A NEW OPTIMAL EIGHTH-ORDER FAMILY OF MULTIPLE ROOT FINDERS

DEJAN ĆEBIĆ AND NEBOJŠA M. RALEVIĆ

Abstract. This paper presents a new optimal three-step eighth-order family of iterative methods for finding multiple roots of nonlinear equations. Different from the all existing optimal methods of the eighth-order, the new iterative scheme is constructed using one function and three derivative evaluations per iteration, preserving the efficiency and optimality in the sense of Kung-Traub’s conjecture. Theoretical results are verified through several standard numerical test examples. The basins of attraction for several polynomials are also given to illustrate the dynamical behaviour and the obtained results show better stability compared to the recently developed optimal methods.

1. Introduction

Approximating the roots of the nonlinear equation \( f(x) = 0 \) is one of the most important tasks in numerical mathematics with many applications in engineering and science. There is a great amount of literature that deals with the problem of determining the simple root (say \( \alpha \)) of the nonlinear equation, but not so many papers address the case when \( \alpha \) is the root of multiplicity \( m > 1 \) (which means that \( f^{(i)}(\alpha) = 0 \) for \( i = 0, 1, \ldots, m - 1 \), and \( f^{(m)}(\alpha) \neq 0 \)).

A very basic multiple root finding method is modified Newton’s method [15] (also known as Rall’s method [14])

\[
x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}, \quad n = 1, 2, \ldots
\]

This one-point method is quadratically convergent and therefore optimal in the sense of Kung-Traub’s conjecture [9] which states that any multipoint iterative scheme that requires \( s \) function/derivative evaluations per iteration can reach at most \( 2^{s-1} \) convergence order.

In the last decade, many researchers have developed the multistep methods of the higher convergence order using method (1) as the first step in their
iterative schemes. For example, such Newton-type methods of optimal fourth-order can be found in [10,11,25] and some more efficient optimal eighth-order methods are described in [1,3,4,7,23,24]. Nevertheless, the majority of those methods could be considered as the generalizations of the optimal two or three-step optimal methods constructed for finding the simple roots. Thus, several well-known fourth and eighth-order multiple root finders have been derived in [13] directly from the previously published methods for simple roots by using a relatively simple technique for generalization. All of the eighth-order methods mentioned above require three function evaluations and one derivative evaluation per iteration.

Very recently, Sharma and Kumar [17] have constructed the optimal eighth-order iterative scheme

$$y_n = x_n - m f(x_n),$$
$$z_n = y_n - m Q(u_n) f(x_n),$$
$$x_{n+1} = z_n - mu_n w_n W(u_n, w_n) f(x_n),$$

for $m > 1$, where $u_n = \left( \frac{f'(y_n)}{f'(x_n)} \right)^{\frac{1}{m-1}}$, $v_n = \left( \frac{f'(z_n)}{f'(x_n)} \right)^{\frac{1}{m-1}}$, $w_n = \frac{v_n}{u_n}$, while $Q(u)$ and $W(u, w)$ are analytic functions in a neighborhood of 0 and $(0,0)$ that satisfy conditions summarized in the theorem [17, page 319]. Despite all existing optimal eighth-order methods, method (2) uses two function and two derivative evaluations per iteration.

In the next section, we present the new three-step family of iterative methods of the optimal eighth-order. Uniqueness of the family lies in the fact that the iterative scheme requires one function and three derivative evaluations per iteration, in contrast to the all other eighth-order methods including Sharma and Kumar’s method (2). In the last two sections, the numerical efficiency and the dynamic behaviour of the new family members are compared to the other recently developed optimal methods.

2. A new iterative family

The first two steps of method (2) are actually the optimal fourth-order derived by Liu and Zhou [11]. They established the following conditions function $Q$ must satisfy to provide the optimal fourth-order of convergence of Liu-Zhou method,

$$Q(0) = 0, \quad Q'(0) = 1, \quad Q''(0) = \frac{4m}{m-1}.$$

Sharma and Kumar have improved the Liu-Zhou method by adding the third step that involves the additional function evaluation $f(z_n)$. In contrast to this, the new iterative family consists of the Liu-Zhou method and the third step
that requires the additional evaluation of $f'(z_n)$. Hence, the general form of the new iterative scheme is

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - mQ(u_n) \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - mG(u_n, w_n) \frac{f(x_n)}{f'(x_n)},$$

where $u_n = \left( \frac{f'(y_n)}{f'(x_n)} \right)^{\frac{1}{m-1}}$ and $w_n = \left( \frac{f'(z_n)}{f'(x_n)} \right)^{\frac{1}{m-1}}$, while $Q(u)$ and $G(u, w)$ are analytic in neighborhoods of 0 and $(0, 0)$, respectively.

