups whose maximal dimension of K-orbits is equal to the dimension of the group [24].
- In 1990, L. A. Vu (the second author of this paper) gave the classification (up to isomorphism) of all MD-algebras of dimension 4 [26, 27].
- In 1995, D. Arnal et al. gave a list of all MD-algebras in which the maximal dimension of K-orbits of corresponding MD-groups is 2 [1].
- In 2012, L. A. Vu and H. V. Hieu (the first author of this paper) et al. gave the classification (up to isomorphism) of MD-algebras of dimension 5 [28, 30].
- In 2016, L. A. Vu, H. V. Hieu et al. classified (up to isomorphism) all of MD-algebras which have the first derived ideal of codimension 1 or 2 [29].
- In 2019, M. Goze and E. Remm used Cartan class to give the classification (up to isomorphism) of all of MD-algebras such that their nontrivial coadjoint orbits are of dimension 4 [11].

Basically, the approaches of the classifying MD-algebras obtained in the above papers can be divided into the following directions: (1) fixing the dimensions of Lie algebras and (or) the dimensions of the derived series [26–30], (2) fixing the maximal dimension of coadjoint orbits [1, 11], and (3) fixing the codimension of coadjoint orbits [24]. Our paper follows the third direction. Specifically, we will classify all MD-algebras whose nontrivial K-orbits are of codimension 1 (i.e., the maximal dimension of K-orbits is 1 less than the dimension of such a Lie algebra).

The paper is organized into three sections, including this introduction. In Section 2 we recall some basic preliminary concepts, notations and properties which will be used throughout the paper. The last section will be devoted to setting and proving the main results of the paper.

2. Preliminaries

Throughout this paper, the following notations will be used.

- An $n \times n$ matrix whose (i, j)-entry is a_{ij} will be written as $(a_{ij})_{n\times n}$. While the (i, j)-entry of a matrix A will be denoted by $(A)_{ij}$. To simplify notation we use the same letter A for the endomorphism of $\mathbb{F}^n$ which is defined by assigning a column vector $v \in \mathbb{F}^n$ to Av, assumed that the entries of A take values in $\mathbb{F}$. Note that $\operatorname{Im} A$ is equal to the

$\mathbb{F}$-vector space spanned by the columns of A. We shall denote by $\mathbf{0}$ the zero matrix of suitable size.

- With the notation $V = \langle v_1, v_2, \ldots, v_n \rangle$, we mean $v_1, v_2, \ldots, v_n$ is a basis of the vector space V. The dual space of V will be denoted by V^*. It is well-know that if $V = \langle v_1, \ldots, v_n \rangle$, then $V^* = \langle v_1^*, \ldots, v_n^* \rangle$, where each v_i^* is defined by $v_i^*(v_j) = \delta_{ij}$ (the Kronecker delta symbol) for $1 \leq i, j \leq n$.
- By $\mathcal{G}$ we always mean a solvable Lie algebra of finite dimension over the real field with Lie bracket $[\cdot, \cdot]$. For any $x \in \mathcal{G}$, we will denote by ad_x the adjoint action of x on $\mathcal{G}$, i.e., ad_x is the endomorphism of $\mathcal{G}$ defined by $\mathrm{ad}_x(y) = [x, y]$ for every $y \in \mathcal{G}$. By ad_x^1, ad_x^2 we mean the restricted maps of ad_x on the first derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ and the second derived ideal $\mathcal{G}^2 := [\mathcal{G}^1, \mathcal{G}^1]$, respectively. Since both $\mathcal{G}^1$ and $\mathcal{G}^2$ are ideals of $\mathcal{G}$, we shall consider ad_x^1 and ad_x^2 as endomorphisms of $\mathcal{G}^1$ and $\mathcal{G}^2$, respectively. Similarly, we will denote by F^1 the restriction $F|_{\mathcal{G}^1}$ of any form $F \in \mathcal{G}^*$ on $\mathcal{G}^1$.

Next, we will recall some definitions, elementary results about coadjoint representation of solvable Lie groups. For more details, we refer reader to [13].

Definition. Let G be a Lie group and let $\mathcal{G}$ be its Lie algebra. If $\mathrm{Ad} : G \to \mathrm{Aut}(\mathcal{G})$ denotes the adjoint representation of G, then the action

$$K: \begin{array}{ccc} G & \to & \mathrm{Aut}(\mathcal{G}^*) \\ g & \mapsto & K_g \end{array}$$

defined by

$$K_g(F)(x) = F\big(\mathrm{Ad}(g^{-1})(x)\big) \text{ for } F \in \mathcal{G}^*,\ x \in \mathcal{G}$$

is called the *coadjoint representation* of G in $\mathcal{G}^*$. Each orbit in the coadjoint representation of G is called a *coadjoint orbit* (or a *K-orbit* for short) of G as well as of $\mathcal{G}$.

For each $F \in \mathcal{G}^*$, the K-orbit that passes through the point F is denoted by Ω_F, i.e.,

$$\Omega_F = \{K_g(F) : g \in G\}.$$

Definition ([13, Section 15.1])**.** Let F be any element in $\mathcal{G}^*$. The Kirillov's bilinear form with respect to F is defined by

$$B_F(x, y) := F([x, y]) \text{ for } x, y \in \mathcal{G}.$$

It is a bilinear skew symmetric form on $\mathcal{G}$.

