ON LIMIT BEHAVIOURS FOR FELLER’S UNFAIR-FAIR-GAME AND ITS RELATED MODEL

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Abstract. Feller introduced an unfair-fair-game in his famous book [3]. In this game, at each trial, player will win $2^k$ yuan with probability $p_k = 1/2^k(k + 1)$, $k \in \mathbb{N}$, and zero yuan with probability $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Because the expected gain is 1, player must pay one yuan as the entrance fee for each trial. Although this game seemed “fair”, Feller [2] proved that when the total trial number $n$ is large enough, player will lose $n$ yuan with its probability approximate 1. So it’s an “unfair” game. In this paper, we study in depth its convergence in probability, almost sure convergence and convergence in distribution. Furthermore, we try to take $2^k = m$ to reduce the values of random variables and their corresponding probabilities at the same time, thus a new probability model is introduced, which is called as the related model of Feller’s unfair-fair-game. We find out that this new model follows a long-tailed distribution. We obtain its weak law of large numbers, strong law of large numbers and central limit theorem. These results show that their probability limit behaviours of these two models are quite different.

1. Introduction

In recent years, many scholars studied the limit behaviours for random variable sequences with their expectations or variances not existed. They applied these conclusions to study some special random variables, such as Pareto-Zipf distribution ([1]), Feller game ([1, 9–11]), St. Petersburg game ([6, 10]), etc., and obtained a lot of interesting and meaningful outcomes.

Feller introduced an unfair-fair-game in his famous book [3]. In this game, at each trial, player will win $2^k$ yuan with probability $p_k$ and zero yuan with...
probability \( p_0 \), where

\[
(1.1) \quad p_k = P \{ X_1 = 2^k \} = \frac{1}{2^k k(k + 1)}, \quad k \in \mathbb{N}, \quad p_0 = P \{ X_1 = 0 \} = 1 - \sum_{k=1}^{\infty} p_k.
\]

Because the expected gain is 1, the player must pay one yuan as the entrance fee for each trial. Although this game seemed “fair”, Feller [2] proved that when the total trial number \( n \) is large enough, player will loss \( n \) yuan with its probability approximate 1. Let \( X_i \) represent the player’s gain at the \( i \)th trial, \( S_n = \sum_{i=1}^{n} X_i \) denote the total gain in its \( n \) trials, this result can be expressed as the following theorem (see [3]).

**Theorem 1.1.** Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution described in (1.1). Then for any \( \varepsilon > 0 \),

\[
(1.2) \quad \lim_{n \to \infty} P \left( S_n - n < -\frac{(1 - \varepsilon)n}{\log_2 n} \right) = 1,
\]

where \( \log_2 n = \max \{ 1, \log_2 n \} \), and \( \log_2 n \) denotes the logarithm to the base 2.

Theorem 1.1 shows that this game is unfavorable for players. So it’s an “unfair” game. We call it as Feller’s unfair-fair-game. For this model, the expectation \( EX_1 = 1 \). But, for any \( \alpha > 1 \), the moment \( EX_\alpha^1 \) doesn’t exist. Therefore, many classical limit theorems don’t hold, which arouses our interest in this subject. Unfortunately, its further limit behaviours have not be investigated in literatures. In this paper, some probability limit properties for this model are studied. Concretely peaking, in Section 2, Theorem 2.1 studies the convergence in probability and its rate of convergence for \( \{ S_n \} \). Theorem 2.2 investigates the convergence in probability and almost sure convergence for \( \{ X_n \} \). Theorem 2.3 obtains the characteristic function of the limit distribution of \( \{ S_n \} \), which is similar to Theorem 1 of [8].

As we known, the probability limit properties of random sequences are often closely related to their moment conditions. For the model of Feller’s unfair-fair-game, the expectation \( EX_1 = 1 \). But, for any \( \alpha > 1 \), the moment \( EX_\alpha^1 \) does not exist as stated above. Now if we reduce the values of random variables and their corresponding probabilities at the same time, does the new probability model have the same limit properties? Driven by this motivation, we try to take \( 2^k = m \), equivalently, \( k = [\log_2 m] \), where \( [a] \) denotes the largest integer no more than the real number \( a \). For any integer \( m \geq 2 \), we define

\[
p_m = P \{ X_1 = m \} = \frac{1}{m \log_2 m (\log_2 m + 1)}, \quad m \geq 2.
\]

Based on this idea, we introduce the following probability model.

Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution

\[
(1.3) \quad p_k = P \{ X_1 = k \} = \frac{1}{c k (\log_a k)^2}, \quad k \geq 1, \quad c = \sum_{k=1}^{\infty} \frac{1}{k (\log_a k)^2}, \quad a > 1,
\]
where \( \log_a n = \max\{1, \log a n\} \). We take the denominator of \( p_k \) as \( ck(\log_a k)^2 \) instead of \( ck(\log_a k)(\log_a k+1) \), for the sake of the convenience of mathematical processing, however, they are not much different. We call the new model of (1.3) as the related model of Feller’s unfair-fair-game, abbreviated as related model of FUFG.