Since the new scheme should provide eighth convergence order, we need proper forms of Taylor’s expansions of $f(x_n)$ and $f'(x_n)$ about $\alpha$, given by

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^m \cdot \left(1 + \sum_{i=1}^{s} c_i e_n^i + O(e_n^9)\right),$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \cdot \left(1 + \sum_{i=1}^{s} \frac{m+i}{m} c_i e_n^i + O(e_n^9)\right),$$

where $e_n = x_n - \alpha$ and $c_i = \left( m!/(m+i)! \right) \cdot \left( f^{(m+i)}(\alpha)/f^{(m)}(\alpha) \right)$ for $i \geq 1$. Let $\hat{e}_n = y_n - \alpha$ and $\tilde{e}_n = z_n - \alpha$ be the errors of the first and second step in the $n$-th iteration. From (5) and (6), the error of the first step $\hat{e}_n$ in terms of $e_n$ equals

$$\hat{e}_n = e_n^2 \left[ \frac{c_1}{m} + (2mc_2 - (1+m)c_1^2) \frac{e_n}{m^2} \right. \left. + \left( (1+m)^2 c_1^3 - m(4+3m)c_1c_2 + 3m^2c_3 \right) \frac{e_n^2}{m^3} \right.$$

$$\left. + \left( - (1+m)^3 c_1^4 + 2m(1+m)(3+2m)c_2^2 \right) \frac{e_n^3}{m^4} \right. \left. - 2m^2(2+m)c_2^3 - 2m^2(3+2m)c_1c_3 + 4m^3c_4 \right) \frac{e_n^4}{m^5} \right.$$ 

$$\left. + \left( (1+m)^4 c_1^5 - m(1+m)^2(8+5m)c_1^3c_2 + 2m^2(1+m)(9+5m)c_1^2c_3 \right. \right. \left. + m^2c_1[(2+m)(6+5m)c_2^2 - m(8+5m)c_4] \right. \left. + m^3(5mc_5 - (12 + 5m)c_2c_3) \right) \frac{e_n^5}{m^6} + \cdots \right] + O(e_n^9).$$

It is clear that

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \cdot \left(1 + \sum_{i=1}^{s} \frac{m+i}{m} c_i \hat{e}_n^i + O(\hat{e}_n^9)\right).$$
Therefore, using (6) and (8), after substituting (7) into (8), we get

\[
 u_n = \int_{y_n}^{z_n} f'(y) \, dy = e_n \frac{c_1}{m} + (2(m-1)c_2 - (1+m)c_1^2) \frac{e_n}{m(m-1)} \\
+ ((-2 - m + 2m^2 + 3m^3 + 2m^4)c_1^2 - 2m^2(-4 + m + 3m^2)c_1c_2 \\
+ 6(m-1)^2m^2c_3) \frac{e_n^2}{2m^3(m-1)^2} \\
+ ((1 + m)^2(6 - 16m + 7m^2 - m^3 + 6m^4)c_1^4 \\
- 6m(4 - m - 8m^2 - 3m^3 + 4m^4 + 4m^5)c_1^2c_2 + 12(m-1)^2m^3c_3 \\
+ 12(m-1)^2m^3((2 + m)c_2 - 2(m-1)c_1)) \frac{e_n^3}{6m^4(m-1)^3} \\
+ \cdots + O(e_n^9).
\]

If the function \( Q(\cdot) \) satisfies conditions (3), then from (5), (6), (9) and Taylor's expansion of \( Q(u_n) \) about 0, the error of the second step \( \tilde{e}_n \) has a form

\[
 \tilde{e}_n = \left( (3(2 + m + 8m^2 + m^3) - (m - 1)^2Q''(0))c_1^2 \\
+ 6m^2(1 - m)c_1c_2 \right) \frac{e_n^4}{6m^3(m-1)^2} \\
+ \left[ - 48m^3(m - 1)^2c_2^2 - 48m^3(m - 1)^2c_1c_3 \\
+ 24m(m-1)(4 + 2 + 24m^2 + 4m^3 - (m - 1)^2Q'''(0))c_1^2c_2 \\
+ (4(1 + m)(12 + 5m + 9m^2 - 63m^3 - 7m^4) \\
+ 4(m - 1)^2(3m^2 + 4m - 1)Q''''(0) \\
- (m - 1)^3Q^{(4)}(0))c_1^4 \right] \frac{e_n^5}{24m^4(m-1)^3} + \cdots + O(e_n^9).
\]

Substituting \( \tilde{e}_n \) instead \( e_n \) into (6) to get \( f'(z_n) \), using (8) and Taylor's expansion of \( m - 1 \)st root, we have

\[
 w_n = \int_{y_n}^{z_n} f'(y) \, dy = (6 + 3m + 24m^2 + m^3 - (m - 1)^2Q''''(0))c_2^2 \\
+ 6m^2(1 - m)c_2 \frac{e_n^2}{6m^2(m-1)^2} \\
+ \left[ 16m(m-1)(3 + 3m + 24m^2 + 3m^3 - (m - 1)^2Q''''(0))c_1c_2 \\
+ (8(1 + m)(3 + 4m - 6m^2 - 21m^3 - 2m^4) \\
+ 8m(2 + m)(m - 1)^2Q''''(0) - (m - 1)^3Q^{(4)}(0))c_1^2 \right] e_n^4 \\
+ \cdots + O(e_n^9).
\]
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\[-48m^3(m - 1)^2c_3 + \cdots + O(e_n^8).\]

