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ON THE SEMI-LOCAL CONVERGENCE OF CONTRAHARMONIC-MEAN NEWTON'S METHOD (CHMN)

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Abstract. The main objective of this work is to investigate the study of the local and semi-local convergence of the contraharmonic-mean Newton's method (CHMN) for solving nonlinear equations in a Banach space. We have performed the semi-local convergence analysis by using generalized conditions. We examine the theoretical results by comparing the CHN method with the Newton's method and other third order methods by Weerakoon et al. using some test functions. The theoretical and numerical results are also supported by the basins of attraction for a selected test function.

1. Introduction

Let $F : D \subset X \to X$ be an operator, where D is an open convex subset of a Banach space X and F is a Fréchet differentiable operator at each point of D. We study the convergence of contraharmonic-mean method defined for $x_0 \in D$ and $n = 0, 1, 2, \ldots$,

$$(1)\quad \begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n + F'(x_n)^{-1}F'(y_n)(F'(y_n) - F'(x_n))(F'(x_n)^2 + F'(y_n)^2)^{-1}F(x_n), \end{cases}$$

where $F'(x)$ is the Fréchet derivative of operator F at the point $x \in D$. We use the method (1) to find a solution x^* of non linear operator equation of the form

$$(2)\qquad F(x) = 0.$$

Newton's method is most commonly used for solving such equations. But it is only of order two under some conditions [1–23]. It is defined as follows:

$$(3)\qquad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots.$$

There are several papers on the variant or modification of Newton's method in a real space ([9, 10, 18, 19]), and in a Banach space ([7, 8]). Generally, we use two types of convergence analysis for the solutions of nonlinear equation

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(2). First is the local convergence analysis in which we start with the assumption of existence of the particular solution, around this solution there exists a neighborhood and starting with any vector in this neighborhood lead to a sequence which converges to the root under some suitable conditions. Second is the semilocal convergence analysis. It does not require the knowledge of the existence of a solution, rather than demands the same conditions around the initial vector.

The semilocal convergence of Newton's method in Banach spaces was established by Kantorovich in [13]. The convergence of the sequence obtained by the iterative expression is derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton's methods (see, for example, [2, 3]). The earlier study of the contraharmonic-mean Newton method for the solution of (2) in a real space uses the higher order dericatives while the method involves only the first-order derivative, hence it limits the applications of the method (1). In this paper we present the semi-local convergence using only the first-order Fréchet derivative and generalized conditions.

The convergence order is found using the following formula:

$$\mu(COC) = \frac{\ln\Big(\|(x_{n+1}-x_*)\|/\|(x_n-x_*)\|\Big)}{\ln\Big(\|(x_n-x_*)\|/\|(x_{n-1}-x_*)\|\Big)}, \tag{4}$$

or

$$\mu_1(ACOC) = \frac{\ln\Big(\|(x_{n+1}-x_n)\|/\|(x_n-x_{n-1})\|\Big)}{\ln\Big(\|(x_n-x_{n-1})\|/\|(x_{n-1}-x_{n-2})\|\Big)}. \tag{5}$$

These do not require the F''' or x_* (in (5)).

Numerical results consist of the comparative study of the proposed method along with the Newton's method and the method of Weerakoon et al. (see [22]) by using the some test functions for nonlinear equations and system of equations. One important aspect is the discussion of the extraneous fixed points and the comparative study of the dynamics of the CHN method with the Newton's method and the method of Weerakoon et al. [22] for the solution of non linear equation.

2. Local convergence analysis

Theorem 1. *Let I be a convex subset of $\Re^i$ and $F : I \to \Re^i$ be a function such that*

(a) *F has a simple root $x^* \in I$,*

(b) *Jacobian matrix $F'(x^*)$ is non singular at the root x^* and,*

(c) *F is a third order Fréchet differential in the convex set I at some neighbourhood S of the root x^*. Then, the iterative method* (1) *has convergence of third order to the root x^*.*

Proof. Let $x^* \in I$ be a simple zero of a function F, $e_n = x_n - x^*$ and $A_k = \left(\frac{1}{k!}\right) F'(x^*)^{-1} F^{(k)}(x^*)$. Using Taylor expansion and taking into account $F(x^*) = 0$, we get

$$F(x_n) = F'(x^*)\left[e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right], \tag{6}$$

$$F'(x_n) = F'(x^*)\left[1 + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + O(e_n^4)\right]. \tag{7}$$

