DIRICHLET EIGENVALUE PROBLEMS UNDER MUSIELAK-ORLICZ GROWTH

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Abstract. This paper studies the eigenvalues of the $G(\cdot)$-Laplacian Dirichlet problem

$$\begin{cases}
-\text{div} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \right) = \lambda \left( \frac{g(x, |u|)}{|u|} u \right) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $g$ is the density of a generalized $\Phi$-function $G(\cdot)$. Using the Lusternik-Schnirelmann principle, we show the existence of a nondecreasing sequence of nonnegative eigenvalues.

1. Introduction

In the fields of partial differential equations and the calculus of variations, there has been much research on non-standard growth problems, such as the eigenvalue problems [5]. The study of eigenvalue problems relies on the Lusternik-Schnirelmann (L-S) theory of critical points for an even functional on a manifold. The presentations of this theory, in both finite and infinite-dimensional spaces, can be found in [1,4,14,15].

A mathematical prototype for nonlinear elliptic eigenvalue problems is expressible by involving the $p$-Laplacian operator

$$\begin{cases}
-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \lambda \left( |u|^{p-2} u \right) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $1 < p < \infty$ and $\Omega$ is a bounded domain of $\mathbb{R}^N$. The problem (1.1) has attracted much attention and has been extensively studied in the literature (see for examples [2,7,9]). One of the important consequences of the Lusternik-Schnirelmann principle is the existence, exactly as for the classical Laplace

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operator \((p = 2),\) of an increasing sequence of eigenvalues 
\[\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \quad \lambda_i \to \infty.\]

Later on, this result has been generalized to variable exponent and Orlicz cases
\[
-\text{div} \left( |\nabla u|^{p(x)} \nabla u \right) = \lambda \left( |u|^{p(x)} - u \right) \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial\Omega\]
and
\[
-\text{div} \left( g(\nabla u) \nabla u \right) = \lambda \left( g(|u|) - u \right) \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial\Omega,
\]
where \(x \to p(x)\) is a continuous function on \(\Omega\) such that \(1 < p(x) < \infty\) and \(t \to g(t)\) is the density of a \(\Phi\)-function \(G\) (see \([6,12,13]\)).

One naturally asks whether a similar result holds in the Musielak-Orlicz case. For this, we consider the following eigenvalue problem under generalized Orlicz growth
\[
-\text{div} \left( g(x,|\nabla u|) \nabla u \right) = \lambda \left( g(x,|u|) - u \right) \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial\Omega,
\]
where \(g(x, \cdot)\) is the right-hand derivative of a \(\Phi\)-function \(G(x, \cdot)\). This situation covers not only the variable exponent \(G(x, t) = t^{p(x)}\) and Orlicz case \(G(x, t) = G(t)\), but also the variable exponent perturbation \(G(x, t) = t^{p(x)} \ln(e + t)\), the double phase \(G(x, t) = t^p + a(x)t^q\) and their various combinations (see \([8]\)). Note that, some particular vector inequalities are helpful in the study of the eigenvalue problem for the \(p\)-Laplacian. In our situation, a lack of these inequalities and homogeneity are a major source of difficulties. To overcome these problems, we developed a method inspired by Lieberman’s pioneering article \([10]\), which allows us to apply the L-S principle for establish the existence of a nondecreasing sequence of nonnegative eigenvalue tending to infinity of the problem \((1.2)\) (see Theorem 3.7).

2. Musielak-Orlicz-Sobolev spaces

To deal with the problem \((1.2)\), we need Musielak-Orlicz-Sobolev spaces. Most of the results concerning these spaces are given in Musielak’s monograph \([11]\), hence the alternative name of Musielak-Orlicz spaces.

**Definition 2.1.** A function \(G : \Omega \times [0, \infty) \to [0, \infty]\) is called a generalized \(\Phi\)-function, denoted by \(G(\cdot) \in \Phi(\Omega)\), if the following conditions hold:

- For each \(t \in [0, \infty)\), the function \(G(\cdot, t)\) is measurable.
- For a.e. \(x \in \Omega\), the function \(G(x, \cdot)\) is a \(\Phi\)-function, i.e.,
  1. \(G(x, 0) = \lim_{t \to 0^+} G(x, t) = 0\) and \(\lim_{t \to \infty} G(x, t) = \infty\);
  2. \(G(x, \cdot)\) is increasing and convex.
Note that, a generalized $\Phi$-function can be represented as

$$G(x, t) = \int_0^t g(x, s) \, ds,$$

where $g(x, \cdot)$ is the right-hand derivative of $G(x, \cdot)$. Furthermore for each $x \in \Omega$, the function $g(x, \cdot)$ is right-continuous and nondecreasing.