The matrix of B_F with respect to a basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathcal{G}$, which is equal to $\big(F([x_i, x_j])\big)_{n \times n}$, is called the Kirillov's matrix of F with respect to that basis. From now on, when a basis of $\mathcal{G}$ is fixed, we will always treat B_F as the Kirillov's matrix of F with respect to the fixed matrix. The set of matrices $\{B_F : F \in \mathcal{G}^*\}$ will be called the Kirillov's matrices of $\mathcal{G}$. The following proposition gives us a connection between dimension of Ω_F and the rank of B_F.

Proposition 2.1 ([13, Section 15.3])**.** *Let F be any element in $\mathcal{G}^*$. Then*

$$\dim \Omega_F = \operatorname{rank} B_F.$$

As a consequence, $\dim \Omega_F$ is always even for every F in $\mathcal{G}^*$ and $\dim \Omega_F > 0$ if and only if F^1 is not equal to zero, i.e., $F|_{\mathcal{G}^1} \neq 0$. It is trivial to see that if $\dim \Omega_F = 0$ for every $F \in \mathcal{G}^*$, then $\mathcal{G}$ is commutative. Our concern will be Lie algebras in which the dimensions of non-trivial K-orbits are equal.

Definition ([24, Introduction])**.** A finite dimensional solvable and simply connected Lie group is called an *MD-group* (in terms of Do Ngoc Diep) if its K-orbits are either of dimension zero (trivial ones) or of the same positive constant dimension. The Lie algebra of an MD-group is called an *MD-algebra*.

Let G be an MD-group. Then the positive constant dimension of K-orbits is also called the maximal dimension of K-orbits of G (as well as of the corresponding Lie algebra $\mathcal{G}$). If this maximal dimension is given by k ($k > 0$), we also say that $\mathcal{G}$ is an MD_k-algebra. Furthermore, if $\dim \mathcal{G} = n$, then $\mathcal{G}$ is also called an $\mathrm{MD}_k(n)$-algebra. In particular, if an MD-group G satisfies one of the following properties: either all K-orbits are trivial or the maximal dimension of K-orbits is equal to the dimension of G, then G is said to be an SMD-group. The Lie algebra of an SMD-group is called an SMD-algebra. It is clear that any commutative Lie algebra is an SMD-algebra and it has no maximal dimension of K-orbits because its K-orbits are all trivial.

In order to classify $\mathrm{MD}_{n-1}(n)$-algebras, we firstly note that all solvable Lie algebras of dimension $n < 4$ are obviously MD-algebras and can be classified easily. Therefore we only take interest in MD-algebras of dimension $n \geq 4$. On the other hand, in 1984 V. M. Son and H. H. Viet gave a bound for the solvable index of MD-algebras which is presented in the following proposition.

Proposition 2.2 ([24, Theorem 4])**.** *If $\mathcal{G}$ is an MD-algebra, then its second derived ideal $\mathcal{G}^2$ is commutative.*

Therefore, if $\mathcal{G}$ is an MD-algebra, then the third derived ideal $\mathcal{G}^3 := [\mathcal{G}^2, \mathcal{G}^2]$ is the trivial vector space. In this paper, a solvable Lie algebra is said to be *i-step solvable* if $\mathcal{G}^i$ is non trivial (i.e., $\mathcal{G}^i \neq \{0\}$) and commutative. By using this term, Proposition 2.2 means that any non-commutative MD-algebra is either 1-step or 2-step solvable. The basis idea of the classification we use in this paper is to classify 1-step solvable $\mathrm{MD}_{n-1}(n)$-algebras firstly and to connect 2-step solvable $\mathrm{MD}_{n-1}(n)$-algebras with 1-step solvable $\mathrm{MD}_{n-1}(n-1)$-algebras afterwards. To deal with 1-step solvable ones, it is necessary to recall the following classical result.

Proposition 2.3 ([30, Lemma 3.1])**.** *If $\mathcal{G}$ is 1-step solvable, then*

$$\mathrm{ad}_x^1 \circ \mathrm{ad}_y^1 = \mathrm{ad}_y^1 \circ \mathrm{ad}_x^1 \quad \forall x, y \in \mathcal{G}.$$

In other words, $\{\mathrm{ad}_x^1 : x \in \mathcal{G}\}$ *is a commuting family of linear endomorphisms of* $\mathcal{G}^1$.

Hence, it is reasonable to consider the set of commuting matrices over the real field. It is well-known that an arbitrary set of commuting matrices over the complex field $\mathbb{C}$ has an invariant subspace of dimension 1. Therefore, it may be simultaneously brought to triangular form by a unitary similarity [18, 20]. Similarly for the triangular form of an arbitrary set of commuting real matrices. We state these classical results in the following proposition.

Proposition 2.4. *Let* $\mathcal{S} \subseteq M_{n\times n}(\mathbb{F})$ *be a set of commuting* $n \times n$ *matrices over a field* $\mathbb{F}$, *i.e.,* $AB = BA$ *for every* $A, B \in \mathcal{S}$.

(i) *If* $\mathbb{F} = \mathbb{C}$, *then* $\mathcal{S}$ *is simultaneously triangularizable by a unitary matrix, i.e., there exists a complex unitary matrix* T *so that* $T^{-1}AT$ *is a upper triangular matrix for all* $A \in \mathcal{S}$.

(ii) *If* $\mathbb{F} = \mathbb{R}$, *then* $\mathcal{S}$ *is block simultaneously triangularizable by a orthogonal matrix, i.e., there is a real orthogonal matrix* T *so that* $T^{-1}AT$ *is a block upper triangular matrix, where each diagonal block is of size either* 1 *or* 2, *for all* $A \in \mathcal{S}$.

To close this section, we recall here the classification of real solvable Lie algebras whose non-trivial coadjoint orbits are all of the same dimension as the algebras, or the classification of real $\mathrm{MD}_n(n)$-algebras (as known as SMD-algebras).