Then, from (6) and (7), we get

$$F'(x_n)^{-1} F(x_n) = e_n - A_2 e_n^2 + \left(2A_2^2 - 2A_3\right) e_n^3 + O(e_n^4).$$

Since $y_n = x_n - F'(x_n)^{-1} F(x_n)$, we obtain

$$y_n = x^* + A_2 e_n^2 + \left(2A_3 - 2A_2^2\right) e_n^3 + (4A_2^3 - 7A_2 A_3 + 3A_4) e_n^4 + O(e_n^5). \tag{8}$$

Hence, we have

$$F(y_n) = F'(x^*)[A_2 e_n^2 - 2(A_2^2 - A_3) e_n^3 + (5A_2^3 - 7A_2 A_3 + 3A_4) e_n^4 + O(e_n)^5], \tag{9}$$

$$F'(y_n) = F'(x^*)[1 + 2C_2^2 e_n^2 + 4C_2(C_3 - C_2^2) e_n^3 + O(e_n^4)]. \tag{10}$$

We have again

$$F'(x_n)^2 = F'(x^*)^2[1 + 4C_2 e_n + (4C_2^2 + 6C_3) e_n^2 + (8C_4 + 12C_2 C_3) e_n^3 + O(e_n^4)], \tag{11}$$

$$F'(y_{(n)})^2 = F'(x^*)^2[1 + 4C_2^2 e_n^2 + (8C_2 C_3 - 8C_2^3) e_n^3 + O(e_n^4)]. \tag{12}$$

Therefore by the CHN method, we have

$$e_{n+1} = (2C_2^2 + \frac{C_3}{2}) e_n^3 + O(e_n^4). \tag{13}$$

Equation (13) confirms that the CHN method (1) converges with third-order to the root of F locally, if there exists a third-order Fréchet differentiable operator in an open convex domain I. □

3. Semi-local convergence analysis

We need the following Ostrowski-type results connecting the iterates of method (1).

Lemma 1. *Suppose that the iterates of method* (1) *are well defined for all* $n = 0, 1, 2 \ldots.$ *and some* $x_0 \in D$. *Then, the following estimates hold*

$$\begin{aligned} y_{n+1} - x_{n+1} = & -F'(x_{n+1})^{-1}\Big[\int_0^1 (F'(x_n + \theta(x_{n+1} - x_n)) \\ & - F'(x_n))(x_{n+1} - x_n) d\theta - F'(x_n)(y_n - x_{n+1})\Big] \end{aligned} \tag{14}$$

and

$$\begin{aligned} x_{n+1} - y_n = & F'(x_n)^{-1} F'(y_n)(F'(y_n) - F'(x_n))(F'(x_n)^2 \\ & + F'(y_n)^2)^{-1} F'(x_0)^2 F'(x_0)^{-1} F'(x_0)^{-1} F(x_{n+1}). \end{aligned} \tag{15}$$

Proof. By the first substep of method (1) we can write

$$F(x_{n+1}) = F'(x_{n+1}) - F(x_n) - F'(x_n)(y_n - x_n).$$

Hence, we further get

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ = & -F'(x_{n+1})^{-1}\Big[F(x_{n+1}) - F(x_n) - F'(x_n)(y_n - x_n)\Big] \\ = & \ F'(x_{n+1})^{-1}\Big[\int_0^1 (F'(x_n + \theta(x_{n+1} - x_n))d\theta - F'(x_n))(x_{n+1} - x_n) \\ & + F'(x_n)(x_{n+1} - y_n)\Big], \end{aligned} \tag{16}$$

which shows estimate (14). Then, estimates (15) follows immediately from the second substep of the method (1). □

Scalar parameters and functions are needed for the semilocal convergence analysis that follows: Let $w_0 : [0, \infty) \to [0, \infty)$ be a nondecreasing and continuous function.