For the related model of FUFG of (1.3), it’s easy to get

\[
 F(x) = P\{X_1 > x\} \sim \frac{\ln a}{c \log a x} \text{ as } x \to +\infty,
\]

where \( \ln a = \log e a \). For any \( t > 0 \), \( F(x+t) \sim F(x) \) as \( x \to +\infty \). Now \( X_1 \) follows a long-tailed distribution (see [4]), which is widely used in many fields, such as machine learning, artificial intelligence, finance theory, insurance theory and so on. Therefore it is meaningful to study in depth the probability limit properties of this model.

Since \( EX_1 = +\infty \), the conditions of classical limit theorem are not satisfied, it’s necessary to specially discuss its limit behaviours. In Section 3, Theorem 3.3, Theorem 3.4 and Theorem 3.6 study the convergence in probability and almost sure convergence for its partial sums \( S_n = \sum_{i=1}^{n} X_i \). Theorem 3.7 obtains its central limit theorem. Our findings show that these two models have completely different probability limit behaviours.

Throughout this paper, \( X_n = o_P(Y_n) \) denotes \( X_n/Y_n \to 0 \) in probability; a.s. is the abbreviation of “almost surely”; i.o. is the abbreviation of “infinitely often”; \( C \) represents a positive constant and its value may be different and unimportant on different occasions; \( \overset{d}{\to} \) means convergence in distribution; \( a_n = o(b_n) \) represents \( a_n/b_n \to 0 \) as \( n \to \infty \); \( a_n \sim b_n \) means \( \lim_{n \to \infty} a_n/b_n = 1 \); \( a_n = O(b_n) \) stands for \( -\infty < \lim \inf_{n \to \infty} a_n/b_n \leq \lim \sup_{n \to \infty} a_n/b_n < \infty \).

2. Limit behaviours for Feller’s unfair-fair-game

Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution described in (1.1). Define \( b_n = n/\log_2 n, m(n) = \sup\{m \in \mathbb{N} : 2^m \leq b_n\} \), \( a_n = m(n)/(m(n)+1) \). From

\[
 2^{m(n)} \leq n/\log_2 n < 2^{m(n)+1},
\]

we can get

\[
 m(n) \leq \log_2 n - \log_2 \log_2 n < m(n) + 1.
\]

Thus

\[
 1 - \frac{\log_2 \log_2 n + 1}{\log_2 n} < \frac{m(n)}{\log_2 n} \leq 1 - \frac{\log_2 \log_2 n}{\log_2 n},
\]

and

\[
 \lim_{n \to \infty} \frac{m(n)}{\log_2 n} = 1, \quad \lim_{n \to \infty} a_n = 1.
\]

We first study the convergence in probability and its rate of convergence for the partial sums \( S_n \) and obtain the following result.
Theorem 2.1. Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.1), $S_n = \sum_{i=1}^{n} X_i$. Then

\begin{equation}
\frac{S_n - na_n}{b_n} \to 0 \text{ in probability,}
\end{equation}

and

\begin{equation}
\frac{S_n - na_n}{b_n} = o_P(\Log_2 n).
\end{equation}

Proof. Define $\tilde{X}_i = X_i I(X_i \leq b_n)$. For $n$ large enough, $m(n) > \Log_2 n/2$, $2^{m(n)} > b_n/2$, so

\begin{equation}
E\tilde{X}_1 = \sum_{k=1}^{m(n)} \frac{1}{k(k+1)} = 1 - \frac{1}{m(n) + 1} = \frac{m(n)}{m(n) + 1} = a_n,
\end{equation}

\begin{equation}
nP\{X_1 > b_n\} = n \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k (k+1)} 
\leq \frac{n}{m(n)^2} \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k}
\leq \frac{n}{m(n)^2 2^{m(n)}}
\leq \frac{4n}{\Log_2 n} \cdot \frac{2\Log_2 n}{n}
\leq C \Log_2 n \to 0 \text{ as } n \to \infty,
\end{equation}

\begin{equation}
E\tilde{X}_2 = \sum_{k=1}^{m(n)} \frac{2^k}{k(k+1)}
\leq \sum_{k=1}^{[m(n)/2]} \frac{2^k}{k(k+1)} + \sum_{k=[m(n)/2]+1}^{m(n)} \frac{2^k}{k(k+1)}
\leq I_1 + I_2.
\end{equation}

We start to estimate the order of $I_1$ and $I_2$, respectively. Firstly,

\begin{equation}
I_1 \leq m(n) 2^{m(n)/2} \leq 2 \Log_2 n \cdot \sqrt{\frac{n}{\Log_2 n}} = 2\sqrt{n \Log_2 n} = o\left(\frac{n}{(\Log_2 n)^2}\right).
\end{equation}