Since \(\tilde{e}_n = z_n - \alpha = O(e_n^4)\) and \(f(x_n)/f'(x_n) = O(e_n)\), from the third step of (4) it is clear that the function \(G(u_n, w_n)\) should be of third-order, which means that, with respect to (9) and (11), in Taylor’s expansion of \(G\) about \((0, 0)\) we have

\[G(0, 0) = 0, \quad \frac{\partial G}{\partial u}(0, 0) = 0, \quad \frac{\partial^2 G}{\partial u^2}(0, 0) = 0 \quad \text{and} \quad \frac{\partial G}{\partial w}(0, 0) = 0.\]

Moreover, to achieve the optimal eighth-order, all coefficients of \(e_n, e_n^2, \ldots, e_n^7\) should vanish in error equation \(e_{n+1} = x_{n+1} - \alpha\). Thus, from the third step of (4), taking into account (5), (6), (10) and Taylor’s expansion of \(G(u_n, w_n)\), after simple computation we get the following conditions which provide optimality of the method:

\[G(0, 0) = \partial G/\partial u(0, 0) = \partial^2 G/\partial u^2(0, 0) = \partial G/\partial w(0, 0) = 0 \quad \text{for} \quad i \in \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad j \in \{1, 2, 3\}\]

\[\begin{align*}
\frac{\partial^3 G}{\partial u^3}(0, 0) &= 4m^2 - 1, \\
\frac{\partial^3 G}{\partial u \partial w^2}(0, 0) &= 6m^2 + 1 + Q^{(m)}(0), \\
\frac{\partial^4 G}{\partial u^4}(0, 0) &= -48(m + 1)(2m^2 + 1) / m(m - 1)^2 + 8m^2 + 1 + Q^{(m)}(0) + Q^{(4)}(0).
\end{align*}\]

Therefore, if conditions (13) are satisfied, it yields

\[x_{n+1} = z_n - mG(u_n, w_n) f(x_n) / f'(x_n) = x_n - \alpha + O(e_n^8),\]

i.e., \(e_{n+1} = x_{n+1} - \alpha = O(e_n^8)\).

The above discussion is summarized in the following theorem.

**Theorem 2.1.** Let \(\alpha\) be a multiple root of known multiplicity \(m\) of a sufficiently differentiable function \(f(x)\). If the initial iteration \(x_0\) is close enough to \(\alpha\), and if functions \(Q\) and \(G\) satisfy conditions (3) and (13) respectively, then family of methods defined by (4) is of optimal eighth convergence order.

**Remark 1.** Some parts of the expressions in the previous analysis are intentionally omitted for the sake of simplicity. All the results have been done and verified with the aid of Mathematica’s symbolic computation system.

**Remark 2.** Since family (4) possesses eighth convergence order, and requires one function and three derivative evaluations per iteration, it is clear that the family is optimal in the sense of Kung-Traub’s conjecture.

Three special cases of (4) based on the different choices of \(Q(u_n)\) and \(G(u_n, w_n)\) have been considered for the numerical comparisons. The first new method
is defined as
\[ y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \]
\[ z_n = y_n - m \left( u_n + \frac{2m}{m-1} u_n^2 \right) \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = z_n - m \frac{\left( u_n + \frac{2}{m(m-1)} u_n^2 \right) w_n}{1 - 2(m+1) m^{-1} u_n + \frac{3(m+1)}{m-1} u_n^2 - w_n} \cdot \frac{f(x_n)}{f'(x_n)}, \]
and denoted by \( \text{NM1} \). The second one, denoted by \( \text{NM2} \), has the following form
\[ y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \]
\[ z_n = y_n - m \left( u_n + \frac{2}{m(m-1)} u_n^2 \right) \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = z_n - m \frac{\left( u_n + \frac{2}{m(m-1)} u_n^2 \right) w_n}{1 - 2(m+1) m^{-1} u_n + \frac{3(m+1)}{m-1} u_n^2 - w_n} \cdot \frac{f(x_n)}{f'(x_n)}. \]

The third one is \( \text{NM3} \) given by
\[ y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \]
\[ z_n = y_n - m \left( u_n + \frac{2}{m(m-1)} u_n^2 \right) \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = z_n - m \frac{\left( u_n + \frac{2}{m(m-1)} u_n^2 \right) w_n + u_n w_n^2}{1 - 2(m+1) m^{-1} u_n + \frac{3(m+1)}{m-1} u_n^2 - w_n} \cdot \frac{f(x_n)}{f'(x_n)}. \]

3. Numerical comparison

In this section, several test examples are employed in order to verify the theoretic results from the previous section and to illustrate the effectiveness of the methods \( \text{NM1}, \text{NM2} \) and \( \text{NM3} \). The results are compared with the very recently developed Newton-type methods of the optimal eighth-order.

