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GENERALIZED PADOVAN SEQUENCES

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Abstract. The Padovan sequence is the third-order linear recurrence $(\mathcal{P}_n)_{n\geq 0}$ defined by $\mathcal{P}_n = \mathcal{P}_{n-2} + \mathcal{P}_{n-3}$ for all $n \geq 3$ with initial conditions $\mathcal{P}_0 = 0$ and $\mathcal{P}_1 = \mathcal{P}_2 = 1$. In this paper, we investigate a generalization of the Padovan sequence called the k-generalized Padovan sequence which is generated by a linear recurrence sequence of order $k \geq 3$. We present recurrence relations, the generalized Binet formula and different arithmetic properties for the above family of sequences.

1. Introduction

There are currently several integer sequences widely studied which are used in almost every field of modern sciences. A classic example is the Fibonacci sequence $F = (F_n)_{n\geq 0}$ which has an extensive bibliography describing its curious properties. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. Cooper-Howard [14] and Dresden-Du [8] investigated a generalization of the Fibonacci sequence given by a recurrence relation of a higher order. They considered, for an integer $k \geq 2$, the k-Fibonacci sequence which is like the Fibonacci sequence but starting with $0, 0, \ldots, 0, 1$ (a total of k terms) and each term afterwards is the sum of the k preceding terms. Many arithmetic properties have recently been studied for generalized Fibonacci sequences. For instance, to cite only a few examples, Fibonacci numbers, and more generally k-Finonacci numbers, which are repdigits were studied in [3,16,17]. Particular representations of k-Fibonacci numbers were treated in [1] and [2]. We refer to [4] and [13] for results on the largest prime factor of k-Fibonacci numbers.

A relatively young sequence as important as the Fibonacci sequence is the Padovan sequence, named after the mathematician R. Padovan who attributed its discovery to Dutch architect Dom Hans van der Laan in his 1994 essay *Dom Hans van der Laan*: *Modern Primitive* [19]. The Padovan sequence, denoted by

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$\mathcal{P} = (\mathcal{P}_n)_{n\geq 0}$ is the ternary recurrence sequence given by $\mathcal{P}_0 = 0$, $\mathcal{P}_1 = \mathcal{P}_2 = 1$ and the recurrence formula $\mathcal{P}_n = \mathcal{P}_{n-2} + \mathcal{P}_{n-3}$ for all $n \geq 3$ (see, sequence A000931 in Sloane's Encyclopedia [20]).

Recently several authors have studied arithmetic problems involving terms of the Padovan sequence. For example, Steward in [21] asked for the intersection of the Fibonacci and Padovan sequences, while De weger in [22] solved this problem by proving that the distance between Fibonacci and Padovan numbers growths exponentially. Diophantine equations involving Padovan numbers have been also considered. For example, García and Hernández found all the repdigits and the powers of 2 that can be written as sums of two Padovan numbers (for more details see [10,11]). Additionally, Ddamulira in [5] and [6] looked for all repdigits that can be written as sums of three Padovan numbers and all Padovan numbers that are concatenations of two repdigits. Other diophantine problems involving the Padovan sequence can be consulted in [7, 12].

In this paper we study a generalization of the Padovan sequence which is generated by a recurrence relation of higher order, i.e., we consider, for an integer $k \geq 3$, the *k-generalized Padovan sequence* or, for simplicity, the *k-Padovan sequence* $\mathcal{P}^{(k)} = (\mathcal{P}_n^{(k)})_{n\geq -(k-3)}$ whose terms satisfy the recurrence relation of order k

$$\mathcal{P}_n^{(k)} = \mathcal{P}_{n-2}^{(k)} + \mathcal{P}_{n-3}^{(k)} + \cdots + \mathcal{P}_{n-k}^{(k)} \quad \text{for all} \quad n \geq 3, \tag{1}$$

with the initial conditions $\mathcal{P}_i^{(k)} = 0$ for $i = 3-k, \ldots, 0$ and $\mathcal{P}_1^{(k)} = \mathcal{P}_2^{(k)} = 1$. We shall refer to $\mathcal{P}_n^{(k)}$ as the *nth k-Padovan number*. We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the Padovan sequence $(\mathcal{P}_n)_{n\geq 0}$ is obtained for $k = 3$. In Table 1 we present the values of these numbers for the first few values of k and $n \geq 1$.