Suppose that equation

$$w_0(t) - 1 = 0 \tag{17}$$

has a minimal positive solution δ_0. Let $a > 0$ and $w : [0, \delta_0) \to [0, \infty)$, $v : [0, \delta_0) \to [0, \infty)$ and $v_0 : [0, \delta_0) \to [0, \infty)$ be continuous and non decreasing functions.

Suppose that equation

$$v_0(t) - 1 = 0 \tag{18}$$

has a minimal positive solution δ_1.

Set $\delta_2 = \min\{\delta_0, \delta_1\}$, $s_0 > 0$,

$$\begin{aligned} t_1 &= s_0 + \frac{a v_0(s_0) w(s_0) b_0}{2(1 - w_0(0))(2 - \frac{1}{2}(v_0 + v_0(s_0)))}, \\ r_1(t) &= \frac{\int_0^1 w(\theta t_1)d\theta + v_0(t)}{1 - w_0(t)}, \\ r_2(t) &= \frac{a v_0(t) w(t) \int_0^1 w(\theta t_1)d\theta + v_0(t)}{2(1 - w_0(t))(1 - v_0(t))}, \\ r(t) &= \max\{r_1(t), r_2(t)\},\ t \in [0, \delta_2), \\ b_0 &= \int_0^1 w(\theta t_1)d\theta t_1 + v_0(0)(t_1 - s_0). \end{aligned}$$

Suppose that equation

$$\delta_0 + \frac{2r(t)t_1}{1 - r(t)} - t = 0$$

has a minimal solution $\delta \in (0, \delta_2)$ satisfying $w_0(\delta) < 1$ and $v(\delta) < 1$. Moreover consider the iterations defined for all $n = 0, 1, 2, \dots$ by

$$t_{n+1} = s_n + \frac{av_0(s_n)w(s_n - t_n)b_n}{2(1 - w_0(t_n))(1 - \frac{1}{2}(v(t_n) + v(s_n)))}, \tag{19}$$

$$s_{n+1} = t_{n+1} + \frac{\int_0^1 w(\theta(t_{n+1} - t_n))d\theta(t_{n+1} - t_n) + v_0(t_n)(t_{n+1} - s_n)}{1 - w_0(t_{n+1})},$$

where

$$b_n = \int_0^1 w(\theta(t_{n+1} - t_n))d\theta(t_{n+1} - t_n) + v_0(t_n)(t_{n+1} - s_n).$$

Then, we have the following results on mazorizing sequences for method (1).

Lemma 2. *Under the prestated hypotheses, the sequences $\{t_n\}$ and $\{s_n\}$ are nondecreasing bounded from above by δ and converge to the common least upper bound t_* satisfying $t_1 \leq t_* \leq \delta$.*

Proof. By the definitions of scalar sequences we have

$$0 \leq s_n \leq t_{n+1} \leq t_{n+1} \leq t_{n+1} \leq \delta,$$

$$0 \leq s_{n+1} - t_{n+1} \leq r(t_{n+1} - t_n) \leq r^{n+1}(t_1 - t_0), \ r = r(\delta)$$

and

$$0 \leq t_{n+1} - s_n \leq r(t_{n+1} - t_n) \leq r^{n+1}(t_1 - t_0).$$

These estimates lead to

$$\begin{aligned} s_{n+1} &\leq t_{n+1} + r^{n+1}t_1 \leq s_n + 2r^{n+1}t_1 \\ &\leq s_{n-1} + 2r^{n+1}t_1 \leq s_0 + 2rt_1 + \cdots + 2r^n t_1 + 2r^{n+1}t_1 \\ &= s_0 + 2rt_1(1 + r + \cdots + r^n) \\ &= s_0 + 2rt_1\frac{1 - r^{n+1}}{1 - r}t_1 \\ &< \delta. \end{aligned} \tag{20}$$

Hence, the sequences $\{t_n\}$ and $\{s_n\}$ are non decreasing bounded from above by δ and as such they converge to their unique least upper bound t_*. □

The following hypotheses (A) are utilized in the semilocal convergence analysis of method (1).

(A_1) $F : D \to Y$ is continuously Fréchet-differentiable and there exists $x_0 \in D$ such that $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq s_0$.

(A_2) There exists a continuous and nondecreasing function $w_0 : [0, \infty) \to [0, \infty)$ such that $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq w_0(\|(x - x_0)\|)$ $\forall x \in D$.