Secondly,

\begin{equation}
I_2 \leq \frac{4}{m(n)^2} \sum_{k=[m(n)/2]+1}^{m(n)} 2^k
\end{equation}
\[ \begin{align*}
\leq & \frac{4}{m(n)^2} \cdot 2^{m(n)+1} \\
\leq & \frac{16}{(\log_2 n)^2} \cdot \frac{2n}{\log_2 n} \\
= & o \left( \frac{n}{(\log_2 n)^2} \right).
\end{align*} \]

Combining (2.5), (2.6) and (2.4), we get that
\[ (2.7) \quad \frac{1}{b_n} \sum_{i=1}^{n} \text{Var} \tilde{X}_i \leq \frac{1}{b_n} \sum_{i=1}^{n} E \tilde{X}_i^2 = n \left( \frac{\log_2 n}{n} \right)^2 \cdot o \left( \frac{n}{(\log_2 n)^2} \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

Applying Theorem 6.3.3 of [5] with (2.4) and (2.7), it yields (2.2).

Because \( E X_1 = 1 \), by the Khintchine’s weak law of large numbers we get \( \frac{S_n - n}{n} \rightarrow 0 \) in probability.

Consequently,
\[ \frac{S_n - na_n}{b_n} = \frac{S_n - n}{n} \cdot \frac{n}{b_n} + \frac{n(1 - a_n)}{b_n} = \frac{S_n - n}{n} \cdot \log_2 n + (1 - a_n) \log_2 n \]
\[ = \left\{ \frac{S_n - n}{n} + (1 - a_n) \right\} \log_2 n = o_P(\log_2 n). \]

Thus (2.3) holds. The proof of Theorem 2.1 is completed.

Using Theorem 2.1, we can prove Theorem 1.1 easily.

Proof of Theorem 1.1. For any \( \varepsilon \in (0, 1) \), taking \( \varepsilon' \in (0, 1) \) such that \( 0 < 1 - \varepsilon + \varepsilon' = 1/\alpha < 1 \) for this \( \alpha > 1 \) and \( n \) large enough, the first part of (2.1) implies \( m(n) + 1 \leq \alpha \log_2 n \). Now
\[ P \left\{ S_n - n < -\frac{(1 - \varepsilon)n}{\log_2 n} \right\} = P \left\{ S_n - n < -\frac{n}{\alpha \log_2 n} + \frac{n \varepsilon'}{\log_2 n} \right\} \]
\[ \geq P \left\{ S_n - n < -\frac{n}{m(n) + 1} + \frac{n \varepsilon'}{\log_2 n} \right\} = P \left\{ S_n - na_n < \varepsilon' b_n \right\} \]
\[ \geq P \left\{ \left| \frac{S_n - na_n}{b_n} \right| < \varepsilon' \right\} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. \]

So (1.2) holds.
Next we study the convergence in probability and almost sure convergence for \( \{X_n\} \).

**Theorem 2.2.** Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution described in (1.1). Then, we have the following limit behaviour

\[
X_n - a_n \frac{b_n}{b_n} \rightarrow 0 \quad \text{in probability},
\]

and

\[
\liminf_{n \to \infty} X_n - a_n \frac{b_n}{b_n} = 0 \quad \text{a.s.,} \quad \limsup_{n \to \infty} X_n - a_n \frac{b_n}{b_n} = +\infty \quad \text{a.s.}.
\]

**Proof.** For any \( \varepsilon > 0 \) and \( n \) large enough, \( a_n \varepsilon n / \log 2n < 0 \) and

\[
P\{X_1 < a_n - \varepsilon n / \log 2n\} = 0,
\]

since \( a_n > 0, a_n \uparrow 1 \) as \( n \to \infty \), \( X_1 \geq 0 \) a.s.. Thus,

\[
P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon\right\} = P\{X_1 < a_n - \varepsilon n / \log 2n\} = 0.
\]

On the other hand, for any \( \varepsilon > 0 \) and \( n \) large enough,

\[
P\left\{\frac{X_n - a_n}{b_n} > \varepsilon\right\} = P\{X_1 > a_n + \varepsilon n / \log 2n\} \to 0 \text{ as } n \to \infty.
\]

Consequently, (2.10) and (2.11) lead to (2.8).

Based on the same reason of (2.10) we know that

\[
\sum_{n=1}^\infty P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon\right\} = \sum_{n=1}^\infty P\{X_1 < a_n - \varepsilon n / \log 2n\} < \infty.
\]

Using the first Borel-Cantelli lemma we can also obtain

\[
P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon, \text{ i.o.}\right\} = 0,
\]

which yields

\[
\liminf_{n \to \infty} \frac{X_n - a_n}{b_n} \geq 0 \quad \text{a.s.}
\]

For any \( \varepsilon > 0 \), \( \{X_1 = 0\} \subset \{X_1 < a_n + \varepsilon b_n\} \), thus

\[
\sum_{n=1}^\infty P\left\{\frac{X_n - a_n}{b_n} < \varepsilon\right\} = \sum_{n=1}^\infty P\{X_1 < a_n + \varepsilon b_n\} \geq \sum_{n=1}^\infty P\{X_1 = 0\} = +\infty.
\]

By the second Borel-Cantelli lemma we obtain

\[
P\left\{\frac{X_n - a_n}{b_n} < \varepsilon \text{ i.o.}\right\} = 1,
\]

which leads to

\[
\liminf_{n \to \infty} \frac{X_n - a_n}{b_n} \leq 0 \quad \text{a.s.}
\]


Combining (2.12) and (2.13), the first part of (2.9) is proved.