Table 1. First k-Padovan numbers

k	Name	First terms with index ≥ 1
3	Padovan	1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, ...
4	4-Padovan	1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, ...
5	5-Padovan	1, 1, 1, 2, 3, 5, 7, 11, 17, 26, 40, 61, 94, 144, 221, ...
6	6-Padovan	1, 1, 1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, 286, ...
7	7-Padovan	1, 1, 1, 2, 3, 5, 8, 13, 20, 32, 51, 81, 129, 205, 326, ...

Here, we investigate the k-Padovan sequences and present recurrence relations, the generalized Binet formula and different arithmetic properties for $\mathcal{P}^{(k)}$. Some interesting identities involving Fibonacci and generalized Padovan numbers are also deduced and some well–known properties of $\mathcal{P}^{(3)}$ are generalized to the sequence $\mathcal{P}^{(k)}$. We also exhibit a good approximation to the *nth* k-Padovan number and show the exponential growth of $\mathcal{P}^{(k)}$.

2. Preliminary results

First of all, we denote the characteristic polynomial of the k-Padovan sequence $\mathcal{P}^{(k)}$ by

$$\Phi_k(x) = x^k - x^{k-2} - x^{k-3} - \cdots - x - 1.$$

In 2015, Iliopoulos [15] showed some interesting properties of the polynomial $\Phi_k(x)$, which we summarize in the following lemma.

Lemma 2.1. *For an integer $k \geq 3$, we have*

(a) *$\Phi_k(x)$ has k simple zeros.*
(b) *If k is odd, then $\Phi_k(x)$ has a unique real zero $\lambda(k) \in (1, \varphi)$ and $k-1$ complex zeros, where $\varphi = (1+\sqrt{5})/2$ is the golden section.*
(c) *If k is even, then the zeros of $\Phi_k(x)$ are $\lambda(k) \in (1, \varphi)$, -1 and $k-2$ complex zeros.*
(d) *For all the complex zeros μ of $\Phi_k(x)$, it holds that $|\mu| < 1$.*

From the above we deduce that $\Phi_k(x)$ has just one real zero located between 1 and φ. Throughout this paper $\lambda := \lambda(k)$ denotes that single zero, and to simplify notation, we shall omit the dependence on k of λ whenever no confusion may arise. If k is odd, λ is a *Pisot number* of degree k since the other zeros of the characteristic polynomial $\Phi_k(x)$ are strictly inside the unit circle. If k is even, λ is a *Salem number* since the other zeros of the characteristic polynomial $\Phi_k(x)$ have absolute value no greater than 1, and at least one of which has module exactly 1. Further, Lemma 2.1 implies that the solution of the generalized recurrence can be approximated by $\mathcal{P}_n^{(k)} \approx C \cdot \lambda^n$ with negligible error term. This important property of λ leads us to call it *the dominant root* of $\mathcal{P}^{(k)}$.

On the other hand, from the results of Mignotte [18] and Dubickas et al. [9] we know that two algebraic conjugates of a Pisot number may have the same absolute value only if these are complex conjugate numbers, and that two non-real algebraic conjugates of a Pisot number can not have the same argument. For the family of k-Padovan sequences, it is known that λ is a Pisot number when k is odd and so the polynomial $\Phi_k(x)$ is irreducible over $\mathbb{Q}[x]$ when k is odd. Now, if k is even, then there exists an irreducible polynomial $g(x) \in \mathbb{Q}[x]$ such that $\Phi_k(x) = (x+1)g(x)$, which implies that λ is a Pisot number with minimal polynomial $g(x)$. From the above we deduce that the roots of $\Phi_k(x)$ have the following geometric properties.