Proposition 2.5 ([24]). *Let* $\mathcal{G}$ *be a real* $MD_n(n)$*-algebra. If* $\mathcal{G}$ *is non-commutative, then* $\mathcal{G}$ *is isomorphic to one of the following forms*:

(i) *the real affine algebra* $\mathrm{aff}(\mathbb{R}) = \langle x, y\rangle$ *with*

$$[x, y] = y,$$

(ii) *the complex affine algebra* $\mathrm{aff}(\mathbb{C}) = \langle x_1, x_2, y_1, y_2\rangle$ *with*

$$[x_1, y_1] = y_1, [x_1, y_2] = y_2, [x_2, y_1] = -y_2, [x_2, y_2] = y_1.$$

3. Main results

In this section, we firstly give some new interest properties of $\mathrm{MD}_k(n)$-algebras, especially 1-step solvable MD-algebras (Theorems 3.1, 3.2 & 3.4). Secondly, we give a connection between 1-step solvable MD-algebras and 2-step solvable MD-algebras (Theorem 3.5). Finally, we will present the complete classification of $\mathrm{MD}_{n-1}(n)$-algebras in the last theorem. It is possible that we can apply Theorem 3.2 and Theorem 3.5 to obtain the classification of $\mathrm{MD}_k(n)$-algebras with $n - k$ small (not necessary to be 1) but we will not develop this point here.

3.1. Some new characteristics of MD-algebras

Theorem 3.1. *Let $\mathcal{G}$ be a non-commutative Lie algebra. Then $\mathcal{G}$ is a decomposable MD$_k$-algebra if and only if it is a trivial extension of an indecomposable MD$_k$-algebra by a finite-dimensional commutative Lie algebra.*

Proof. Let's first prove the "if" part. Assume that $\mathcal{G}$ is a direct sum of an MD$_k$-algebra with a finite-dimensional commutative algebra $\mathbb{R}^r$, i.e., $\mathcal{G} = \mathcal{K} \oplus \mathbb{R}^r$, where $\mathcal{K}$ is an MD$_k$-algebra. If so, we have

$$\mathcal{G}^1 = \mathcal{K}^1. \tag{1}$$

Denote by n the dimension of $\mathcal{G}$. Let's fix a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathcal{G}$ so that $\mathcal{K} = \langle x_1, \dots, x_{n-r}\rangle$. For any $F = F_\mathcal{K} + F_r \in \mathcal{G}^*$ with $F_\mathcal{K} \in \mathcal{K}^* \subseteq \mathcal{G}^*$ and $F_r \in (\mathbb{R}^r)^* \subseteq \mathcal{G}^*$, we denote by $\bar{\Omega}_{F_\mathcal{K}}$ the K-orbit of $F_\mathcal{K}$ in $\mathcal{K}^* \subset \mathcal{G}^*$. It is elementary to see that

$$B_F = \begin{bmatrix} \bar{B}_{F_\mathcal{K}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{2}$$

where $\bar{B}_{F_\mathcal{K}}$ is the Kirillov's matrix of $F_\mathcal{K}$ in $\mathcal{K}$ with respect to the basis $\{x_1, \dots, x_{n-r}\}$. Therefore, we get $\operatorname{rank} B_F = \operatorname{rank} \bar{B}_{F_\mathcal{K}}$. Since $\mathcal{K}$ is an MD$_k$-algebra, we have

$$\dim \Omega_F = \dim \bar{\Omega}_{F_\mathcal{K}} = \begin{cases} 0 & \text{if } F|_{\mathcal{K}^1} = \mathbf{0}, \\ k & \text{elsewhere.} \end{cases} \tag{3}$$

This proves $\mathcal{G}$ is an MD$_k$-algebra, as required.

It remains to prove the "only if" part. By contradiction, assume that $\mathcal{G} = \mathcal{K} \oplus \mathcal{L}$ in which both $\mathcal{K}$ and $\mathcal{L}$ are non-commutative sub-algebras of $\mathcal{G}$. Let's denote by n and m the dimensions of $\mathcal{G}$ and $\mathcal{K}$, respectively. Since $\mathcal{G} = \mathcal{K} \oplus \mathcal{L}$, there is a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathcal{G}$ so that

$$\begin{cases} \mathcal{K} = \langle x_1, x_2, \dots, x_m\rangle, \\ \mathcal{L} = \langle x_{m+1}, \dots, x_n\rangle. \end{cases} \tag{4}$$

Because both $\mathcal{K}$ and $\mathcal{L}$ are non-commutative, there exist two non-zero elements $F_1, F_2 \in (\mathcal{G})^*$ so that

$$\begin{cases} F_1|_{\mathcal{K}^1} \neq \mathbf{0}, & F_1|_{\mathcal{L}^1} = \mathbf{0}, \\ F_2|_{\mathcal{K}^1} = \mathbf{0}, & F_2|_{\mathcal{L}^1} \neq \mathbf{0}. \end{cases} \tag{5}$$

If so, it is elementary to see that

$$B_{F_1} = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ and } B_{F_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix}, \tag{6}$$

where M and N are two non-zero matrices of dimension $m \times m$ and $(n-m) \times (n-m)$, respectively. We thus get

$$0 < \operatorname{rank} B_{F_1} < \operatorname{rank} B_{F_1} + \operatorname{rank} B_{F_2} = \operatorname{rank} B_{F_1+F_2}.$$

In other words,

$$0 < \dim \Omega_{F_1} < \dim \Omega_{F_1+F_2},$$

a contradiction to the fact that $\mathcal{G}$ is an MD-algebra. This completes the proof. □

Theorem 3.2. *Let $\mathcal{G}$ be a 1-step solvable $MD_k(n)$-algebra so that $\dim \mathcal{G}^1 > n-k$. Then we have the following assertions.*

(i) $\sum_{y \in \mathcal{G}} \operatorname{Im} \operatorname{ad}_y^1 = \mathcal{G}^1$.