Such existing method is the one proposed by Zafar et al. [22], denoted by \( \text{ZCJT} \), with the following structure
\[ y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \]
\[ z_n = y_n - m \frac{1 + 8u_n + 11u_n^2}{1 + 6u_n} \cdot \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = z_n - m w_n \left( 1 + t_n + \frac{1}{2} t_n^2 + u_n (2 + 4t_n) \right) \frac{f(x_n)}{f'(x_n)}. \]
where \( u_n = \left( \frac{f(y_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), \( t_n = \left( \frac{f(z_n)}{f'(y_n)} \right)^{\frac{1}{4}} \) and \( w_n = \left( \frac{f(z_n)}{f'(y_n)} \right)^{\frac{1}{4}} \).

Behl et al. [2] have developed an efficient family (14) and tested several special cases. We choose two members of this family with the best performance in the original research. This family has the following general form

\[
y_n = x_n - m \frac{f(x_n)}{f'(x_n)},
\]
\[
z_n = y_n - mH(v_n) \frac{f(x_n)}{f'(x_n)},
\]
\[
x_{n+1} = z_n - w_n u_n \left( G(u_n) + \frac{mw_n}{1 - 4u_n} \right) \frac{f(x_n)}{f'(x_n)},
\]

where \( u_n = \left( \frac{f(y_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), \( w_n = \left( \frac{f(z_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), \( v_n = \frac{1 + au_n}{4} \) for some real numbers \( \alpha \neq \beta \) and \( H \) and \( G \) are analytic functions in neighborhoods of 1 and 0. The first chosen special case is denoted by \textbf{BAASA1} for and \( \alpha = 1/2, \beta = -3/2 \) and functions

\[
H(v_n) = m \frac{\alpha - \beta + 2v_n - 2}{\alpha - \beta}, \quad G(u_n) = m \left( 1 + 2u_n + (1 - 2\beta)u_n^2 + 2(\beta^2 - 2\beta - 2)u_n^4 \right).
\]

The second special case \textbf{BAASA2} uses \( \alpha = 0, \beta = -2 \) and functions

\[
H(v_n) = m \frac{\alpha - \beta + 2v_n - 2}{\alpha - \beta}, \quad G(u_n) = \frac{m(2\beta^2u_n + \beta(2 - 4u_n^2) - (3u_n + 1)^2)}{2\beta^2u_n + \beta(2 - 4u_n^2) - 4u_n - 1}.
\]

Kumar et al. [8] have constructed the eighth-order family

\[
y_n = x_n - m \frac{f(x_n)}{f'(x_n)},
\]
\[
z_n = y_n - m u_n (1 + 2u_n - u_n^2) \frac{f(x_n)}{f'(x_n)},
\]
\[
x_{n+1} = z_n - m (1 + u_n) v_n H(v_n) \frac{f(x_n)}{f'(x_n)} - m (u_n + w_n) v_n G(u_n) \frac{f(x_n)}{f'(x_n)},
\]

where \( u_n = \left( \frac{f(y_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), \( v_n = \left( \frac{f(z_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), \( w_n = \left( \frac{f(z_n)}{f'(y_n)} \right)^{\frac{1}{4}} \), while \( H \) and \( G \) are analytic functions in neighborhood of 0. The special case \textbf{KKSdA} with the best performance in the original research is the one where

\[
H(v_n) = \frac{1}{1 - 4v_n} \quad \text{and} \quad G(u_n) = \frac{1 + 6u_n}{1 + 6u_n + 6u_n^2}.
\]

Finally, the variant of Sharma-Kumar’s family (2) that we use for the comparison has been established in [17] for

\[
Q(u_n) = u_n + \frac{2m}{m - 1} u_n^2 + \frac{6m^4 + m^3 - 5m^2 - 3m - 3}{3(m - 1)^2(m^2 - m - 1)} u_n^3;
\]
\[
W(u_n, w_n) = 1 + 2u_n + \frac{m - 1}{m} w_n + \frac{u_n}{3} \left( \frac{k_1 w_n}{m^2} + \frac{k_2 w_n}{m^3 - 2m^2 + 1} \right),
\]
where \( k_1 = 6(2m^2 - 2m - 1) \) and \( k_2 = 9m^3 - 8m^2 - 5m + 6 \). This method is denoted by SK.

Table 1 displays the test functions used for the comparison, with appropriate root \( \alpha \) and its multiplicity \( m \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( \alpha )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = (x - x^3 \cos \frac{2\pi}{5} + \frac{1}{1+x^2} - 30.1)(x - 3)^3 )</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>( f_2(x) = \exp \left( \frac{((x-0.5)^2 + 3)^2}{x^3 + \cos((x-0.5)^2 + 3)} \right) - 1 )</td>
<td>0.5 + \sqrt{3}i</td>
<td>2</td>
</tr>
<tr>
<td>( f_3(x) = x^4 + 11.3x^3 + 47.49x^2 + 83.06325x + 51.23266875 )</td>
<td>-2.85</td>
<td>2</td>
</tr>
<tr>
<td>( f_4(x) = (\cos x - x)^3 )</td>
<td>0.7390851...</td>
<td>3</td>
</tr>
<tr>
<td>( f_5(x) = (\arcsin(x^3 - 1) + e^x - 3)^2 )</td>
<td>1.0579494...</td>
<td>2</td>
</tr>
<tr>
<td>( f_6(x) = (2x - e^{-x} + \sin x^2 - 3)^3 )</td>
<td>3.8173523...</td>
<td>5</td>
</tr>
</tbody>
</table>