Lemma 2.2. *Let $k \geq 3$ be integer and assume that α and β are two distinct roots of $\Phi_k(x)$. Then, α and β have different arguments. Moreover, if $|\alpha| = |\beta|$, then we have $\alpha = \overline{\beta}$.*

We now consider for each integer $k \geq 3$ the function $h_k(x)$ defined by

$$h_k(x) = (x-1)\Phi_k(x) = x^{k+1} - x^k - x^{k-1} + 1. \tag{2}$$

Since $\mathcal{P}^{(k)}$ is a linear recurrence of order k with characteristic polynomial $\Phi_k(x)$ and $\Phi_k(x)$ divides $h_k(x)$, we deduce that $\mathcal{P}^{(k)}$ is also a linear recurrence of order

$k+1$ with characteristic polynomial $h_k(x)$. Hence, we obtain our first result of the paper which is a "shift formula" that will be used in the sequel.

Theorem 2.3. *Let $k \geq 3$ be integer. Then*

$$\mathcal{P}_n^{(k)} = \mathcal{P}_{n-1}^{(k)} + \mathcal{P}_{n-2}^{(k)} - \mathcal{P}_{n-k-1}^{(k)} \quad \textit{for all} \quad n \geq 4.$$

As an application of Theorem 2.3, one can prove by using induction that

$$\mathcal{P}_n^{(k)} = F_{n-1} \quad \text{for all} \quad 2 \leq n \leq k+1. \tag{3}$$

3. Main results

We summarize the main results in the following theorem.

Theorem 3.1. *Let $k \geq 3$ be an integer. Then*

(a) *For all $n \geq 3-k$, we have*

$$\mathcal{P}_n^{(k)} = \sum_{i=1}^{k} q_k(\lambda_i)\lambda_i^{n-1} \quad \textit{and} \quad |\mathcal{P}_n^{(k)} - q_k(\lambda)\lambda^{n-1}| < 5/2,$$

where $\lambda := \lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of characteristic polynomial $\Phi_k(x)$ and

$$q_k(z) := \frac{z^2-1}{(k+1)z^2 - kz - k + 1}. \tag{4}$$

(b) *For all $n \geq 1$, we have*

$$\lambda^{n-3} \leq \mathcal{P}_n^{(k)} \leq \lambda^{n-1}. \tag{5}$$

In order to prove Theorem 3.1, we establish some lemmas which give us interesting properties of the dominant root of $\mathcal{P}^{(k)}$, and we believe are of independent interest.

3.1. Generalized Binet formula

We begin by considering the generating function for the sequence $\mathcal{P}^{(k)}$ as the formal power series whose coefficients are the k-Padovan numbers themselves

$$\mathcal{P}(x) = \sum_{n=0}^{\infty} \mathcal{P}_n^{(k)} x^n.$$

Then, it is not difficult to see that

$$\mathcal{P}(x) = \frac{x(x+1)}{1 - x^2 - x^3 - \cdots - x^k}. \tag{6}$$

On the other hand, since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the zeros of $\Phi_k(x)$, we can write

$$\Phi_k(x) = \prod_{i=1}^{k} (x - \lambda_i). \tag{7}$$

Now, if we define $T(x) = x^k\Phi_k(1/x) = 1 - x^2 - x^3 - \cdots - x^k$, by (7) we get that

$$T(x) = x^k \prod_{i=1}^{k}\left(\frac{1}{x} - \lambda_i\right) = \prod_{i=1}^{k}(1-\lambda_i x).$$

From the above, using the generating function (6) and partial fractions, we deduce

$$\mathcal{P}(x) = \frac{x(x+1)}{T(x)} = \sum_{j=1}^{k}\frac{A_j}{1-\lambda_j x}$$

for some unique complex constants $A_1, A_2, \ldots, A_n$. Then,

$$x(x+1) = \sum_{j=1}^{k} A_j \prod_{\substack{i=1\\ i\neq j}}^{k}(1-\lambda_i x).$$