**Assumptions.** We say that $G(\cdot)$ satisfies

$(SC)$ : If for a.e. $x \in \Omega$, the function $t \to g(x, t)$ is a $C^1(\mathbb{R}^+)$ and there exist two constants $g_0, g^0 > 0$ such that,

$$g_0 \leq \frac{tg'(x, t)}{g(x, t)} \leq g^0.$$

$(A_0)$ : If there exists a constant $c_0 > 1$ such that,

$$\frac{1}{c_0} \leq G(x, 1) \leq c_0 \text{ for a.e. } x \in \Omega.$$

$(A_1)$ : If there exists $C > 0$ such that, for every $x, y \in B_R \subset \Omega$ with $R \leq 1$, we have

$$G_{B_R}(x, t) \leq CG_{B_R}(y, t) \text{ when } G_{B_R}^-(t) \in \left[1, \frac{1}{R^N}\right],$$

where $G_{B_R}^-(t) := \inf_{B_R} G(x, t)$.

**Remark 2.2.** 1) Note that $(A_1)$ corresponds to local log-Holder continuity in the variable exponent case (see Proposition 7.1.2 in [8]). In the double phase case $G(x, t) = t^p + a(x)t^q$, condition $(A_1)$ is equivalent to $a(y) - a(x) \leq C|y - x|^\frac{p}{q-p}$ (see Proposition 7.2.2 in [8]).

2) The condition $(SC)$ implies

$$g_0 + 1 \leq \frac{tg(x, t)}{G(x, t)} \leq g^0 + 1.$$ 

So, we have the following inequalities [3]

(2.1) \[ \sigma^{g_0+1}G(x, t) \leq G(x, \sigma t) \leq \sigma^gG(x, t) \text{ for } x \in \Omega, \ t \geq 0 \text{ and } \sigma \geq 1. \]

(2.2) \[ \sigma^gG(x, t) \leq G(x, \sigma t) \leq \sigma^{g_0+1}G(x, t) \text{ for } x \in \Omega, \ t \geq 0 \text{ and } \sigma \leq 1. \]

**Definition 2.3.** We define $G^*(\cdot)$ the conjugate $\Phi$-function of $G(\cdot)$, by

$$G^*(x, s) := \sup_{t \geq 0} (st - G(x, t)) \text{ for } x \in \Omega \text{ and } s \geq 0.$$ 

Note that $G^*(\cdot)$ is also a generalized $\Phi$-function and can be represented as

$$G^*(x, t) = \int_0^t g^{-1}(x, s) \, ds,$$

with $g^{-1}(x, s) := \sup\{t \geq 0 : g(x, t) \leq s\}$. 
The functions \( G(\cdot) \) and \( G^*(\cdot) \) satisfies the following Young inequality

\[
stt \leq G(x,t) + G^*(s,t)\quad \text{for } x \in \Omega \quad \text{and } s, t \geq 0.
\]

Further, we have the equality if \( s = g(x,t) \) or \( t = g^{-1}(x,s) \).

**Definition 2.5.** Let \( G(\cdot) \in \Phi(\mathbb{R}^N) \), the generalized Orlicz space, also called Musielak-Orlicz space, is defined as the set

\[
L^{G(\cdot)}(\Omega) := \{ u \in L^0(\Omega) : \lim_{\lambda \to 0} \int_{\Omega} G(x, \lambda|u|) \, dx = 0 \},
\]

where \( L^0(\Omega) \) is the set of measurable functions in \( \Omega \). If \( G(\cdot) \) satisfies \( (SC) \), then

\[
L^{G(\cdot)}(\Omega) = \{ u \in L^0(\Omega) : \int_{\Omega} G(x, |u|) \, dx < \infty \}.
\]

**Remark 2.6.** On the generalized Orlicz space, we define the following norms:

- Luxembourg norm: \( \|u\|_{G(\cdot)} = \inf \{ \lambda > 0 : \int_{\Omega} G(x, \frac{|u|}{\lambda}) \, dx \leq 1 \} \).
- Orlicz norm: \( \|u\|_{G(\cdot)}^0 = \sup \{ \int_{\Omega} u(x)v(x) \, dx : \int_{\Omega} G^*(x, |v|) \, dx \leq 1 \} \).