Tables 2-7 present the number of iterations (\( it \)) required to satisfy stopping criterion \( |f(x_n)| < 10^{-1000} \). Along with that, errors \( |x_n - \alpha| \) and residual errors \( |f(x_n)| \) are given for each method after the third iteration. The tables also show the computational order of convergence [21] given by

\[
\text{COC} = \frac{\log |(x_n - \alpha)/(x_{n-1} - \alpha)|}{\log |(x_{n-1} - \alpha)/(x_{n-2} - \alpha)|},
\]

which has been used to numerically check the convergence order of the proposed methods. The last columns display CPU time computed as the average of 25 performances of each method. If an algorithm fails to find the root within 100 iterations, it is denoted by “-”.

All computations have been done using Mathematica program package with the aid of SetPrecision function with 10000 precision digits. The performances of the computer have been 64-bit Windows 10 Pro operating system and AMD Ryzen 7 1700 eight-core CPU 3.00 GHz processor.

| method | it | \( |x_3 - \alpha| \) | \( |f(x_3)| \) | COC | CPU |
|---|---|---|---|---|---|
| ZCJT | 3 | \( 1.4577 \cdot 10^{-818} \) | \( 1.8392 \cdot 10^{-1088} \) | 8.0000 | 0.0312 |
| BAASA1 | 3 | \( 4.5649 \cdot 10^{-863} \) | \( 5.5386 \cdot 10^{-4011} \) | 8.0000 | 0.0306 |
| BAASA2 | 3 | \( 2.3604 \cdot 10^{-863} \) | \( 2.0472 \cdot 10^{-4012} \) | 8.0000 | 0.0294 |
| KKSdA | 3 | \( 9.0274 \cdot 10^{-860} \) | \( 1.6751 \cdot 10^{-3994} \) | 8.0000 | 0.0325 |
| SK | 3 | \( 1.4194 \cdot 10^{-796} \) | \( 1.6096 \cdot 10^{-957} \) | 8.0000 | 0.0412 |
| NM1 | 3 | \( 1.0260 \cdot 10^{-857} \) | \( 3.1769 \cdot 10^{-4284} \) | 8.0000 | 0.0469 |
| NM2 | 3 | \( 1.5370 \cdot 10^{-865} \) | \( 2.3963 \cdot 10^{-4323} \) | 8.0000 | 0.0488 |
| NM3 | 3 | \( 8.9639 \cdot 10^{-782} \) | \( 1.6170 \cdot 10^{-3904} \) | 8.0000 | 0.0506 |
Table 3. Numerical results for $f_2(x)$ and $x_0 = 0.495 + 1.72i$

| method  | it | $|x_3 - \alpha|$ | $|f(x_3)|$ | COC | CPU  |
|---------|----|-----------------|-------------|-----|------|
| ZCJT    | 5  | $7.1869 \cdot 10^{-216}$ | $3.0946 \cdot 10^{-431}$ | 6.0104 | 0.155 |
| BAASA1  | 4  | $1.7861 \cdot 10^{-229}$ | $1.9113 \cdot 10^{-485}$ | 14.050 | 0.132 |
| BAASA2  | 5  | $9.1599 \cdot 10^{-123}$ | $5.1996 \cdot 10^{-245}$ | 6.0250 | 0.157 |
| KKSdA   | 4  | $4.2299 \cdot 10^{-413}$ | $1.1072 \cdot 10^{-825}$ | 4.0000 | 0.142 |
| SK      | 4  | $2.7323 \cdot 10^{-199}$ | $4.4729 \cdot 10^{-398}$ | 4.0000 | 0.156 |
| NM1     | 3  | $1.3399 \cdot 10^{-808}$ | $1.0757 \cdot 10^{-1616}$ | 8.0000 | 0.140 |
| NM2     | 3  | $1.3120 \cdot 10^{-790}$ | $1.0313 \cdot 10^{-1580}$ | 8.0000 | 0.143 |
| NM3     | 3  | $2.1424 \cdot 10^{-804}$ | $2.7499 \cdot 10^{-1608}$ | 8.0000 | 0.143 |

Table 4. Numerical results for $f_3(x)$ and $x_0 = -3.4$

| method  | it | $|x_3 - \alpha|$ | $|f(x_3)|$ | COC | CPU  |
|---------|----|-----------------|-------------|-----|------|
| ZCJT    | 13 | $12.452$         | $23981$     | 6.0026 | 0.0950 |
| BAASA1  | -- | --              | --          | --  | --   |
| BAASA2  | -- | --              | --          | --  | --   |
| KKSdA   | 11 | $1.4981$         | $0.29865$   | 13.981 | 0.0844 |
| SK      | 4  | $3.3798 \cdot 10^{-64}$ | $2.3988 \cdot 10^{-127}$ | 8.0000 | 0.0081 |
| NM1     | 4  | $6.4848 \cdot 10^{-181}$ | $8.8311 \cdot 10^{-361}$ | 8.0000 | 0.0069 |
| NM2     | 4  | $3.0560 \cdot 10^{-229}$ | $1.9612 \cdot 10^{-457}$ | 8.0000 | 0.0075 |
| NM3     | 4  | $2.8531 \cdot 10^{-267}$ | $1.7095 \cdot 10^{-533}$ | 8.0000 | 0.0081 |