Evaluating the above expression at $x = 1/\lambda_t$ for $t \in \{1,2,\ldots,k\}$, we get the relation

$$A_t = \frac{1+1/\lambda_t}{\lambda_t \prod_{\substack{i=1\\ i\neq t}}^{k}(1-\lambda_i/\lambda_t)} = \frac{-(1+1/\lambda_t)}{T'(1/\lambda_t)} = \frac{\lambda_t^{k-2}(1+1/\lambda_t)}{\Phi_k'(\lambda_t)},$$

where we used that $T'(1/\lambda_t) = -\Phi_k'(\lambda_t)/\lambda_t^{k-2}$. By using (2), we also have that

$$\Phi_k'(\lambda_i) = \frac{h_k'(\lambda_t)}{\lambda_t - 1} = \frac{\lambda_t^{k-2}((k+1)\lambda_t^2 - k\lambda_t - k + 1)}{\lambda_t - 1}.$$

So, the relation

$$A_t = \frac{(\lambda_t - 1)(1+1/\lambda_t)}{(k+1)\lambda_t^2 - k\lambda_t - k + 1} \quad \text{holds for all} \quad t \in \{1,2,\ldots,k\}.$$

Thus,

$$\begin{aligned}\mathcal{P}(x) &= \sum_{j=1}^{k} A_j \left(\sum_{n=0}^{\infty}\lambda_j^n x^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{k} A_j\lambda_j^n\right)x^n\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{k}\frac{\lambda_j^2 - 1}{(k+1)\lambda_j^2 - k\lambda_j - k + 1}\lambda_j^{n-1}\right)x^n.\end{aligned}$$

Consequently,

$$\mathcal{P}_n^{(k)} = \sum_{j=1}^{k}\frac{\lambda_j^2 - 1}{(k+1)\lambda_j^2 - k\lambda_j - k + 1}\lambda_j^{n-1}.$$

This proves the first part of (a) in Theorem 3.1.

3.2. Properties of the dominant root

First of all, if we consider the function $q_k(x)$ defined in (4) as a function of a real variable, then it is not difficult to see that $q_k(x)$ has a vertical asymptote in

$$\beta(k) := \frac{k + \sqrt{5k^2 - 4}}{2(k+1)},$$

and is positive and continuous in $(\beta(k), +\infty)$. By Ilopoulos's work [15, p. 4] it is known that $\beta(k) < \lambda(k) < \varphi$ for all $k \geq 3$ and hence $\lim_{k\to\infty} \lambda(k) = \varphi$. Further,

$$q_k'(x) = -\frac{k(x^2+1) - 4x}{((k+1)x^2 - kx + 1)^2}$$

is negative in $(\beta(k), +\infty)$, so $q_k(x)$ is decreasing in $(\beta(k), +\infty)$. Put

$$a_k = \beta(k) + \frac{1}{2k} \quad \text{for all} \quad k \geq 3.$$

We shall show that the sequence $(a_k)_{k\geq 3}$ is increasing and bounded. To do this, let f be the real function defined by

$$f(x) = \frac{x + \sqrt{5x^2 - 4}}{2(x+1)} + \frac{1}{2x}.$$

It is easy to show that

$$f'(x) = \frac{(5x+4)x^2 - (2x+1)\sqrt{5x^2-4}}{2x^2(x+1)^2\sqrt{5x^2-4}} > 0$$

for all $x \geq 3$, since $(5x+4)x^2 > (2x+1)\sqrt{5x^2-4}$ holds for all $x \geq 3$. This, of course, tells us that $(a_k)_{k\geq 3}$ is an increasing sequence. In addition, note that

$$\lim_{k\to\infty} \beta(k) = \lim_{k\to\infty} \frac{k + \sqrt{5k^2-4}}{2(k+1)} = \varphi,$$

and so

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(\beta(k) + \frac{1}{2k}\right) = \varphi.$$

Thus, $\beta(k) \leq \varphi - 1/2k$ for all $k \geq 3$. Finally, taking into account the fact that $2k < \varphi^{(k-2)/2}$ for all $k \geq 17$, it is easy to see that $\varphi - 1/2k < \varphi(1 - \varphi^{-k/2})$ for all $k \geq 17$.