(A_3) There exist $a > 0$ and continuous and nondecreasing functions $w : [0, \delta_0) \to [0, \infty)$, $v : [0, \delta_0) \to [0, \infty)$, $v_0 : [0, \delta_0) \to [0, \infty)$ such that for all $x, y \in B(x_0, s_0)$

$$\|F'(x_0)^{-1}\| < a,$$

$$\|F'(y) - F'(x)\| \leq w\|(y - x)\|,$$
$$\|F'(x_*)^{-2}(F'(x)^2 - F'(x_*)^2)\| \leq v(\|x - x_*\|),$$
$$\|F'(x_0)^{-1}F'(x)\| \leq v_0(\|x - x_0)\|.$$

(A_4) Hypothesis of Lemma 2 hold.

(A_5) $\bar{B}(x_0, t_*) \subset D$, where $t_* = \lim_{n\to\infty} t_n$.

(A_6) There exists $t_{**} \geq t_*$ such that $\int_0^1 w_0(\theta t_{**})d\theta < 1$.

Set $B_1 = D \cap \bar{B}(x_0, t_{**})$.

Then, using the proceeding notations and hypothesis (A), we show the main semilocal convergence analysis result for the method (1).

Theorem 2. *Under the hypothesis* (A)*, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by the method CHN are well defined in* $U(x_0, t_*)$ *remain in* $U(x_0, t_*)$ *and converge to a solution* x_* *of equation* $F(x) = 0$*, so that*

$$\|y_n - x_n\| \leq s_n - t_n,$$
$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n,$$
$$\|x_n - x_*\| \leq t_* - t_n, \text{ and}$$
$$\|y_n - x_*\| \leq t_* - s_n.$$

Proof. The preceding estimates follow immediately by mathematical induction the triangle inequality, Lemma 1, Lemma 2, and consequences of the Banach lemma on invertible operators [13] for $x_0 \in B(x_0, t_*)$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq w_0\|(x - x_0)\| \leq W_0(t_*) < 1,$$

so

$$\begin{aligned}
&\|F'(x_0)^{-1}F'(x)\| \leq \frac{1}{(1 - w_0)\|(x - x_0)\|},\\
&\|(2F'(x_0)^2)^{-1}(F'(x)^2 + F'(y)^2 - 2F'(x_0)^2)\|\\
\leq{}& \frac{1}{2}[\|(F'(x_0)^2)^{-1}(F'(x)^2 - F'(x_0)^2)\| + \|(F'(x_0)^2)^{-1}(F'(y)^2 - F'(x_0)^2)\|]\\
\leq{}& \frac{1}{2}\Big(v(\|x - x_0\|) + v(\|y - x_0\|)\Big)\\
\leq{}& \frac{1}{2}\Big(v(t_*) + v(t_*)\Big)\\
={}& v(t_*)\\
<{}& 1,
\end{aligned}$$

so

$$\|(F'(x)^2 + F'(y)^2)^{-1}F'(x_0)^2\| \leq \frac{1}{2(1 - \frac{1}{2}v(\|x - x_0\|) + v(\|y - x_0\|))}.$$

These estimates lead to

$$\|y_{n+1} - x_{n+1}\| \leq \frac{\bar{b}_n}{1 - w_0(\|x_{n+1} - x_0\|)} \leq s_{n+1} - t_{n+1},$$

where

$$\bar{b}_n = \int_0^1 w(\theta\|x_{n+1} - x_n\|)d\theta\|x_{n+1} - x_n\| + v_0(\|x_n - x_0\|)\|y_n - x_{n+1}\|$$

$$\leq b_n = \int_0^1 w(\theta(t_{n+1} - t_n))d\theta(t_{n+1} - t_n) + v_0(t_n)(t_{n+1} - s_n) \tag{21}$$

and

$$\begin{aligned}
&\|x_{n+1} - y_n\| \\
&= F'(x_n)^{-1}F'(y_n)(F'(y_n) - F'(x_n))(F'(x_n)^2 + F'(y_n)^2)^{-1} \\
&\quad F'(x_0)^2F'(x_0)^{-1}F'(x_0)^{-1}F(x_n) \\
&\leq \frac{av_0(\|y_n - x_0\|)w(\|y_n - x_n\|)\bar{b}_n}{2(1 - w_0(\|x_n - x_0\|))(1 - \frac{1}{2}(v(\|x_n - x_0\|)) + v(\|y_n - x_0\|))} \\
&\leq \frac{av_0(s_n)w(s_n - t_n)\bar{b}_n}{2(1 - w_0t_n)(1 - \frac{1}{2}(v(t_n) + v(s_n)))} \\
&= t_{n+1} - s_n.
\end{aligned}$$