These norms are equivalent, precisely, we have

\[
\|u\|_{G(\cdot)} \leq \|u\|_{G(\cdot)}^0 \leq 2\|u\|_{G(\cdot)}.
\]

The functions \( G(\cdot) \) and \( G^*(\cdot) \) satisfy the Hölder inequality

\[
\left| \int_{\Omega} u(x)v(x) \, dx \right| \leq 2\|u\|_{G(\cdot)}\|v\|_{G^*(\cdot)} \quad \text{for } u \in L^{G(\cdot)}(\Omega) \text{ and } v \in L^{G^*(\cdot)}(\Omega).
\]

The following lemmas establish properties of convergent sequences in generalized Orlicz spaces (see [8]).

**Lemma 2.7.** Let \( G(\cdot) \in \Phi(\Omega) \). For any sequence \( \{u_i\}_i \) in \( L^{G(\cdot)}(\Omega) \), we have the following properties: If \( G(\cdot) \) satisfies \( (SC) \), then

\[
\|u_i\|_{G(\cdot)} \to 0 \quad (\text{resp. } 1; \infty) \iff \int_{\Omega} G(x, |u_i(x)|) \, dx \to 0 \quad (\text{resp. } 1; \infty).
\]

**Lemma 2.8.** Let \( G(\cdot) \in \Phi(\Omega) \) and \( \{u_i\}_i \) be a sequence of measurable functions \( u_i \). Assume the sequence \( \{u_i\}_i \) converges almost everywhere to a measurable function \( u \), and is dominated by a function \( h \in L^{G^*(\cdot)}(\Omega) \). Then all \( u_i \) as well as \( u \) are in \( L^{G(\cdot)}(\Omega) \) and the sequence \( \{u_i\}_i \) converges to \( u \) in \( L^{G(\cdot)}(\Omega) \).

**Lemma 2.9.** Let \( G(\cdot) \in \Phi(\Omega) \) satisfies \( (SC) \) and \( \{u_i\}_i \) be a sequence in \( L^{G(\cdot)}(\Omega) \). If \( u_i \to u \) in \( L^{G(\cdot)}(\Omega) \), then there exist a subsequence \( \{u_{i_j}\}_j \) and a function \( h \in L^{G^*(\cdot)}(\Omega) \) such that \( u_{i_j} \to u \), for a.e. in \( \Omega \) and for all \( j \), \( |u_{i_j}(x)| \leq h(x) \) for a.e. in \( \Omega \).
Proof. Since $u_i \to u$ in $L^{G(\cdot)}(\Omega)$, then by Lemma 2.7, we have
\[ \int_{\Omega} G(x, |u_i(x) - u(x)|) \, dx \to 0. \]
So, for any $\delta > 0$, let $A_{\delta,i} := \{ x \in \Omega : |u_i(x) - u(x)| > \delta \}$. Using the inequalities (2.1), (2.2) and the condition $(A_0)$, we get
\[ \frac{1}{c_0} \min(\delta \rho_0, \delta^p)|A_{\delta,i}| \leq \int_{A_{\delta,i}} G(x, \delta) \, dx \leq \int_{A_{\delta,i}} G(x, |u_i - u|) \, dx \leq \int_{\Omega} G(x, |u_i - u|) \, dx. \]
Hence, $|A_{\delta,i}| \to 0$, and $u_i \to u$ in measure. Therefore, there exists a subsequence $(u_{i,j})$ of $(u_i)$ which converge to $u$, for a.e. in $\Omega$.

Then we can define
\[ h := \sum_{j=1}^{\infty} |u_{i,j} - u| \]
and it can be seen that $h \in L^{G(\cdot)}(\Omega)$ and $h$ is a dominating function of $(u_{i,j})$ for all $j$.

Definition 2.10. We define the generalized Orlicz-Sobolev space by
\[ W^{1,G(\cdot)}_0(\Omega) := \{ u \in L^{G(\cdot)}(\Omega) \cap L^1_{\text{loc}}(\Omega) : |\nabla u| \in L^{G(\cdot)}(\Omega) \} \]
equipped with the norm
\[ ||u||_{1,G(\cdot)} = ||u||_{G(\cdot)} + ||\nabla u||_{G(\cdot)}. \]

Definition 2.11. $W^{1,G(\cdot)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,G(\cdot)}_0(\Omega)$.