For any $M > 0$, put $k(n) = \inf \{ k : 2^k \geq (M + 1)b_n \}$. Similar to (2.1) we can easily get $k(n) \sim \log_2 n$ as $n \to \infty$. From the knowledge of mathematical analysis,

\[
\int_{k}^{k+1} \frac{1}{2^i x(x+1)} \, dx \leq \frac{1}{2^k k(k+1)} \leq \int_{k-1}^{k} \frac{1}{2^i x(x+1)} \, dx, \quad k = 1, 2, \ldots.
\]

This leads to

\[
\int_{k(n)}^{\infty} \frac{1}{2^i x(x+1)} \, dx \leq \sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)} \leq \int_{k(n)-1}^{\infty} \frac{1}{2^i x(x+1)} \, dx.
\]

Using the law of L’Hospital, it’s easy to verify that

\[
\int_{u}^{\infty} \frac{1}{2^i x(x+1)} \, dx \sim \frac{1}{2^i u(u+1) \ln 2} \quad \text{as} \quad u \to \infty.
\]

Combining (2.14) and (2.15), we can obtain that

\[
\sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)} = O \left( \frac{1}{\log_2 n(k(n)+1)} \right) \quad \text{as} \quad n \to \infty.
\]

Consequently,

\[
\sum_{n=1}^{\infty} P \left\{ \frac{X_n - a_n}{b_n} > M \right\} = \sum_{n=1}^{\infty} P \left\{ X_1 > a_n + Mb_n \right\}
\]

\[
\geq \sum_{n=1}^{\infty} P \left\{ X_1 > (M + 1)b_n \right\} \quad \text{(since} \quad a_n \to 1) \]

\[
= \sum_{n=1}^{\infty} \sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)}
\]

\[
\geq C \sum_{n=1}^{\infty} \frac{1}{2^k(n)(k(n)+1)}
\]

\[
\geq C \sum_{n=1}^{\infty} \frac{1}{n/\log_2 n \cdot (\log_2 n)^2}
\]

\[
= C \sum_{n=1}^{\infty} \frac{1}{n \log_2 n} = +\infty.
\]

By the second Borel-Cantelli lemma we have

\[
P \left\{ \frac{X_n - a_n}{b_n} > M, \ i.o. \right\} = 1,
\]

which yields

\[
\limsup_{n \to \infty} \frac{X_n - a_n}{b_n} = +\infty \quad \text{a.s.}
\]
Thus the second part of (2.9) holds. The proof of Theorem 2.2 is completed. □

Let \( B_n^2 = n \left\{ \sum_{k=1}^{m(n)} 2^k / k(k + 1) - a^2 \right\} \), \( \tilde{X}_i = X_i I(X_i \leq b_n) \), \( \tilde{S}_n = \sum_{k=1}^{n} \tilde{X}_i \).

It’s easy to get

\[
P \left\{ \tilde{S}_n \neq S_n \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Applying the Lindeberg-Lévy-Feller central limit theorem, we can prove that

\[
\frac{\tilde{S}_n - na_n}{B_n^2} \overset{d}{\rightarrow} N(0, 1)
\]

does not hold, where \( N(0, 1) \) denotes the standard normal random variable.

Consequently

\[
\frac{S_n - na_n}{B_n^2} \overset{d}{\rightarrow} N(0, 1)
\]

does not hold.

In order to study the limit distribution for the partial sums, \( S_n \), inspired by Anderson [8], the following theorem obtains its characteristic function of the limit distribution of \( S_n \).

**Theorem 2.3.** Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution described in (1.1), \( N_n = 2^n \), \( M_n = n^2 2^n \). Then

\[
\frac{S_{M_n} - M_n}{N_n} = \frac{S_{M_n}}{N_n} - n^2 \overset{d}{\rightarrow} S,
\]

where the characteristic function of random variable \( S \) is \( \exp\{g(t)\} \), and

\[
g(t) = \sum_{k=-\infty}^{0} \frac{e^{it2^k} - 1 - it2^k}{2^k} + \sum_{k=1}^{\infty} \frac{e^{it2^k} - 1}{2^k}.
\]

**Proof.** We use the method of the proof of Theorem 1 in Anderson [8] to prove this theorem.

The characteristic function of \( X_1 \) is

\[
f(t) = E \exp\{itX_1\} = p_0 + \sum_{k=1}^{\infty} \frac{\exp\{it2^k\}}{2^k k(k + 1)} = 1 + \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k k(k + 1)}.
\]