We summarize what we have proved so far in the following lemma.

Lemma 3.2. *Keep the above notation and let $k \geq 3$ be an integer. Then*

(a) *The function $q_k(x)$ is positive, decreasing and continuous in the interval $(\beta(k), \infty)$ and $g_k(x)$ has a vertical asymptote in $\beta(k)$.*

(b) *If $k \geq 3$, then $\beta(k) \leq \varphi - 1/2k$. In addition, if $k \geq 17$, then the inequalities*

$$\beta(k) \leq \varphi - 1/2k < \varphi(1 - \varphi^{-k/2})$$

hold.

Recall that each choice of k produces a distinct k-Padovan sequence which in turn has an associated dominant root $\lambda(k)$. For the convenience of the reader, let us denote by $(\lambda(k))_{k\geq 3}$ the sequence of the dominant roots of the k-Padovan family of sequences. We have the following lemma in which we prove that this dominant root is strictly increasing as k increases. We also prove, in the second part of the lemma, that this dominant root approaches φ as k approaches infinity, and it is larger than $\varphi(1-\varphi^{-k/2})$. The rest of the statements of the lemma are some technical results that will be used later.

Lemma 3.3. *Let $k, \ell \geq 3$ be integers. Then*

(a) *If $k > \ell$, then $\lambda(k) > \lambda(\ell)$.*
(b) $\varphi(1-\varphi^{-k/2}) < \lambda(k) < \varphi$.
(c) *If $k \geq 5$, then $\beta(k) < \varphi - 1/2k < \lambda(k)$.*
(d) $q_k(\varphi) = \varphi/(\varphi+2) = 1/\sqrt{5} \approx 0.4472135\ldots$.

Proof. To prove (a) we proceed by contradiction by assuming that $\lambda(k) \leq \lambda(\ell)$; hence $(1/\lambda(\ell))^i \leq (1/\lambda(k))^i$ holds for all $i \geq 1$. Taking into account that $\Phi_\ell(\lambda(\ell)) = 0$, one has that

$$(\lambda(\ell))^\ell = (\lambda(\ell))^{\ell-2} + (\lambda(\ell))^{\ell-3} + \cdots + \lambda(\ell) + 1,$$

and, of course, the same conclusion remains valid for $\lambda(k)$. From this, we get that

$$1 = \frac{1}{(\lambda(\ell))^2} + \frac{1}{(\lambda(\ell))^3} + \cdots + \frac{1}{(\lambda(\ell))^\ell} < \frac{1}{(\lambda(k))^2} + \frac{1}{(\lambda(k))^3} + \cdots + \frac{1}{(\lambda(k))^k} = 1$$

which is a contradiction. Another way to prove (a) is by applying Descartes' rule of signs, which tells us that $\Phi_k(x)$ has exactly one positive real zero $\lambda(k)$, and using the fact that $\Phi_{k+1}(x) = x\Phi_k(x) - 1$. Indeed, it follows from the above that $\Phi_{k+1}(\lambda(k)) = -1 < 0$ and $\Phi_{k+1}(\varphi) = 1/(\varphi-1) > 0$ giving that $\lambda(k+1) \in (\lambda(k), \varphi)$. Thus, $\lambda(k) < \lambda(k+1)$.