Hence, the sequences $\{x_n\}$ and $\{y_n\}$ are fundamental in a Banach space X, and as such they converge to some $x_* \in \bar{B}(x_0, t_*)$, since $\bar{B}(X_0, t_*)$ is closed. By letting $n \to \infty$ in the first estimate of Lemma 1 and using continuity of F we obtain $F(x_*) = 0$.

Finally to show the uniqueness part let $Q = \int_0^1 F'(x_* + \theta(x_{**} - x_*))d\theta$ for some $x_{**} \in B_1$ with $F(x_{**}) = 0$. Using (A_2), (A_5), and definition of B_1, we have in turn

$$\begin{aligned}
\|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq \int_0^1 w_0(\theta\|(x_* - x_0)\|) + (1 - \theta\|(x_{**} - x_0)\|)d\theta \\
&\leq \int_0^1 w_0(\theta t_* + (1 - \theta)t_{**})d\theta \\
&< 1. \qquad (22)
\end{aligned}$$

So Q^{-1} exists. But then, we can write

$$0 = F'(x_{**}) - F'(x_*) = Q(x_{**} - x_*),$$

leading to $x_{**} = x_*$ by the invertibility of Q. □

4. Numerical results

In this section, we study the efficiency of iterative method (1), We have done the comparative study of the CHN method along with the classical Newton's method (3), and third order Weerakoon et al. method [22] whose iterative expression is given by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = x_n - (F'(x_n) + F'(y_n))^{-1}2F(x_n), \quad n = 0, 1, 2, \ldots. \end{cases} \tag{23}$$

TABLE 1. Comparison of the different methods for Example 1

Method	N	x	$f(x)$
Newton Method	1	1.00000000	1.71828182845905
	2	0.36787944117144	0.44466786100977
Cpu 0.01560 Sec	3	0.06008006872679	0.06192156984951
	4	0.00176919944264	0.00177076539934
	5	1.564110789898428e-006	1.564112013019425e-006
	6	1.223321565989411e-012	1.223243728531998e-012
	7	7.783745890945912e-017	0.00000000
Weerakoon Method	1	1.00000000	1.71828182845905
	2	0.17448830438386	0.19063681707241
Cpu 0.01760 Sec	3	0.00158415756260	0.00158541300305
	4	1.323865209948080e-009	1.323865239655220e-009
	5	−2.883082976183813e-017	0.00000000
CHMN Method	1	1.00000000	1.71828182845905
	2	0.24514254757422	0.27780344800498
Cpu 0.01680 Sec	3	0.00698738950784	0.00701185827164
	4	1.979112795331278e-007	1.979112991268295e-007
	5	−9.258546569522333e-018	0.00000000

Example 1. Let $X = R$, $D = (-1, 1)$ and $F : D \to R$ be a function defined by

$$F(x) = e^x - 1, \ \forall x \in D.$$

Then, F is Fréchet differentiable and its Fréchet derivative $F'(x)$ at any point $x \in D$ is given by

$$F'(x) = e^x.$$

We have computed the numerical results with the help of MATLAB 2007 and the stopping criterion used for the computation is $|x_{n+1} - x_*| + |f(x_{n+1})| < 10^{-14}$.

The initial approximation is 1.0 and approximate solution is 0. The numerical solution of Example 1 by 2^{nd} order Newton's method (2), 3^{rd} order method of Weerakoon et al. [22] and 3^{rd} order CHMN method (1) is given in Table 1. Numerical results in Table 1 reveals that starting with the point 1.0, the CHMN method is well competing to the other method in converging to root 0.