Hence the characteristic function of \( S_{M_n}/N_n - n^2 \) is

\[
f_{N_n}(t) = E \exp\left\{it \left( \frac{S_{M_n}}{N_n} - n^2 \right) \right\}
\]

\[
= E \exp\left\{itS_{M_n}/N_n \right\} \exp\{-tn^2\}
\]

\[
= \left\{ f \left( \frac{t}{N_n} \right) \right\}^{M_n} \exp\{-tn^2\}.
\]
Next, we will decompose \( f \left( \frac{t}{N_n} \right) - 1 \) into two parts.

\[
(2.17) \quad f \left( \frac{t}{N_n} \right) - 1 = \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k k(k+1)} - \sum_{k=-n+1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} \\
= \frac{1}{N_n} \left\{ \frac{\sum_{k=-n+1}^{0} \exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} + \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} \right\} \\
= \frac{1}{N_n} \left( I_{(1)}^n + I_{(2)}^n \right).
\]

Since

\[
\left| n^2 \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} \right| \leq \sum_{k=1}^{\infty} \frac{|\exp\{it2^k\} - 1|}{2^k(k+n)(1+k+1)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty,
\]

applying the Lebesgue’s controlled convergent theorem, we have

\[
(2.18) \quad I_{(2)}^n = \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} \sim \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{\exp\{it2^k\} - 1}{2^k} \quad \text{as } n \to \infty.
\]

On the other hand,

\[
(2.19) \quad I_{(1)}^n = \sum_{k=-n+1}^{0} \frac{\exp\{it2^k\} - 1}{2^k(k+n)(k+n+1)} \\
= \sum_{k=-n+1}^{0} \frac{\exp\{it2^k\} - 1 - it2^k}{2^k(k+n)(k+n+1)} + \sum_{k=-n+1}^{0} \frac{it}{(k+n)(k+n+1)} \\
= \frac{1}{n^2} \sum_{k=-n+1}^{0} \frac{\exp\{it2^k\} - 1 - it2^k}{2^k(1+\frac{k}{n})(1+\frac{k+1}{n})} + it \left( 1 - \frac{1}{n+1} \right).
\]

Since

\[
\frac{\exp\{it2^{-n+1}\} - 1 - it2^{-n+1}}{2^{-n+1}(1+\frac{-n+1}{n})(1+\frac{-n+2}{n})} = O \left( \frac{t^2 n^2}{2^n} \right) \quad \text{as } n \to \infty,
\]

we have

\[
(2.20) \quad \lim_{n \to \infty} \sum_{k=-n+1}^{0} \frac{\exp\{it2^k\} - 1 - it2^k}{2^k(1+\frac{k}{n})(1+\frac{k+1}{n})} = \sum_{k=-\infty}^{0} \frac{\exp\{it2^k\} - 1 - it2^k}{2^k}.
\]
Combining (2.17), (2.18), (2.19) and (2.20), we get
\[
\begin{align*}
f\left(\frac{t}{N_n}\right) - 1 &= \frac{1}{n^2 N_n} \left\{ \sum_{k=0}^{\infty} \exp\left\{ \frac{i t 2^k}{2^k} \right\} - 1 - \frac{i t 2^k}{2^k} \right\} + i n^2 t + \sum_{k=1}^{\infty} \exp\left\{ \frac{i t 2^k}{2^k} \right\} - 1 + o(1) \\
&= \frac{1}{n^2 N_n} \{ g(t) + i n^2 t + o(1) \} \text{ as } n \to \infty.
\end{align*}
\]
By (2.16) we can also get
\[
(2.21) \quad f_{N_n}(t) = \left\{ 1 + \frac{g(t) + i n^2 t + o(1)}{n^2 N_n} \right\}^{n^2 N_n} \cdot \exp\{-i n^2 t\}
\]
\[
= \exp\{g(t)\} + o(1) \text{ as } n \to \infty.
\]
Thus Theorem 2.3 follows from (2.16), (2.21) and the continuity theorem of characteristic function.

3. Limit behaviours for the related model of FUFG

In this section, we will consider the related model of Feller’s unfair-fair-game (1.3) and study its probability limit behaviours. Our findings show that the limit properties of model (1.3) and model (1.1) are quite different. Since the new model follows a long-tailed distribution (see [4]), which is widely used in many fields, therefore it is meaningful to study in depth the probability limit behaviours of this new model.

To prove the next theorems, we will apply the concepts of regularly varying function, slowly varying function and their key properties.

Definition (see [7]). Let $U, V$ be positive monotone functions on $[0, \infty)$ to $[0, \infty)$. We say that $U$ is a regularly varying function (at $+\infty$) with exponent $\alpha \in \mathbb{R}$ if $U(x) = x^\alpha V(x)$ where $V$ is a slowly varying function (at $+\infty$), that is
\[
\lim_{x \to \infty} \frac{V(tx)}{V(x)} = 1
\]
for every $t > 0$.

Obviously, regularly variation and slowly variation are tail behaviours for functions so that they are independent of their initial values. $\ln x$, $\ln^2 x$ are slowly varying functions on $(0, \infty)$. Every slowly varying function varies regularly with exponent 0.