We next prove (b). First, we turn back to expression (2), which we rewrite here as follows:

$$h_k(x) = (x-1)\Phi_k(x) = x^{k-1}(x^2-x-1) + 1. \tag{8}$$

By using the fact that φ is a root of x^2-x-1 and evaluating expression (8) at λ, we get the relations

$$\varphi^2 - \varphi - 1 = 0 \quad \text{and} \quad \lambda^2 - \lambda - 1 = -1/\lambda^{k-1}.$$

Subtracting the two expressions above and rearranging some terms, one obtains

$$(\varphi - \lambda)(\varphi + \lambda - 1) = 1/\lambda^{k-1}.$$

From this, and using the facts that $\varphi + \lambda - 1 > 1/\varphi^{1/2}$ and $\varphi^{1/2} < \lambda(3) \leq \lambda$, which are easily seen, we get that $\varphi - \lambda < \varphi/\varphi^{k/2}$ and so $\varphi(1-\varphi^{-k/2}) < \lambda$. This finishes the proof of (b).

The proof of (c) can be checked computationally when $5 \leq k \leq 16$. The case when $k \geq 17$ is a direct combination of the second part of this lemma and Lemma 3.2(b). Finally, to prove (d) we observe that

$$q_k(\varphi) = \frac{\varphi^2 - 1}{(k+1)\varphi^2 - k\varphi - (k-1)} = \frac{\varphi}{\varphi + 2} = \frac{1}{\sqrt{5}},$$

where we used the fact that $\varphi^2 = \varphi + 1$. This finishes the proof of the lemma. □

We finish this subsection by giving two immediate consequences of the above lemma which will be needed later.

Lemma 3.4. *Keep the above notation and let $k \geq 3$ be an integer. Then*

$$0.44 < q_k(\lambda(k)) < 0.73 \quad \textit{and} \quad |q_k(\lambda_i)| < 2/(k-2) \quad \textit{for} \ \ 2 \leq i \leq k,$$

where, as before, $\lambda := \lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $\Phi_k(x)$. Consequently,

$$|q_k(\lambda_i)| < 1 \quad \textit{for} \ \ 1 \leq i \leq k.$$

Proof. We begin by noting that $q_k(x)$ is decreasing in the interval $(\beta(k), \infty)$ and that the inequality $\beta(k) < \varphi - 1/2k < \lambda(k) < \varphi$ holds for all $k \geq 5$. After simple transformations we get

$$0.44 < \frac{1}{\sqrt{5}} = q_k(\varphi) < q_k(\lambda(k)) < q_k\left(\varphi - \frac{1}{2k}\right).$$

But

$$q_k(\varphi - 1/2k) = \frac{4\varphi(k^2 - k) + 1}{10k^2 + (1 - 4\varphi)k + 1} < \frac{2\varphi}{5} = 0.6472\ldots.$$

Hence, $0.44 < q_k(\lambda(k)) < 0.65$ holds for all $k \geq 5$. Finally, computationally we get that $q_3(\lambda(3)) = 0.722\ldots$ and $q_4(\lambda(4)) = 0.611\ldots$. This proves the first part of the lemma.

For the second part we consider the function $h_k(x)$ defined by (8) in the proof of Lemma 3.3. Evaluating $h_k(x)$ at λ_i for $2 \leq i \leq k$, and rearranging some terms of the resulting expression, we get the relation $\lambda_i^2 - \lambda_i - 1 = -1/\lambda_i^{k-1}$ and so

$$k(\lambda_i^2 - \lambda_i - 1) + \lambda_i^2 + 1 = \lambda_i^2 + 1 - \frac{k}{\lambda_i^{k-1}}.$$

Hence,

$$|k(\lambda_i^2 - \lambda_i - 1) + \lambda_i^2 + 1| = \left| \frac{k}{\lambda_i^{k-1}} - (\lambda_i^2 + 1) \right| \geq \frac{k}{|\lambda_i|^{k-1}} - |\lambda_i^2 + 1| \geq k - 2,$$

where we used the fact that $|\lambda_i| \leq 1$ because $2 \leq i \leq k$. Consequently,

$$|q_k(\gamma_i)| = \frac{|\lambda_i^2 - 1|}{|k(\lambda_i^2 - \lambda_i - 1) + \lambda_i^2 + 1|} \leq \frac{2}{k-2}.$$

Finally, $|q_k(\gamma_i)| \leq 2/(k-2) < 1$ for all $k \geq 5$ while the cases $k = 3$ and 4 can be checked computationally, which finishes the proof of the lemma. □