(ii) *There is x in $\mathcal{G}$ so that ad_x^1 is an automorphism on the vector space $\mathcal{G}^1$.*

(iii) *$\mathcal{G}$ can be expressed as the semidirect sum $\mathcal{L} \oplus_\rho \mathcal{G}^1$ of a commutative subalgebra $\mathcal{L}$ of $\mathcal{G}$ with the first derived ideal $\mathcal{G}^1$ by the representation $\rho : \mathcal{L} \to Der(\mathcal{G}^1)$, which is defined by $\rho(x) = \operatorname{ad}_x^1$ for every $x \in \mathcal{L}$.*

In order to prove this theorem, we will need the following lemma.

Lemma 3.3. *Let $\mathcal{S} = \{A_1, A_2, \ldots, A_s\}$ be a set of complex upper triangular matrices of size $m \times m$. If $\mathcal{S}$ is commutative (i.e., $AB = BA$ for every $A, B \in \mathcal{S}$) and*

$$\sum_{i=1}^{s} \operatorname{Im} A_i = \mathbb{C}^m, \tag{7}$$

then there is a linear combination of elements of $\mathcal{S}$ so that it is nonsingular.

Proof. The proof is by induction on m. It is trivial for $m = 1$. Since A_i is of upper triangular form for every i, the equality (7) implies that at least one of $\{(A_i)_{mm} : i = 1, \ldots, s\}$ is non-zero (if not, the vector $(0, 0, \ldots, 1)^t \notin \sum_{i=1}^s \operatorname{Im} A_i = \mathbb{C}^m$, a contradiction). By re-indexing if necessary, we may assume that $(A_1)_{mm} \neq 0$. Furthermore, the proof is still correct if we replace $\mathcal{S}$ by $\mathcal{S}' = \{A_1, A_2 - \beta_2 A_1, \ldots, A_s - \beta_s A_1\}$ for every $\beta_2, \ldots, \beta_s \in \mathbb{C}$. Therefore, we may assume from beginning that

$$\begin{cases} (A_i)_{mm} = 0 & \forall i \geq 2, \\ (A_1)_{mm} \neq 0. \end{cases} \tag{8}$$

Since $A_1 A_i = A_i A_1$ for every i, we have

$$(A_1 A_i)_{jm} = (A_i A_1)_{jm} \text{ for every } i, j. \tag{9}$$

Therefore, for every i, the last column of A_i is a linear combination of the following columns: all the columns of A_1 and the first $(m-1)$ columns of A_i. It follows that $\sum_{i=1}^s \operatorname{Im} A_i$ is spanned by the last column of A_1 and the first $m-1$ columns of $A_1, \ldots, A_s$. We conclude from the equality (7) that $\mathbb{C}^{m-1}$ is spanned by the first $(m-1)$ columns of $A_1, \ldots, A_s$.

In other words, if we denote by B_i the $(m-1) \times (m-1)$ matrix obtained by deleting the m-th column and m-th row of A_i for each $1 \leq i \leq s$, then $\{B_1, B_2, \ldots, B_s\}$ satisfies the following properties:

- it is a set of commuting (complex) matrices of dimension $(m-1) \times (m-1)$,
- each B_i is of upper triangular form for every i, and

- $\mathbb{C}^{m-1}$ is spanned by the columns of $B_1, B_2, \ldots, B_s$, i.e.,

$$\sum_{i=1}^{s} \operatorname{Im} B_i = \mathbb{C}^{m-1}. \tag{10}$$

By induction, there are $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}$ so that $\alpha_1 B_1 + \cdots + \alpha_s B_s$ is nonsingular. Remark that we can always choose $\alpha_1 \neq 0$ (if $\alpha_1 = 0$, then $\alpha_2 B_2 + \cdots + \alpha_s B_s$ is nonsingular and hence, there exists $\alpha_1' \neq 0$ so that $\alpha_1' B_1 + \alpha_2 B_2 + \cdots + \alpha_s B_s$ is nonsingular because the equation $\det\big(B_1 + x(\alpha_2 B_2 + \cdots + \alpha_s B_s)\big) = 0$ has finite solutions). With the assumption $\alpha_1 \neq 0$ in hand, it follows immediately from the equation (8) and from the nonsingularity of $\alpha_1 B_1 + \cdots + \alpha_s B_s$ that

$$\alpha_1 A_1 + \cdots + \alpha_s A_s$$

is nonsingular, which proves the lemma. □

Proof of Theorem 3.2. Set $m = \dim \mathcal{G}^1$. Throughout this proof, we fix a basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathcal{G}$ so that $\mathcal{G}^1 = \langle x_{n-m+1}, x_{n-m+2}, \ldots, x_n \rangle$.

(i) By contradiction, assume that $\sum\limits_{y \in \mathcal{G}} \operatorname{Im} \mathrm{ad}_y^1 \subsetneq \mathcal{G}^1$. Then there is an element $F \in (\mathcal{G})^*$ so that

$$\begin{cases} F^1 \neq 0, \\ F|_{\sum\limits_{y \in \mathcal{G}} \operatorname{Im} \mathrm{ad}_y^1} = 0. \end{cases} \tag{11}$$

If so, it is clear that

$$F([x_i, x_j]) = 0, \text{ unless } 1 \leq i, j \leq n - m. \tag{12}$$

It follows from Proposition 2.1 that $k = \dim \Omega_F \leq n - m < k$, a contradiction. This proves the first item.