Lemma 3.1 (Main Karamata Theorem, see [7]). Let $H$ be positive monotone on $[0, \infty)$ and set
\[
U_s(x) = \int_0^x y^s H(y)dy, \quad V_s(x) = \int_x^\infty y^s H(y)dy.
\]
(i) If $H$ varies regularly with exponent $\alpha \leq -s - 1$ and $V_s(x) < \infty$, then, as $x \to \infty$,
\[ \frac{x^{s+1}H(x)}{V_s(x)} \to c = -(s + \alpha + 1) \geq 0. \]
Conversely, if this limit exists and is positive, then $V_s$ and $H$ vary regularly with exponents $-c = s + \alpha + 1$ and $\alpha$, respectively, while if this limit is 0, then $V_s$ is a slowly varying function.

(ii) If $H$ varies regularly with exponent $\alpha \geq -s - 1$, then, as $x \to \infty$,
\[ \frac{x^{s+1}H(x)}{U_s(x)} \to c = s + \alpha + 1 \geq 0. \]
Conversely, if this limit exists and is positive, then $U_s$ and $H$ vary regularly with exponents $c = s + \alpha + 1$ and $\alpha$, respectively, while if this limit is 0, then $U_s$ is a slowly varying function.

Lemma 3.2. If $s \geq 0$, then
\[ \sum_{k=1}^{n} \frac{k^s}{(\log_a k)^2} = O \left( \frac{n^{s+1}}{(\log_a n)^2} \right) \quad \text{as } n \to \infty. \]

Proof. For every $s > 0$ and $x$ large enough, $x^s/(\log_a x)^2$ is monotonously increasing. So, there are constants $C_1$ and $C_2$ satisfying
\[ C_1 \int_{1}^{n} \frac{x^s}{(\log_a x)^2} \, dx \leq \sum_{k=2}^{n} \frac{k^s}{(\log_a k)^2} \leq C_2 \int_{2}^{n+1} \frac{x^s}{(\log_a x)^2} \, dx. \]

Since $(\log_a x)^{-2}$ is regularly varying with exponent $\alpha = 0$ (is also slowly varying), by the (ii) of Lemma 3.1, we can obtain
\[ \int_{1}^{n} \frac{x^s}{(\log_a x)^2} \, dx = O \left( \frac{n^{s+1}}{(\log_a n)^2} \right) = \int_{2}^{n+1} \frac{x^s}{(\log_a x)^2} \, dx \quad \text{as } n \to \infty. \]
Combining (3.2) and (3.3), it’s easy to see that (3.1) holds.

If $s = 0$, then we only change the direction of inequalities (3.2) to know that (3.1) is also true. □

With the above preparations, now we start to research the limit properties of model (1.3). Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.3), we obtain the following results about its weak law of large numbers, strong law of large numbers and central limit theorem, respectively.

Theorem 3.3. Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.3), $b_n = (\log_a n)^n$. Then
\[ \frac{S_n}{b_n} \to 0 \quad \text{in probability.} \]
Proof. Let $m_n = \lfloor b_n \rfloor$, it’s easy to verify that $\log_a m_n \sim n \log_a \log_a n$ as $n \to \infty$. Suppose that $X_i = X_i I(X_i \leq b_n), \bar{S}_n = \sum_{i=1}^{n} \bar{X}_i$. We have

(3.5) \[ P \left\{ \bar{S}_n \neq S_n \right\} = P \left\{ \bigcup_{i=1}^{n} \{ \bar{X}_i \neq X_i \} \right\} \]

\[ \leq n P \{ X_1 > b_n \} \]

\[ \leq n \sum_{k=m_n+1}^{\infty} \frac{1}{ck(\log k)^2} \]

\[ \leq \frac{Cn}{\log_a m_n} \]

\[ \leq \frac{C}{\log_a \log_a n} \to 0 \text{ as } n \to \infty. \]

By Lemma 3.2 we also have

\[ \tilde{\mu}_1 = E\bar{X}_1 = \sum_{k=1}^{m_n} \frac{1}{c(\log k)^2} = O \left( \frac{b_n}{(n \log_a \log_a n)^2} \right). \]

Applying the Chebyshev’s inequality and Lemma 3.2 we can get

(3.6) \[ P \left\{ \left| \frac{\bar{S}_n - n\tilde{\mu}_1}{b_n} \right| > \varepsilon \right\} = P \left\{ \left| \sum_{i=1}^{n} (\bar{X}_i - \tilde{\mu}_1) \right| > \varepsilon b_n \right\} \]

\[ \leq \frac{1}{\varepsilon^2 b_n^2} E \left| \sum_{i=1}^{n} (\bar{X}_i - \tilde{\mu}_1) \right|^2 \]

\[ \leq \frac{1}{\varepsilon^2 b_n^2} E \left( \sum_{i=1}^{n} \bar{X}_i^2 \right) \]

\[ \leq \frac{n}{\varepsilon^2 b_n^2} \sum_{k=1}^{m_n} \frac{k}{c(\log k)^2} \]

\[ \leq \frac{Cn}{b_n^2} \frac{m_n^2}{(\log_a m_n)^2} \]

\[ \leq \frac{C}{n(\log_a \log_a n)^2} \to 0 \text{ as } n \to \infty. \]