Table 5. Numerical results for $f_4(x)$ and $x_0 = 1$

| method  | it | $|x_3 - \alpha|$ | $|f(x_3)|$ | COC | CPU  |
|---------|----|-----------------|-------------|-----|------|
| ZCJT    | 3  | $1.3542 \cdot 10^{-496}$ | $1.1642 \cdot 10^{-1487}$ | 8.0000 | 0.0575 |
| BAASA1  | 3  | $3.3886 \cdot 10^{-592}$ | $1.8240 \cdot 10^{-1774}$ | 8.0000 | 0.0581 |
| BAASA2  | 7  | $1.6438 \cdot 10^{-39}$  | $2.0821 \cdot 10^{-116}$  | 2.0000 | 0.1540 |
| KKSdA   | 3  | $2.8142 \cdot 10^{-483}$ | $1.0448 \cdot 10^{-1447}$ | 8.0000 | 0.0594 |
| SK      | 3  | $1.7382 \cdot 10^{-492}$ | $2.4620 \cdot 10^{-1475}$ | 8.0000 | 0.0431 |
| NM1     | 3  | $3.2879 \cdot 10^{-501}$ | $1.6661 \cdot 10^{-1501}$ | 8.0000 | 0.0569 |
| NM2     | 4  | $1.9335 \cdot 10^{-193}$ | $3.3884 \cdot 10^{-578}$  | 2.2831 | 0.0663 |
| NM3     | 3  | $5.5417 \cdot 10^{-527}$ | $7.9779 \cdot 10^{-1579}$ | 8.0000 | 0.0494 |

Table 6. Numerical results for $f_5(x)$ and $x_0 = 0.9$

| method  | it | $|x_3 - \alpha|$ | $|f(x_3)|$ | COC | CPU  |
|---------|----|-----------------|-------------|-----|------|
| ZCJT    | 5  | $1.7022 \cdot 10^{-35}$ | $7.2776 \cdot 10^{-66}$ | 14.183 | 0.097 |
| BAASA1  | 6  | $1.2364 \cdot 10^{-7}$  | $3.8304 \cdot 10^{-13}$ | 8.0000 | 0.188 |
| BAASA2  | 8  | $9.8845 \cdot 10^{-6}$  | $2.4539 \cdot 10^{-9}$  | 6.0159 | 0.159 |
| KKSdA   | 6  | $2.7924 \cdot 10^{-5}$  | $1.9585 \cdot 10^{-8}$  | 8.0000 | 0.119 |
| SK      | 4  | $2.9970 \cdot 10^{-144}$ | $2.2559 \cdot 10^{-286}$ | 3.4290 | 0.086 |
| NM1     | 4  | $7.2622 \cdot 10^{-341}$ | $1.3246 \cdot 10^{-679}$ | 8.0000 | 0.093 |
| NM2     | 4  | $1.6300 \cdot 10^{-416}$ | $6.6730 \cdot 10^{-831}$ | 8.0000 | 0.094 |
| NM3     | 4  | $5.6748 \cdot 10^{-423}$ | $8.0881 \cdot 10^{-844}$ | 8.0000 | 0.095 |
is defined as a set $\{z \in F \text{ }$fixed point if $\exists \alpha \in \mathbb{C}$ such that $F(\alpha) = \alpha$. The set of such points whose orbits tend to the attractor $\alpha$ is called the Fatou set, while the Julia set is its complementary set and it establishes the borders between different basins of attraction. Any attractor is called the Fatou set, while the Julia set is its complementary set. The orbit of any point $z$ is defined as a set $\{z, F(z), F^2(z), \ldots \}$, and if there exist some point $\tilde{z}$ and $k \in \mathbb{N}$ where $F^k(\tilde{z}) = \tilde{z}$ and $F^s(\tilde{z}) \neq \tilde{z}, s < k$, then such point $\tilde{z}$ is called periodic with period $k$. Therefore, the fixed point is periodic with period 1.

If $\alpha$ is an attracting fixed point of $F$, then its corresponding basin of attraction can be defined as a set $A(\alpha)$ given by

$$A(\alpha) = \{z_0 \in \mathbb{C} : F^n(z_0) \to \alpha, n \to \infty\},$$

which means that the basin of attraction consists of the starting points whose orbits tend to the attractor $\alpha$. The set of such points whose orbits converge to any attractor is called the Fatou set, while the Julia set is its complementary set and it establishes the borders between different basins of attraction.