Example 2. Let $D = X = Y = \mathbb{R}^2$. Consider an operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x) = \big(-x^2 + 1/3, -y^2 + 1/3 \big), \ \forall x = (x, y) \in \mathbb{R}^2.$$

The starting vector is $[0.5, 0.5]$ and approximate solution is $[0.57735, 0.57735]$. The numerical solution of Example 2 by 2^{nd} order Newton's method (2), 3^{rd} order method of Weerakoon et al. [22] and CHMN method (1) is given in Table

TABLE 2. Comparison of the different methods for Example 2

Method	N	x	y	$f(x,y)$	$g(x,y)$
	1	0.500000	0.500000	0.0833333	0.0833333
Newton Method	2	0.583333	0.583333	-0.00694444	-0.00694444
	3	0.577381	0.577381	-0.0000354308	-0.0000354308
Cpu 0.1560 Sec	4	0.57735	0.57735	-9.41408×10^{-10}	-9.41408×10^{-10}
	5	0.57735	0.57735	0.000000	0.000000
	1	0.500000	0.500000	0.0833333	0.0833333
Weerakoon Method	2	0.576923	0.576923	0.000493097	0.000493097
	3	0.57735	0.57735	6.75901×10^{-11}	6.75901×10^{-11}
Cpu 0.2360 Sec	4	0.57735	0.57735	0.000000	0.000000
	1	0.500000	0.500000	0.0833333	0.0833333
CHMN Method	2	0.576471	0.576471	0.00101499	0.00101499
	3	0.57735	0.57735	1.18221×10^{-9}	1.18221×10^{-9}
Cpu 0.2160 Sec	4	0.57735	0.57735	0.000000	0.000000

2. Numerical results show that the CHMN method is converging to root in a very well manner.

5. Corresponding conjugacy maps for quadratic polynomials

In this section, we will discuss the rational map $R(z)$ arising from various methods applied to a generic polynomial with simple roots.

Theorem 3 (Newton's method)**.** *For a rational map $R(z)$ arising from Newton's method applied to $P(z) = (z-a)(z-b)$, $a \neq b$, $R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z-a)/(z-b)$ to*

$$S(z) = MoRoM^{-1}(z) = M\Big(R\big(\frac{zb-a}{z-1}\big)\Big),$$

$$S(z) = z^2.$$

Theorem 4 (Weerakoon et al. method [22])**.** *For a rational map $R(z)$ arising from Weerakoon et al. method* [22] *equation* (23) *applied to $P(z) = (z-a)(z-b)$, $a \neq b$, $R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z-a)/(z-b)$ to*

$$S(z) = z^3 M(z),$$

where, $M(z) = 1$.

Theorem 5 (Proposed method CHMN)**.** *For a rational map $R(z)$ arising from proposed method CHMN* (1) *applied to $P(z) = (z-a)(z-b)$, $a \neq b$, $R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z-a)/(z-b)$ to*

$$S(z) = z^3 M(z),$$

where, $M(z) = (2+z+z^2)/(1+z+2z^2)$.

Theorem 6 (Newton like method)**.** *For a rational map $R(z)$ arising from Newton like method applied to $P(z) = (z - a)(z - b)$, $a \neq b$, $R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z - a)/(z - b)$ to*

$$S(z) = z^p M(z),$$

where, $M(z)$ is either unity or a rational function and p is the order of the Newton like method

6. Extraneous fixed points

The Newton like iterative methods discussed in earlier sections can be written in the fixed-point iteration form as

$$x_{n+1} = x_n - E_f(x_n)\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots. \tag{24}$$

Clearly, the root x_* of $f(x) = 0$ is a fixed point of the method. However, the points $\xi \neq x_*$ at which $E_f(\xi) = 0$ are also fixed points of the method as, with $E_f(\xi) = 0$, second term on right side of (24) vanishes. These points are called extraneous fixed points (see [20]). In this section, we will discuss the extraneous fixed points of some Newton like method for the polynomial $z^3 - 1$.