(ii) Let $\mathcal{G}_{\mathbb{C}} = \langle x_1, \ldots, x_n \rangle_{\mathbb{C}}$ be the complexification of $\mathcal{G}$. By Proposition 2.3, $\{\mathrm{ad}_{x_i}^1 : i = 1, \ldots, n - m\}$ is a family of commuting endomorphisms of the real type. Hence, there is a basis of $\mathcal{G}_{\mathbb{C}}^1$ so that the matrix of $\mathrm{ad}_{x_i}|_{\mathcal{G}_{\mathbb{C}}^1}$ with respect to that basis is of upper triangular form for every $1 \leq i \leq n - m$ (by Proposition 2.4). It follows from the first item and Lemma 3.3 that there is a complex linear combination x of $x_1, x_2, \ldots, x_{n-m}$ so that $\mathrm{ad}_x|_{\mathcal{G}_{\mathbb{C}}^1}$ is isomorphic. Therefore, by using the same notation $\mathrm{ad}_{x_i}^1 (i = 1, \ldots, n - m)$ for their matrices with respect to that the basis $\{x_1, \ldots, x_n\}$, we easily see that the polynomial

$$\det(\zeta_1 \mathrm{ad}_{x_1}^1 + \cdots + \zeta_{n-m} \mathrm{ad}_{x_{n-m}}^1)$$

of variables $\zeta_1, \ldots, \zeta_{n-m}$ does not vanish. Hence, there is $\alpha_1, \ldots, \alpha_{n-m} \in \mathbb{R}$ so that

$$\det(\alpha_1 \mathrm{ad}_{x_1}^1 + \cdots + \alpha_{n-m} \mathrm{ad}_{x_{n-m}}^1) \neq 0. \tag{13}$$

This prove the existence of a real linear combination x' of $x_1, x_2, \ldots, x_{n-m}$ so that $\mathrm{ad}_{x'}^1$ is nonsingular, as required.

(iii) It follows from the second item proven above that we may assume (after changing basis if necessary) $\mathrm{ad}^1_{x_1}$ is nonsingular. Hence, for each $j \geq 2$, there are $\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jm} \in \mathbb{R}$ so that

$$[x_1, x_j] + \alpha_{j1}[x_1, x_{n-m+1}] + \cdots + \alpha_{jm}[x_1, x_n] = 0. \tag{14}$$

Therefore, by basis changing $x'_j = x_j + \sum_{i=1}^{m} \alpha_{ji} x_{n-m+i}$ if necessary, we can assume from beginning that $[x_1, x_j] = 0$ for all $1 \leq j \leq n-m$. With this assumption in hand, we may apply the Jacobi's identity for (x_1, x_j, x_k) to obtain

$$[x_1, [x_j, x_k]] = 0 \quad \text{for all } 1 \leq j, k \leq n-m. \tag{15}$$

It follows from the nonsingularity of $\mathrm{ad}^1_{x_1}$ that $[x_j, x_k] = 0$ for all $1 \leq j, k \leq n-m$. Hence, $\mathcal{G}$ is isomorphic to the semidirect sum of $\mathcal{G}^1$ with the commutative subalgebra $\mathcal{L} := \langle x_1, \dots, x_{n-m} \rangle \cong \mathbb{R}^{n-m}$ of $\mathcal{G}$, as required. □

Theorem 3.2 gains in interest if we realize that, by fixing a basis $\{x_1, x_2, \dots, x_n\}$ of the Lie algebra $\mathcal{L} \oplus_\rho \mathcal{G}^1$ so that $\mathcal{L} = \langle x_1, \dots, x_{n-m} \rangle$ and $\mathcal{G}^1 = \langle x_{n-m+1}, \dots, x_n \rangle$, the Kirilov's matrix of any $F \in (\mathcal{L} \oplus_\rho \mathcal{G}^1)^*$ is of the following form

$$\begin{bmatrix} \mathbf{0} & M_F \\ -(M_F)^t & \mathbf{0} \end{bmatrix},$$

where M_F is some matrix of size $(n-m) \times m$. It turns out that the dimension of each Ω_F depends on the rank of M_F. More precisely, $\dim \Omega_F = 2 \operatorname{rank} M_F$.

Besides, it is easily seen that the i-th row of M_F is exactly

$$\left(F^1 \circ \mathrm{ad}^1_{x_i}(x_{n-m+1}), F^1 \circ \mathrm{ad}^1_{x_i}(x_{n-m+2}), \dots, F^1 \circ \mathrm{ad}^1_{x_i}(x_n)\right),$$

or the (row) matrix of the form $F^1 \circ \mathrm{ad}^1_{x_i}$ on $\mathcal{G}^1$ with respect to the basis $\{x_{n-m+1}, \dots, x_n\}$. Therefore, we can connect the rank of M_F, as well as the dimension of Ω_F, with the maximum number of linearly independent matrices in the set

$$\{\mathrm{ad}^1_{x_1}, \mathrm{ad}^1_{x_2}, \dots, \mathrm{ad}^1_{x_{n-m}}\}.$$

Remark that the vector space spanned by the above set and the vector space spanned by the set $\mathcal{S} := \{\mathrm{ad}^1_x : x \in \mathcal{L} \oplus_\rho \mathcal{G}^1\}$ coincide. It follows that

$$\dim \Omega_F = 2 \operatorname{rank} M_F = 2 \dim F^1(\langle \mathcal{S} \rangle), \tag{16}$$

where $\langle \mathcal{S} \rangle$ denotes the vector space spanned by S, and $F^1(\langle \mathcal{S} \rangle)$ denotes the vector space $\{F^1 \circ f : f \in \langle \mathcal{S} \rangle\}$. In the following theorem, we will use the maximum number mentioned above, which is equal to the dimension of $\langle \mathcal{S} \rangle$, to determine some interest properties of $\mathcal{G}$, including the decomposition.