By (3.5) and (3.6) we can also get

(3.7) \[ \frac{S_n - n\tilde{\mu}_1}{b_n} \to 0 \text{ in probability.} \]

Therefore (3.4) follows from (3.7) and $n\tilde{\mu}_1/b_n \to 0$ as $n \to \infty$. The proof of Theorem 3.3 is completed. \qed
Theorem 3.4. Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.3), $b_n = (\log a n)^n$. Then

$$ \liminf_{n \to \infty} \frac{S_n}{b_n} = 0 \text{ a.s., and } \limsup_{n \to \infty} \frac{S_n}{b_n} = +\infty \text{ a.s..} $$

Proof. Since $X_i \geq 0$ a.s. thus

$$ \liminf_{n \to \infty} \frac{S_n}{b_n} \geq 0 \text{ a.s..} $$

From Theorem 3.3 we know that there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$ \lim_{k \to \infty} \frac{S_{n_k}}{b_{n_k}} = 0 \text{ a.s..} $$

Therefore the first part of (3.8) holds.

To prove the second part of (3.8) we first prove

$$ \limsup_{n \to \infty} \frac{X_n}{b_n} = +\infty \text{ a.s..} $$

In fact, for every $M > 0$,

$$ \sum_{n=1}^{\infty} P\{X_n > Mb_n\} = \sum_{n=1}^{\infty} \sum_{k=\lfloor Mb_n \rfloor + 1}^{\infty} \frac{1}{ck (\log a k)^2} $$

$$ \geq C \sum_{n=1}^{\infty} \frac{1}{\log([Mb_n] + 1)} $$

$$ \geq C \sum_{n=1}^{\infty} \frac{1}{n \log a n} = +\infty. $$

By the second Borel-Cantelli lemma we know that

$$ P\left\{ \frac{X_n}{b_n} > M, \text{ i.o.} \right\} = 1. $$

So (3.9) holds. Since $\lim_{n \to \infty} b_{n-1}/b_n = 0$, it follows that

$$ \limsup_{n \to \infty} \frac{S_n}{b_n} = \limsup_{n \to \infty} \left( \frac{X_n}{b_n} + \frac{S_{n-1}}{b_{n-1}} \cdot \frac{b_{n-1}}{b_n} \right) $$

$$ \geq \limsup_{n \to \infty} \frac{X_n}{b_n} + \liminf_{n \to \infty} \frac{S_{n-1}}{b_{n-1}} \cdot \lim_{n \to \infty} \frac{b_{n-1}}{b_n} $$

$$ = +\infty. $$

The last step above follows from (3.9) and the first part of (3.8) proved just now. The proof of Theorem 3.4 is completed. □

Remark 3.5. Since $X_n \geq 0$ a.s., for any $\epsilon > 0$,

$$ \sum_{n=1}^{\infty} P\{X_n < \epsilon b_n\} = \sum_{n=1}^{\infty} P\{X_1 < \epsilon b_n\} = +\infty, $$
by the second Borel-Cantelli lemma we know \( P\{X_n < \epsilon b_n \text{ i.o.}\} = 1 \), consequently

\[
\lim \inf_{n \to \infty} \frac{X_n}{b_n} = 0 \text{ a.s.}
\]

**Theorem 3.6.** Let \( X_1, X_2, \ldots \) be independent random variables with the same distribution described in (1.3), \( t_n = a^{n(\log_a n)^\gamma} \), \( \gamma > 1 \). Then

\[
\lim_{n \to \infty} \frac{S_n}{t_n} = 0 \text{ a.s.}
\]

**Proof.** Let \( \tilde{X}_i = X_i I (X_i < t_n) \). Since

\[
\sum_{n=1}^{\infty} P\{X_n > t_n\} = \sum_{n=1}^{\infty} P\{X_1 > t_n\}
= \sum_{n=1}^{\infty} \sum_{k = [t_n] + 1}^{\infty} \frac{1}{k(\log_a k)^2}
\leq C \sum_{n=1}^{\infty} \frac{1}{\log_a t_n}
\leq C \sum_{n=1}^{\infty} \frac{1}{n(\log_a n)^\gamma} < \infty,
\]

by the Borel-Cantelli lemma we know

\[
P\{\tilde{X}_n \neq X_n \text{ i.o.}\} = 0.
\]

Let \( \tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i \). By Lemma 3.2, it’s easy to get

\[
\tilde{\mu}_1 = E\tilde{X}_1 = \sum_{k=1}^{[t_n]} \frac{1}{c(\log_a k)^2} = O \left( \frac{t_n}{(\log_a t_n)^2} \right) = O \left( \frac{t_n}{n^2 (\log_a n)^{2\gamma}} \right).
\]