3.3. Sequence of errors

For an fixed integer $k \geq 3$ and $n \geq 3-k$, define $E_n^{(k)}$ to be the error of the approximation of the *nth* k-Padovan number with the dominant term of the Binet–style formula of $\mathcal{P}^{(k)}$ given in Theorem 3.1(a), i.e.,

$$E_n^{(k)} = \mathcal{P}_n^{(k)} - q_k(\lambda)\lambda^{n-1} \tag{9}$$

for λ the dominant root of $\Phi(x)$ and $q_k(x)$ defined as in (4).

Given a polynomial f, it is well known that the set of all possible linear recurrence sequences satisfying the characteristic equation $f(x) = 0$ is a vector space over the real numbers. Since $\mathcal{P}^{(k)}$ and $(\lambda^n)_n$ satisfy the characteristic equation $\Phi_k(x) = 0$, it follows from (9) that the sequence $(E_n^{(k)})_n$ satisfies the same recurrence relation as the k-Padovan sequence. We record this as follows.

Lemma 3.5. *Let $k \geq 3$ be an integer. Then*

$$E_n^{(k)} = E_{n-2}^{(k)} + E_{n-3}^{(k)} + \cdots + E_{n-k}^{(k)} \quad \textit{for all} \quad n \geq 3.$$

Furthermore, if $n \geq 4$, then

$$E_n^{(k)} = E_{n-1}^{(k)} + E_{n-2}^{(k)} - E_{n-k-1}^{(k)}.$$

The last result of this subsection is the following.

Lemma 3.6. *For a fixed integer $k \geq 3$ we have*

$$\lim_{n\to\infty} E_n^{(k)} = 0.$$

Proof. We begin by observing that

$$|E_n^{(k)}| \leq \sum_{j=2}^{k} |q_k(\lambda_j)||\lambda_j|^{n-1}. \tag{10}$$

In order to prove the statement, we will consider two cases on the integer k. If k is even, then there exists $2 \leq j_0 \leq k$ such that $\lambda_{j_0} = -1$ and so $q_k(\lambda_{j_0}) = 0$. Without loss of generality, we may assume that $j_0 = 2$. Then, from (10) and using the fact that $\lim_{n\to\infty} |\lambda_j|^n = 0$ for all $3 \leq j \leq k$, we get that $\lim_{n\to\infty} |E_n^{(k)}| = 0$. Now, when k is odd we have that $\lim_{n\to\infty} |\lambda_j|^n = 0$ for all $2 \leq j \leq k$ and so $\lim_{n\to\infty} |E_n^{(k)}| = 0$. In any case, we can conclude that $\lim_{n\to\infty} E_n^{(k)} = 0$. ☐

To conclude this subsection, we prove the second part of Theorem 3.1(a). With the notation above, we have to prove that

$$|E_n^{(k)}| < 5/2 \quad \text{for all} \quad k \geq 3 \quad \text{and} \quad n \geq 3-k.$$

Indeed, from inequality (10) and Lemma 3.4, we get that

$$|E_n^{(k)}| \leq \sum_{j=2}^{k} |q_k(\lambda_j)| \leq \frac{2(k-1)}{k-2}.$$

Here, we can check computationally that $|E_n^{(k)}| < 5/2$ for each $3 \leq k \leq 5$. But, this also holds for all $k \geq 6$ since $|E_n^{(k)}| \leq 2(k-1)/(k-2) \leq 5/2$.

3.4. Exponential growth

We begin by mentioning that for the Fibonacci sequence and the Padovan sequence (namely, the case $k = 3$), it is well–known that

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1} \quad \text{holds for all} \quad n \geq 1, \tag{11}$$

and

$$\lambda^{n-3} \leq \mathcal{P}_n \leq \lambda^{n-1} \quad \text{holds for all} \quad n \geq 1, \tag{12}$$

exhibiting an exponential growth of the Fibonacci and Padovan numbers. In the above expression (12), the value of λ is $\lambda(3) = 1.3247\ldots$. We finally prove (5) by using induction on n.