Next, we recall the norm version of the Poincare inequality, which will be investigated in this work (see [8]).

Theorem 2.12. Let $\Omega$ be a bounded set of $\mathbb{R}^N$ and $G(\cdot) \in \Phi(\Omega)$ satisfy $(SC)$, $(A_0)$ and $(A_1)$. For every $u \in W^{1,G(\cdot)}_0(\Omega)$, we have
\[ ||u||_{G(\cdot)} \leq C||\nabla u||_{G(\cdot)}. \]
In particular, $||\nabla u||_{G(\cdot)}$ is a norm on $W^{1,G(\cdot)}_0(\Omega)$ and it is equivalent to the norm $||u||_{1,G(\cdot)}$.

The following compact embedding theorem for Musielak-Sobolev spaces is given by P. Hasto [8].

Theorem 2.13. Let $G(\cdot) \in \Phi(\mathbb{R}^N)$ satisfy $(SC)$, $(A_0)$ and $(A_1)$ and let $\Omega$ be bounded. Then
\[ W^{1,G(\cdot)}_0(\Omega) \hookrightarrow L^{G(\cdot)}(\Omega). \]
3. Eigenvalue problems for the $G(\cdot)$-Laplacian

3.1. Lusternik-Schnirelmann principle (L-S principle)

We recall here a version of the Lusternik-Schnirelmann principle, which Browder discussed in [4] and Zeidler in [14,15].

**Theorem 3.1.** Let $X$ be a real reflexive Banach space. If $A, B$ are two functionals on $X$ satisfying the following properties:

1. $(LS_1)$: $A, B : X \to \mathbb{R}$ are even functionals and that $A, B \in C^1(X, \mathbb{R})$ with $A(0) = B(0) = 0$.
2. $(LS_2)$: $A'$ is strongly continuous (i.e., $u_i \rightharpoonup u$ in $X$ implies $A'(u_i) \to A'(u)$) and,
   \[
   \langle A'(u), u \rangle = 0, \quad u \in \text{co}S_B \quad \text{implies} \quad A(u) = 0,
   \]
   \[
   A(u) = 0, \quad u \in \text{co}S_B \quad \text{implies} \quad u = 0,
   \]
   where $\text{co}S_B$ is the closed convex hull of $S_B := \{u \in X, B(u) = 1\}$.
3. $(LS_3)$: $B'$ is continuous, bounded and satisfies $(S_0)$, i.e., as $i \to \infty$,
   \[
   u_i \rightharpoonup u, \quad B'(u_i) \to v, \quad \langle B'(u_i), u_i \rangle \to \langle v, u \rangle \quad \text{implies} \quad u_i \to u.
   \]
4. $(LS_4)$: The level set $S_B$ is bounded and $u \neq 0$ implies
   \[
   \langle B'(u), u \rangle > 0, \quad \lim_{t \to \infty} B(tu) = \infty, \quad \inf_{u \in S_B} \langle B'(u), u \rangle > 0.
   \]

Then the eigenvalue problem
\[
A'(u) = \mu B'(u), \quad u \in S_B, \quad \mu \in \mathbb{R},
\]
admits a sequence of eigenpairs $(u_i, \mu_i)$ such that $u_i \rightharpoonup u$, $\mu \to 0$ as $i \to \infty$ and $\mu_i \neq 0$ for all $i$.

3.2. Application of L-S principle in $W_0^{1,G(\cdot)}(\Omega)$

In the sequel, let $G(\cdot) \in \Phi(\mathbb{R}^N)$ satisfy $(SC)$, $(A_0)$, $(A_1)$, $G(x, \cdot) \in C^1([0, \infty))$ for every $x \in \Omega$ and $C$ a generic constant which may change from line to line. We consider the following eigenvalue problem
\[
\begin{aligned}
-\text{div} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \right) &= \lambda \left( \frac{g(x, |u|)}{|u|} u \right) \quad \text{in } \Omega, \\
\quad \quad \quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $g(x, \cdot)$ is the derivative of $G(x, \cdot)$. The existence of a weak solution to the Dirichlet-Sobolev problem associated of (3.2) has been studied in [3].