Theorem 7. *There are no extraneous fixed points for Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \dots$.*

Proof. For Newton's method, we have $E_f(x_n) = 1$. Hence, it has no extraneous fixed point. □

Theorem 8. *There are no extraneous fixed points for method of Weerakoon et al.* [22] *given by equation* (23)*.*

Proof. For method of (23), $E_f(x_n)$ given by the following equation:

$$(18z^6)/(1 + 4z^3 + 13z^6).$$

In this equation numerator is of degree 6 but it has no extraneous fixed points. □

Theorem 9. *There are* 6 *extraneous fixed points for the proposed method CHMN* (1)*.*

Proof. For the proposed method CHMN (1) we have $E_f(x_n)$ given by the following equation.

$$(9z^6(1 + 4z^3 + 13z^6))/(1 + 8z^3 + 24z^6 + 32z^9 + 97z^12).$$

In this equation numerator is of degree 12. The proposed method CHMN (1) has 6 extraneous fixed points.

$$\begin{aligned} z = & -0.617460638817014801375177189 4392 \\ & -0.20983977171802813185600539068 16i, \\ z = & -0.617460638817014801375177189 4392 \end{aligned}$$

$$+\,0.2098397717180281318560053906816i,$$
$$z = 0.1270037463763676588679558109348$$
$$-\,0.6396564849115167404893886044665i,$$
$$z = 0.1270037463763676588679558109348$$
$$+\,0.6396564849115167404893886044665i,$$
$$z = 0.4904568924406471425072213785045$$
$$-\,0.4298167131934886086333832137849i,$$
$$z = 0.4904568924406471425072213785045$$
$$+\,0.4298167131934886086333832137849i.$$

These fixed points are repelling (the derivative at these points has its magnitude > 1). □

Remark. Similarly we may calculate the extraneous fixed points for other Newton like method. These fixed points are repelling (the derivative at these points has its magnitude > 1). These fixed points can be seen in the basin of attractions plot for Example 3 (z^3-1), Figure 2 (see dynamics of methods Subsection 7.2).

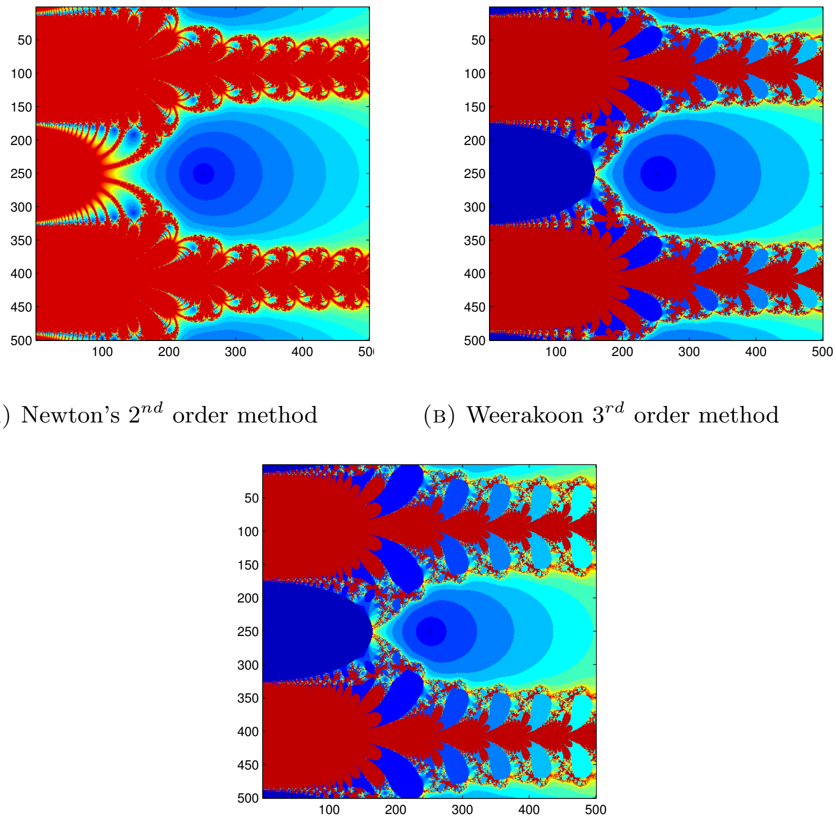
7. Dynamics of methods

We have studied dynamics and fractal patterns of the function $f(x) = (e^x - 1)$ by using different iterative methods. The dynamics of the function by iterative methods usually help us to study the convergence and stability of the methods. The basic definitions and dynamical concepts of function can be found in [1, 7].