Theorem 3.4. *Let $\mathcal{G}$ satisfy the hypothesis of Theorem 3.2 and let $\mathcal{S}$ be the set of adjoint actions of all points in $\mathcal{G}$, i.e., $\mathcal{S} = \{\mathrm{ad}^1_x : x \in \mathcal{G}\}$. Denote by m the dimension of $\mathcal{G}^1$ and by r the dimension of the vector space spanned by $\mathcal{S}$. Then the following assertions hold.*

(i) $k \leq 2r \leq 2(n-m)$,

(ii) *$r = n - m$ if and only if $\mathcal{G}$ is indecomposable,*

(iii) *$2r = k$ if and only if every non-zero element in the vector space spanned by $\mathcal{S}$ is an automorphism on the vector space $\mathcal{G}^1$.*

Proof. Followed by Theorem 3.2, $\mathcal{G}$ is a direct sum of two commutative subalgebras $\mathcal{L}$ and $\mathcal{G}^1$. By the second assertion in that theorem, we can fix a basis $\{x_1, \ldots, x_n\}$ of $\mathcal{G}$ so that

$$\left\{ \begin{array}{l} \mathcal{L} = \langle x_1, \ldots, x_{n-m} \rangle, \\ \mathcal{G}^1 = \langle x_{n-m+1}, \ldots, x_n \rangle, \end{array} \right. \tag{17}$$

and $\mathrm{ad}^1_{x_1}$ is an automorphism on the vector space $\mathcal{G}^1$.

(i) Since $\mathcal{G}$ is spanned by $\{x_1, x_2, \ldots, x_n\}$, the real vector space spanned by $\mathcal{S}$, $\langle \mathcal{S} \rangle$, is also spanned by $\{\mathrm{ad}^1_{x_i} : i = 1, 2, \ldots, n\}$. Moreover, the commutation of $\mathcal{G}^1$ implies that $\mathrm{ad}^1_{x_i} = \mathbf{0}$ for every $n - m + 1 \le i \le n$. Hence,

$$r \le n - m. \tag{18}$$

On the other hand, in the light of the equation (16), we easily see that, for any $F \in \mathcal{G}^*$ which does not vanish on $\mathcal{G}^1$,

$$0 \ne \dim \Omega_F = 2 \dim F^1(\langle \mathcal{S} \rangle) \le 2 \dim \langle \mathcal{S} \rangle = 2r. \tag{19}$$

The inequalities (18) and (19) establish the formula in the first item.

(ii) By Theorem 3.1, we conclude that $\mathcal{G}$ is decomposable if and only if $\mathcal{G} = \mathcal{K} \oplus \mathbb{R}$ for some sub-algebra $\mathcal{K} \subseteq \mathcal{G}$. Hence, $\mathcal{G}$ is decomposable if and only if there is an element $x \in \mathcal{G} \setminus \mathcal{G}^1$ so that it is contained in the center of $\mathcal{G}$. Because $x \notin \mathcal{G}^1$, we get

$$x = y + z \tag{20}$$

for some $0 \ne y \in \mathcal{L}$ and for some $z \in \mathcal{G}^1$. Since x belongs to the center of $\mathcal{G}$ and $\mathcal{G}^1$ is commutative, we obtain

$$\mathrm{ad}^1_y = \mathbf{0}. \tag{21}$$

Therefore, $\mathcal{G}$ is decomposable if and only if the dimension of the vector space spanned by

$$\{\mathrm{ad}^1_{x_1}, \ldots, \mathrm{ad}^1_{x_{n-m}}\}$$

is strictly less than $n - m$, or $r < n - m$, which is our assertion.

(iii) In light of the equation (16), what we need to show is the following statement: $\dim F^1(\langle \mathcal{S} \rangle) = \dim(\langle \mathcal{S} \rangle)$ for every $F \in (\mathcal{G})^*$ with $F^1 \ne 0$ if and only if every non-zero element of $\langle \mathcal{S} \rangle$ is an automorphism on the vector space $\mathcal{G}^1$. Equivalently, $\dim F^1(\langle \mathcal{S} \rangle) < \dim(\langle \mathcal{S} \rangle)$ for some $F \in (\mathcal{G})^*$ with $F^1 \ne 0$ if and only if there is a non-zero element f of $\langle \mathcal{S} \rangle$ which is not an automorphism on the vector space $\mathcal{G}^1$.

Indeed, the existence of a non-zero element $f \in \langle \mathcal{S} \rangle$ which is not an automorphism on the vector space $\mathcal{G}^1$ is equivalent to the existence of an element

$0 \neq x \in \mathcal{G}^1$ so that $f(x) = 0$, hence is equivalent to the existence of an element $F \in (\mathcal{G})^*$ so that

$$\begin{cases} F^1 \neq \mathbf{0}, \\ F^1 \circ f = \mathbf{0}. \end{cases} \tag{22}$$

If the equation (22) holds, then by choosing a basis for $\langle \mathcal{S} \rangle$ which contains f, we can see easily that the dimension of $F^1(\langle \mathcal{S} \rangle)$ is strictly less than the dimension of $\langle \mathcal{S} \rangle$.