In the ideal cases, if a function has several distinct roots, every initial point should converge to the nearest root applying iterative method, and consequently, the basins boundaries should have smooth form. Nevertheless, for concrete functions and multiple iterative schemes, the dynamical behaviour of the methods are not so predictable, the overlapping of the basins of attraction can be defined as a set

$$A = \{z \in F \text{ }$$and $\exists \alpha \in \mathbb{C}$ such that $F(\alpha) = \alpha$. The set of such points whose orbits tend to the attractor $\alpha$ is called the Fatou set, while the Julia set is its complementary set. The orbit of any point $z$ is defined as a set $\{z, F(z), F^2(z), \ldots \}$, and if there exist some point $\tilde{z}$ and $k \in \mathbb{N}$ where $F^k(\tilde{z}) = \tilde{z}$ and $F^s(\tilde{z}) \neq \tilde{z}, s < k$, then such point $\tilde{z}$ is called periodic with period $k$. Therefore, the fixed point is periodic with period 1.

In the ideal cases, if a function has several distinct roots, every initial point should converge to the nearest root applying iterative method, and consequently, the basins boundaries should have smooth form. Nevertheless, for concrete functions and multiple iterative schemes, the dynamical behaviour of the methods are not so predictable, the overlapping of the basins of attraction

| method | it | $|x_3 - \alpha|$ | $|f(x_3)|$ | COC | CPU |
|--------|----|-----------------|-------------|-----|-----|
| ZCJT   | 7  | $1.1372 \cdot 10^{-24}$ | $3.3738 \cdot 10^{-118}$ | 1.9960 | 0.301 |
| BAASA1 | 6  | $5.9075 \cdot 10^{-8}$ | $1.2765 \cdot 10^{-34}$ | 8.0000 | 0.271 |
| BAASA2 | 6  | $5.9072 \cdot 10^{-8}$ | $1.2762 \cdot 10^{-34}$ | 8.0000 | 0.269 |
| KKSdA  | 4  | $1.2927 \cdot 10^{-28}$ | $6.4040 \cdot 10^{-138}$ | 7.9998 | 0.188 |
| SK     | 5  | $1.8431 \cdot 10^{-16}$ | $3.7735 \cdot 10^{-77}$ | 14.045 | 0.243 |
| NM1    | 4  | $7.5331 \cdot 10^{-93}$ | $4.3038 \cdot 10^{-459}$ | 8.0000 | 0.127 |
| NM2    | 4  | $4.0614 \cdot 10^{-93}$ | $1.9604 \cdot 10^{-460}$ | 8.0000 | 0.122 |
| NM3    | 4  | $6.9922 \cdot 10^{-27}$ | $2.9652 \cdot 10^{-129}$ | 7.9995 | 0.121 |

Obviously, the values of the COC columns confirm eighth-order of convergence of the new proposed methods. According to the numerical results computed for the inner three columns, all new methods are very competitive compared to the existing ones.

4. Dynamical comparison

Another very frequent way of comparing the iterative methods is the analysis of their basins of attraction in the complex plane (see, for example [6,12,16,18]). Through the basins of attraction analysis, researchers are able to visualize the areas of convergence of particular roots of $f(x)$ in the complex plane for the iterative methods under consideration. Here we only give a brief review of some basic concepts related to basins of attraction, while the underlying ideas as well as the description of the dynamical behaviour of the methods in more details can be found in [5,18–20].

For a function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (where $\hat{\mathbb{C}}$ is the Riemann sphere), $z_0 \in \hat{\mathbb{C}}$ is a fixed point if $F(z_0) = z_0$. Fixed point can be attracting, repelling or neutral, if $|F'(z_0)| < 1$, $|F'(z_0)| > 1$ or $|F'(z_0)| = 1$, respectively. The orbit of any point $z$ is defined as a set $\text{orb}(z) = \{z, F(z), F^2(z), \ldots \}$, and if there exist some point $\tilde{z}$ and $k \in \mathbb{N}$ where $F^k(\tilde{z}) = \tilde{z}$ and $F^s(\tilde{z}) \neq \tilde{z}, s < k$, then such point $\tilde{z}$ is called periodic with period $k$. Therefore, the fixed point is periodic with period 1.

In the ideal cases, if a function has several distinct roots, every initial point should converge to the nearest root applying iterative method, and consequently, the basins boundaries should have smooth form. Nevertheless, for concrete functions and multiple iterative schemes, the dynamical behaviour of the methods are not so predictable, the overlapping of the basins of attraction...
is noticeable as well as the chaotic structure of the basins boundaries. Hence, iterative methods with less fractal ‘decorations’ along boundaries are considered as more desirable ones.