To begin with, we show that inequality (5) holds for $n = 1, 2, \ldots, k$. It is clear that the result is true for $n = 1, 2$ because $\lambda > 1$. For $n = 3, \ldots, k$ we know, by (3), that $\mathcal{P}_n^{(k)} = F_{n-1}$, so we need to show that

$$\lambda^{n-3} \leq F_{n-1} \leq \lambda^{n-1} \quad \text{for} \quad 3 \leq n \leq k. \tag{13}$$

By Lemma 3.3(b) and (11), we get

$$\lambda^{n-3} < \varphi^{n-3} \leq F_{n-1}$$

and therefore the left–hand side of the above inequality (13) holds. Then, it remains to prove that

$$F_{n-1} \leq \lambda^{n-1} \quad \text{holds for} \quad 3 \leq n \leq k. \tag{14}$$

Computationally one checks that the inequality (14) holds for $3 \leq k \leq 11$, so we may assume that $k \geq 12$. Now, by making use of the famous Binet formula for the Fibonacci numbers, we get

$$F_{n-1} = \frac{\varphi^{n-1} - (-1)^{n-1}\varphi^{-(n-1)}}{\sqrt{5}} = \frac{\varphi^{n-1}}{\sqrt{5}} \left(1 + \frac{\varepsilon}{\varphi^{2n-2}}\right),$$

where $\varepsilon \in \{\pm 1\}$. Since $\varphi^{n-1}(1 - \varphi^{-k/2})^{n-1} < \lambda^{n-1}$ because $\varphi(1 - \varphi^{-k/2}) < \lambda$ by Lemma 3.3(b), it suffices to prove that

$$\frac{\varphi^{n-1}}{\sqrt{5}} \left(1 + \frac{\varepsilon}{\varphi^{2n-2}}\right) \leq \varphi^{n-1}(1 - \varphi^{-k/2})^{n-1},$$

which is equivalent to

$$1 + \frac{\varepsilon}{\varphi^{2n-2}} \leq \sqrt{5}\,(1 - \varphi^{-k/2})^{n-1}. \tag{15}$$

Using the fact that the function $x \to (1 - \varphi^{-x/2})^{x-1}$ is increasing for $x \geq 12$ and taking into account that $3 \leq n \leq k$ and $k \geq 12$, we deduce that

$$\sqrt{5}\,(1 - \varphi^{-k/2})^{n-1} \geq \sqrt{5}\,(1 - \varphi^{-k/2})^{k-1} \geq \sqrt{5}\,(1 - \varphi^{-6})^{11} = 1.1902\ldots.$$

whereas

$$1 + \frac{\varepsilon}{\varphi^{2n-2}} \leq 1 + \frac{1}{\varphi^4} = 1.1459\ldots.$$

This proves inequality (15). Thus, we have proved that inequality (5) holds for $1 \leq n \leq k$.

Finally, suppose that (5) holds for all terms $\mathcal{P}_m^{(k)}$ with $m \leq n-1$ for some $n > k$. It then follows from the recurrence relation of $\mathcal{P}^{(k)}$ that

$$\lambda^{n-5} + \lambda^{n-6} + \cdots + \lambda^{n-k-3} \leq \mathcal{P}_n^{(k)} \leq \lambda^{n-3} + \lambda^{n-4} + \cdots + \lambda^{n-k-1}.$$

So

$$\lambda^{n-k-3}(\lambda^{k-2} + \lambda^{k-3} + \cdots + 1) \leq \mathcal{P}_n^{(k)} \leq \lambda^{n-k-1}(\lambda^{k-2} + \lambda^{k-3} + \cdots + 1),$$

which combined with the fact that $\lambda^k = \lambda^{k-2} + \lambda^{k-3} + \cdots + 1$ gives the desired result. Thus, inequality (5) holds for all positive integers n. So, the proof of Theorem 3.1 is now complete.

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