Conversely, if $\dim F^1(\langle \mathcal{S} \rangle) < \dim(\langle \mathcal{S} \rangle)$, then there is a basis $\{f_1, f_2, \ldots, f_r\}$ of $\langle \mathcal{S} \rangle$ so that

$$F^1(\alpha f_1 + \cdots + \alpha_r f_r) = \mathbf{0} \tag{23}$$

for some non-zero $(\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$. This proves the existence of a non-zero element $f \in \langle \mathcal{S} \rangle$ so that the equation (22) holds. The proof is completed. □

To close this subsection, let's present the connection between 2-step MD-algebras with 1-step MD-algebras in the next theorem.

Theorem 3.5. *Let $\mathcal{G}$ be an $MD_k(n)$-algebra. Then $\dim \mathcal{G}^2 \leq n - k$. Furthermore, the quotient Lie algebra $\mathcal{H} := \mathcal{G}/\mathcal{G}^2$ is also an MD_k-algebra.*

Proof. The assertion of the theorem is obviously true when $\mathcal{G}^2$ is trivial. Therefore, we only need to consider the case $\dim \mathcal{G}^2 = p > 0$. Denote by m the dimension of $\mathcal{G}^1$ $(m > p)$. Let's fix a basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathcal{G}$ so that $\mathcal{G}^1 = \langle x_{n-m+1}, \ldots, x_n \rangle$ and $\mathcal{G}^2 = \langle x_{n-p+1}, \ldots, x_n \rangle$. Since $\mathcal{G}^2$ is an ideal of $\mathcal{G}$, $x^*_{n-m+1}([x_i, x_j]) = 0$ for every i, j with $1 \leq i \leq n$ and $n - p + 1 \leq j \leq n$. Therefore, by setting $F = x^*_{n-m+1}$ we get

$$B_F = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p \times p} \end{bmatrix} \tag{24}$$

for some square matrix M of order $n - p$. It follows that the rank of B_F is at most $n - p$. On the other hand, because $\mathcal{G}$ is an $\mathrm{MD}_k(n)$-algebra and $F^1 \neq 0$, we have

$$\dim \Omega_F = k. \tag{25}$$

It follows from Proposition 2.1 that

$$k = \dim \Omega_F = \operatorname{rank} B_F \leq n - p. \tag{26}$$

In other words,

$$\dim \mathcal{G}^2 \leq n - k. \tag{27}$$

This proves the first part of the theorem.

To prove the second part, we still assume $\{x_1, x_2, \ldots, x_n\}$ is a basis of $\mathcal{G}$ so that $\{x_{n-m+1}, \ldots, x_n\}$ and $\{x_{n-p+1}, \ldots, x_n\} (p < m)$ are bases of $\mathcal{G}^1$ and $\mathcal{G}^2$, respectively. It is standard to check that

$$x^*_q([x_i, x_j]) = \overline{x^*_q}([x_i + \mathcal{G}^2, x_j + \mathcal{G}^2]) \text{ for any } 1 \leq i, j \leq n, \text{ and } 1 \leq q \leq n - p, \tag{28}$$

where $\overline{x_q^*}$ is the corresponding element of x_q^* in $(\mathcal{H}^1)^*$. Therefore, the dimensions of the coadjoint orbits $\Omega_{\overline{F}}$ (of $\mathcal{H}$) and Ω_F (of $\mathcal{G}$) are equal for any $F \in (\mathcal{G})^*$ with $\overline{F}|_{\mathcal{H}^1} \neq 0$. This completes the proof. □

3.2. The complete classification of all $\mathrm{MD}_{n-1}(n)$-algebras

Now, we will state the last main result of the paper, which gives the complete classification of all indecomposable $\mathrm{MD}_{n-1}(n)$-algebras for $n \geq 4$.

Theorem 3.6. *Let $\mathcal{G}$ be an indecomposable $MD_{n-1}(n)$-algebra with $n \geq 4$. Then $\mathcal{G}$ is isomorphic to one of the followings*:

(i) *The real Heisenberg Lie algebra*

$$\mathfrak{h}_{2m+1} = \langle x_i, y_i, z : i = 1, \ldots, m \rangle,$$

where non-zero Lie brackets are given by $[x_i, y_i] = z$ for every $1 \leq i \leq m$.

(ii) *The Lie algebra*

$$\mathfrak{s}_{5,45} = \langle x_1, x_2, y_1, y_2, z \rangle,$$

where non-zero Lie brackets are given by

$$[x_1, y_1] = y_1, [x_1, y_2] = y_2, [x_1, z] = 2z, [x_2, y_1] = y_2, [x_2, y_2] = -y_1, [y_1, y_2] = z.$$

Proof. The proof will be divided into 3 cases.

• **Case 1.** Assume $\dim \mathcal{G}^1 = 1$. Followed by the classifications of solvable real Lie algebras having 1-dimensional derived ideal in [29], we easily see that $\mathcal{G}$ is isomorphic to $\mathfrak{h}_{2m+1}$ with $2 \leq m \in \mathbb{N}$ because $\mathcal{G}$ is indecomposable and $n \geq 4$.

• **Case 2.** Assume that $\mathcal{G}^1$ is commutative and $\dim \mathcal{G}^1 \geq 2$. We will show that this case is excluded.