7.1. For Example 1

We have taken a square $R \times R = [-5.0, 5.0] \times [-5.0, 5.0]$ of 500×500 points to study the dynamics of function $f(x) = (e^x - 1)$. If with every starting point $z_{(0)}$ in the above squares our numerical iterative methods generate a sequence that converges to a zero z^* of the function with a tolerance- $f(z_n) < 5 \times 10^{-2}$ and a maximum of 21 iterations, then we say that $z_{(0)}$ will lie in the basin of attraction of this zero, and we assign a fixed color to this point $z_{(0)}$. we have described the basins of attraction for Newton's method, method of Weerakoon et al. [22] and CHMN method for finding complex roots of above mentioned functions (Figure 1).

(i) The basin of attraction for all the iterative methods contains fractal Julia set and basin of all the methods look almost similar.
(ii) The Julia set with the red colour show the failure of the method. The Newton's method contains the largest area of Julia set with red color while the CHMN method have smallest red color area.



(A) Newton's 2^{nd} order method (B) Weerakoon 3^{rd} order method

(C) CHN 3^{rd} order method

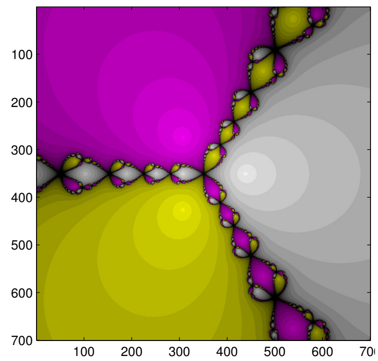
FIGURE 1. Basin of attraction for $e^x - 1$ by different methods

(iii) Again the fatou set with blue color shows the basins of the methods. The blue color area shows that CHMN method contains the largest fatou set.

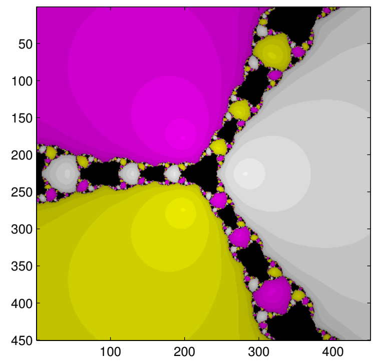
7.2. For Example 3

We have also considered Example 3 for the illustrations of the dynamics of the iterative methods under the same previous conditions. We have plotted the fractal patterns graph of Example 3 ($F(z) = z^3 - 1$) for the different iterative methods with a fixed different color to each root of the basins of attraction.

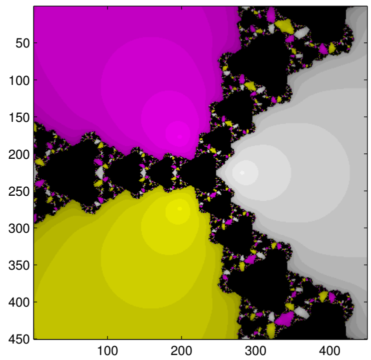
We can see the extraneous fixed points for Newton like methods in the basins of attraction for Example 3 ($z^3 - 1$) Figure 2. These fixed points are repelling (the derivative at these points has its magnitude > 1). Clearly, there is no



(A) Newton's 2^{nd} order method



(B) Weerakoon 3^{rd} order method



(C) Proposed CHMN 3^{rd} order method

FIGURE 2. Basin of attraction for $f_2 = z^3 - 1$ by different methods

extraneous fixed points for Newton method and Weerakoon et al. 3^{rd} order method [22]. The proposed method CHMN has 6 extraneous fixed points.

8. Conclusion

We discussed a third- order Newton-like method for solving nonlinear equations in Banach space. We performed the two convergence analyses for the method CHMN. Local convergence analysis demands the third- order differentiability but the semilocal convergence analysis needs only the first-order Fréchet derivative. We have studied about the extraneous fixed points and they are repulsive. Theoretical results are checked by the numerical examples and numerical results are examined with the basin of attractions for a selected example. All the results (Theoretical, numerical, dynamical) are fruitful for the further study of Newton like methods.

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