For any \( \epsilon > 0 \), applying the Chebyshev’s inequality and Lemma 3.2 we have

\[
\sum_{n=1}^{\infty} P\left\{ \frac{\tilde{S}_n - \epsilon \tilde{\mu}_1}{t_n} > \epsilon \right\} = \sum_{n=1}^{\infty} P\left\{ \tilde{S}_n > \epsilon \tilde{\mu}_1 + \epsilon t_n \right\}
\leq \sum_{n=1}^{\infty} P\left\{ \tilde{S}_n > \epsilon t_n \right\}
\leq \sum_{n=1}^{\infty} \frac{1}{\epsilon^2 t_n^2} E\tilde{S}_n^2
= \sum_{n=1}^{\infty} \frac{n}{\epsilon^2 t_n^2} E\tilde{X}_1^2
= \sum_{n=1}^{\infty} \frac{n}{\epsilon^2 t_n^2} \sum_{k=1}^{[t_n]} \frac{k}{c(\log_a k)^2}
\]
From the Borel-Cantelli lemma it follows that

\[
P\left\{ \frac{\tilde{S}_n - n\tilde{\mu}_1}{t_n} > \varepsilon \text{ i.o.} \right\} = 0,
\]

thus

\[
\lim_{n \to \infty} \frac{\tilde{S}_n - n\tilde{\mu}_1}{t_n} = 0 \text{ a.s.}
\]

Since

\[
\frac{n\tilde{\mu}_1}{t_n} = O\left( \frac{1}{n (\log a n)^{2\gamma}} \right) \to 0,
\]

we have

\[
\lim_{n \to \infty} \frac{\tilde{S}_n}{t_n} = 0 \text{ a.s.}
\]

It follows from (3.10) that there exists a positive integer \(N\) such that \(\tilde{X}_i = X_i\) a.s. for all \(i > N\). Now, for \(n > N\),

\[
\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_N + X_{N+1} + \cdots + X_n = \tilde{S}_N + S_n - S_N \text{ a.s.}
\]

Because \(\lim_{n \to \infty} \tilde{S}_N/t_n = 0 \text{ a.s. and } \lim_{n \to \infty} S_n/t_n = 0 \text{ a.s., so}
\]

\[
\lim_{n \to \infty} \frac{S_n}{t_n} = \lim_{n \to \infty} \frac{\tilde{S}_n}{t_n} - \lim_{n \to \infty} \frac{\tilde{S}_N}{t_n} + \lim_{n \to \infty} \frac{S_N}{t_n} = 0 \text{ a.s.}
\]

The proof of Theorem 3.6 is completed.

**Theorem 3.7.** Let \(X_1, X_2, \ldots\) be independent random variables with the same distribution described in (1.3). \(\tilde{X}_i = X_i I (X_i \leq n),\) \(\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i,\) \(\tilde{\mu}_1 = E\tilde{X}_1,\) \(\tilde{\sigma}_1^2 = \text{Var}\tilde{X}_1.\) Then

\[
\frac{\tilde{S}_n - n\tilde{\mu}_1}{\tilde{\sigma}_1 \sqrt{n}} \xrightarrow{d} N(0, 1).
\]

**Proof.** Using Lemma 3.2 and the \(C_r\)-inequality, we can get

\[
\tilde{\mu}_1 = E\tilde{X}_1 = \sum_{k=1}^n \frac{1}{c(\log a k)^2} = O\left( \frac{n}{(\log a n)^2} \right),
\]

\[
\tilde{\sigma}_1^2 = \text{Var}\tilde{X}_1 = E\tilde{X}_1^2 - \tilde{\mu}_1^2 = \sum_{k=1}^n \frac{k}{c(\log a k)^2} - \tilde{\mu}_1^2 = O\left( \frac{n^2}{(\log a n)^2} \right),
\]

\[
E|\tilde{X}_1 - \tilde{\mu}_1|^3 \leq 4 \left( E\tilde{X}_1^3 + \tilde{\mu}_1^3 \right) = 4 \sum_{k=1}^n \frac{k^2}{c(\log a k)^2} + 4\tilde{\mu}_1^3 = O\left( \frac{n^3}{(\log a n)^2} \right).
\]
Therefore
\[
\hat{\rho} = \frac{E|\hat{X}_1 - \hat{\mu}_1|^3}{\hat{\sigma}_1^3} = O \left( \frac{n^3}{(\log_2 n)^2} \cdot \frac{(\log_2 n)^3}{n^3} \right) = O(\log_2 n).
\]

By the Esseen’s inequality (see [12]) we know
\[
\sup_{-\infty < x < \infty} \left| P \left\{ \frac{\hat{S}_n - n\hat{\mu}_1}{\hat{\sigma}_1 \sqrt{n}} \leq x \right\} - \Phi(x) \right| \leq \frac{C\hat{\rho}}{\sqrt{n}} \leq \frac{C\log_2 n}{\sqrt{n}} \to 0 \text{ as } n \to \infty,
\]
where \(\Phi(x)\) denotes the distribution function of the standard normal random variable \(N(0, 1)\). The proof of Theorem 3.7 is completed. \(\square\)

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References

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