The following functions with associated multiple roots and their multiplicity are observed:

1. \( p_1(z) = (z^2 - 1)^2; \alpha_1 = 1, \alpha_2 = -1, m = 2 \)
2. \( p_2(z) = (z^3 + 4z^2 - 10)^3; \alpha_1 \approx 1.3652, \alpha_{2,3} \approx -2.6826 \pm 0.3582i, m = 3 \)
3. \( p_3(z) = (z^3 - z)^4; \alpha_1 = 0, \alpha_{2,3} = \pm 1, m = 4 \)

In those examples, we consider the region \([-3, 3] \times [-3, 3]\) of the complex plane, with 256 \times 256 equally distributed initial points. We picture the dynamical planes for every method described in the previous sections where each initial point is colored associated to the root which it converges to. If the method does not converge (here, this means that the distance after at most 100 iterations is still greater than \(10^{-5}\) to any of the roots), then that point is marked black. The intensity of the color suggests the number of iterations (fewer iterations — lower intensity).
Table 8. The number of black points (in %) for $p_1, p_2$ and $p_3$

<table>
<thead>
<tr>
<th>method</th>
<th>$p_1(z)$</th>
<th>$p_2(z)$</th>
<th>$p_3(z)$</th>
<th>total average</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZCJT</td>
<td>0</td>
<td>1.794</td>
<td>1.448</td>
<td>1.081</td>
</tr>
<tr>
<td>BAASA1</td>
<td>0.003</td>
<td>0.027</td>
<td>0.018</td>
<td>0.016</td>
</tr>
<tr>
<td>BAASA2</td>
<td>1.511</td>
<td>0</td>
<td>0</td>
<td>0.504</td>
</tr>
<tr>
<td>KKSdA</td>
<td>0</td>
<td>0.397</td>
<td>0.366</td>
<td>0.254</td>
</tr>
<tr>
<td>SK</td>
<td>0.629</td>
<td>10.948</td>
<td>4.834</td>
<td>5.470</td>
</tr>
<tr>
<td>NM1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>NM2</td>
<td>0</td>
<td>0</td>
<td>0.024</td>
<td>0.008</td>
</tr>
<tr>
<td>NM3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Tables 8, 9 and 10 show the computed values related to the depiction of the basins of attraction given in Figures 1, 2 and 3. Table 8 displays the percentage of the black points (out of 65536 starting points) for each graph. The values displayed in Table 9 represent the average number of iterations per starting point calculated without black starting points, which means that the average

![Basins of attraction of different methods for polynomial $p_2$](image)

**Figure 2.** Basins of attraction of different methods for polynomial $p_2$ (the first row: ZCJT(left), BAASA1(middle), BAASA2(right), the second row: KKSdA(left), SK(right), the third row: NM1(left), NM2(middle), NM3(right))
does not take into account the number of iterations for initial points that do not reach the neighborhood of any root within 100 iterations. Table 10 presents the CPU time required for the depiction of each graph.

Table 9. The average number of iterations for $p_1, p_2$ and $p_3$

<table>
<thead>
<tr>
<th>method</th>
<th>$p_1(z)$</th>
<th>$p_2(z)$</th>
<th>$p_3(z)$</th>
<th>total average</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZCJT</td>
<td>6.552</td>
<td>15.136</td>
<td>12.767</td>
<td>11.458</td>
</tr>
<tr>
<td>BAASA1</td>
<td>5.565</td>
<td>8.721</td>
<td>8.385</td>
<td>7.557</td>
</tr>
<tr>
<td>BAASA2</td>
<td>3.895</td>
<td>5.361</td>
<td>5.937</td>
<td>5.065</td>
</tr>
<tr>
<td>KKSdA</td>
<td>4.458</td>
<td>9.276</td>
<td>8.426</td>
<td>7.387</td>
</tr>
<tr>
<td>SK</td>
<td>11.179</td>
<td>20.128</td>
<td>15.625</td>
<td>15.644</td>
</tr>
<tr>
<td>NM1</td>
<td>6.609</td>
<td>7.794</td>
<td>5.353</td>
<td>6.585</td>
</tr>
<tr>
<td>NM2</td>
<td>3.568</td>
<td>3.959</td>
<td>5.177</td>
<td>4.235</td>
</tr>
<tr>
<td>NM3</td>
<td>3.577</td>
<td>5.347</td>
<td>5.166</td>
<td>4.697</td>
</tr>
</tbody>
</table>
According to these results, all new methods are very competitive with the previously developed methods. For example, methods NM2 and NM3 are the best performers in terms of the total average of iterations, followed by BAASA2 and NM1. Note that NM1 and NM3 are the only methods without black initial points. Furthermore, the best CPU time results are associated with the new methods.

5. Conclusion

In this paper, we have considered a new optimal three-step iterative family of multiple root finders and compared some special members of the family to several recently published Newton-type optimal methods of eighth order. The construction of the new iterative scheme is based on one function and three derivative evaluations per iteration, which is a unique structure of the algorithm of the eighth convergence order. The eighth-order is empirically checked in the numerical section. The advantage of the proposed methods is their good dynamical performance. For some special cases of the new family, a dynamical analysis suggests better stability and wider basins of attraction compared to existing methods.
A NEW OPTIMAL EIGHTH-ORDER FAMILY OF MULTIPLE ROOT FINDERS 1081


**Dejan Čebić**

Department of Applied Mathematics and Informatics

Faculty of Mining and Geology

University of Belgrade

Djušina 7, Belgrade 11000, Serbia

*Email address:* cebicd@gmail.com, dejan.cebic@rgf.bg.ac.rs

**Nebojša M. Ralević**

Department of Mathematics

Faculty of Technical Sciences

University of Novi Sad

Trg Dositeja Obradovića 7, Novi Sad 21000, Serbia

*Email address:* nralevic@uns.ac.rs