We first claim that $\dim \mathcal{G}$ is exactly five. Indeed, according to Theorem 3.2, one can see that $\mathcal{G} = \mathcal{L} \oplus_\rho \mathcal{G}^1$ where $\mathcal{L}$ is a commutative subalgebra of $\mathcal{G}$ with $\mathcal{L} \cap \mathcal{G}^1 = \{0\}$. Hence, we can choose a basis $\mathfrak{b} = \{x_1, x_2, \ldots, x_n\}$ of $\mathcal{G}$ so that

$$\begin{cases} \mathcal{G}^1 = \langle x_{n-m+1}, x_{n-m+2}, \ldots, x_n \rangle, \\ \mathcal{L} = \langle x_1, x_2, \ldots, x_{n-m} \rangle. \end{cases} \tag{29}$$

Because both $\mathcal{G}^1$ and $\mathcal{L}$ are commutative, we have

$$[x_i, x_j] = 0 \quad \text{if } \begin{cases} 1 \leq i, j \leq n-m, \\ n-m+1 \leq i, j \leq n. \end{cases} \tag{30}$$

On the other hand, the commutation of $\mathcal{G}^1$ implies that $\{\mathrm{ad}_x^1 : x \in \mathcal{L}\}$ is a family of commuting endomorphisms (Proposition 2.3). As a consequence of the second item of Proposition 2.4, we can assume that the matrix of $\mathrm{ad}_{x_i}^1$ with respect to the basis $\mathfrak{b}$ is of block upper triangular form in which each block is of size up to 2, for every $1 \leq i \leq n-k$. With this assumption in hand, the equality (30) implies that

$$x_n^*([x_i, x_j]) = 0 \quad \text{if} \left\{ \begin{array}{l} 1 \leq i, j \leq n-2, \\ n-1 \leq i, j \leq n. \end{array} \right. \tag{31}$$

Therefore,

$$0 \neq \dim \Omega_{x_n^*} = \operatorname{rank} B_{x_n^*} \leq 4. \tag{32}$$

Hence, $n - 1 = \dim \Omega_{x_n^*} \leq 4$, or $n \leq 5$. Since $\dim \Omega_F$ is even for every $F \in \mathcal{G}^*$ and $n \geq 4$, n must be equal to 5.

To the best of our knowledge, all 1-step solvable real Lie algebras of dimension 5 are completely classified by Jacqueline Dozias [8, 23]. In particular, L. A. Vu and K. P. Shum gave the classification of 1-step solvable MD-algebras of dimension 5 in [30]. According to their classification, there is no indecomposable 1-step solvable MD-algebra of dimension 5 so that its maximal dimension of K-orbits is 4. Therefore, this case is excluded.

• **Case 3.** Assume that $\mathcal{G}^1$ is non-commutative. If so, $\dim \mathcal{G}^1 \geq 2$ and $\dim \mathcal{G}^2 \geq 1$. It follows from Theorem 3.5 that $\dim \mathcal{G}^2 \leq \dim \mathcal{G} - \dim \Omega_F = 1$ for every $0 \neq F \in (\mathcal{G}^1)^*$. Hence, $\dim \mathcal{G}^2 = 1$ and $\mathcal{H} = \mathcal{G}/\mathcal{G}^2$ is a solvable real Lie algebra in which $\dim \mathcal{H} = n-1 = \dim \Omega_F$ for every $0 \neq F \in (\mathcal{G}^1)^*$. In other word, $\mathcal{H}$ is an indecomposable SMD-algebra. It follows from [24, Theorem 1] that $n - 1 = \dim \mathcal{H} \leq 4$. Hence, $n \leq 5$. Once again, by using the classification of 5-dimensional Lie algebras in [8, 23], especially of those in which the first derived ideal is non-commutative in [29], we get exactly one 5-dimensional Lie algebra satisfying the requirement which is defined in item 3.6. Remark that this algebra is denoted as $\mathfrak{s}_{5,45}$ in [23] and the proof is complete. □

Remark 3.7. To sharpen Theorem 3.6, using Theorem 3.1, we complete the paper by the following remarks.

(i) All solvable Lie algebras of dimension 3 are obviously MD-algebras. Therefore, except $\mathbb{R}^3$ (the real 3-dimensional abelian Lie algebra), all the remaining solvable non-abelian ones are $\mathrm{MD}_2(3)$-algebras. One can find their classification in [23].

(ii) Any decomposable $\mathrm{MD}_{n-1}(n)$-algebra is a direct sum of $\mathbb{R}$ and an indecomposable non-abelian SMD-algebra of dimension $n-1$. Therefore, by applying directly the classification of SMD-algebras in [24, Theorem 1], we easily see that any decomposable $\mathrm{MD}_{n-1}(n)$-algebra is isomorphic to either $\mathrm{aff}(\mathbb{R}) \oplus \mathbb{R}$ for $n = 3$ or $\mathrm{aff}(\mathbb{C}) \oplus \mathbb{R}$ for $n = 5$ where
   – $\mathrm{aff}(\mathbb{R}) = \langle x, y \rangle$ is defined by $[x, y] = y$;
   – $\mathrm{aff}(\mathbb{C}) = \langle x_1, x_2, y_1, y_2 \rangle$ is defined by $[x_1, x_2] = [y_1, y_2] = 0$, $[x_1, y_1] = y_1$, $[x_1, y_2] = y_2$, and $[x_2, y_1] = y_2$, $[x_2, y_2] = -y_1$.

(iii) The nilradical of the Lie algebra $\mathfrak{s}_{5,45}$ is equal to its derived algebra $\mathfrak{s}_{5,45}^1$, which is also the unique Heisenberg Lie algebra of dimension 3. Therefore, $\mathfrak{s}_{5,45}$ is a solvable extension of the Heisenberg Lie algebra

$\mathfrak{h}_3$. For more details about solvable extensions of nilradicals, we refer readers to [15–17, 22].

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