**Definition 3.2.** Let $\lambda \in \mathbb{R}$ and $u \in W_0^{1,G(\cdot)}(\Omega)$. $(u, \lambda)$ is called a solution of the problem (3.2) if
\[
\int_{\Omega} g(x, |\nabla u|) \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} g(x, |u|) uv \, dx, \quad \forall v \in W_0^{1,G(\cdot)}(\Omega).
\]
If $(u, \lambda)$ is a solution of the problem (3.2) and $u \neq 0$, as usual, we call $\lambda$ and $u$ an eigenvalue and an eigenfunction corresponding to $\lambda$ of (3.2), respectively.
In what follows we use the previous version of the L-S principle in order to prove the existence of a sequence of eigenvalues for problem (3.2). For this reason, we define on $X = W^{1,G(\cdot)}_0(\Omega)$ the functionals

\begin{align}
(3.4) \quad A(u) &= \int_{\Omega} G(x,|u|) \, dx, \\
(3.5) \quad B(u) &= \int_{\Omega} G(x,|\nabla u|) \, dx.
\end{align}

**Lemma 3.3.** Let $A$ and $B$ be defined in (3.4), (3.5). Then $A$ and $B$ satisfies (LS$_1$).

**Proof.** Let $G(\cdot) \in \Phi(\mathbb{R}^N)$. By definition of a generalized $\Phi$-function, we have $G(x,0) = 0$ which implies $A(0) = B(0) = 0$. The $C^1$-smooth regularity of the functionals $A$ and $B$ follows by computing the Gateaux derivatives of $A$ and $B$ at $u \in W^{1,G(\cdot)}_0(\Omega)$ in the direction $v \in W^{1,G(\cdot)}_0(\Omega)$. Precisely, for every $u, v \in W^{1,G(\cdot)}_0(\Omega)$, we have

\begin{align}
(3.6) \quad \langle A'(u), v \rangle &= \int_{\Omega} \frac{g(x,|u|)}{|u|} uv \, dx, \\
(3.7) \quad \langle B'(u), v \rangle &= \int_{\Omega} \frac{g(x,|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx.
\end{align}

Then $A$ and $B$ satisfies (LS$_1$). \hfill $\Box$

**Lemma 3.4.** Let $A$ be defined in (3.4). Then $A'$ satisfies (LS$_2$).

**Proof.** Let $u \in W^{1,G(\cdot)}_0(\Omega)$, by the condition (SC), we have

$$A(u) = \int_{\Omega} G(x,|u|) \, dx \leq \frac{1}{g_0 + 1} \int_{\Omega} g(x,|u|)|u| \, dx \leq \langle A'(u), u \rangle.$$ 

Then, if $\langle A'(u), u \rangle = 0$ implies $A(u) = 0$.

Next, we assume $A(u) = 0$. By Proposition 2.2.7 in [8], there exists $\tilde{G}(\cdot) \in \Phi(\Omega)$ with $\tilde{G}(\cdot) \approx G(\cdot)$ which is a strictly increasing. So, we have

$$0 \leq \int_{\Omega} \tilde{G}(x,|u|) \, dx \leq C \int_{\Omega} G(x,|u|) \, dx = 0 \text{ implies } u = 0.$$ 

To end the proof of Lemma 3.4, it remains for us to prove that $A'$ is strongly continuous. Let $u_i \rightharpoonup u$ in $W^{1,G(\cdot)}_0(\Omega)$, we need to show that

$$A'(u_i) \rightharpoonup A(u) \text{ in } W^{1,G(\cdot)}_0(\Omega)^*.$$ 

For every $v \in W^{1,G(\cdot)}_0(\Omega)$, using the Holder inequality, we get

\begin{align}
(3.8) \quad \int_{\Omega} \frac{g(x,|u_i|)}{|u_i|} u_i v \, dx &\leq \left| \int_{\Omega} \frac{g(x,|u_i|)}{|u_i|} u_i v \, dx - \int_{\Omega} \frac{g(x,|u|)}{|u|} uv \, dx \right| \\
&\leq \int_{\Omega} \frac{g(x,|u_i|)}{|u_i|} u_i - \frac{g(x,|u|)}{|u|} u \, dx \, dx
\end{align}

\[ \leq \frac{1}{g_0 + 1} \int_{\Omega} |u_i - u| \, dx \]
Lemma 3.5. Let $H$ be defined in (3.5). Then $B'$ satisfies $(LS_3)$.

Proof. Using the Hölder inequality, we have

$$
\|B'\| \left( W_{\alpha}^{1, G(\cdot)}(\Omega) \right)^* = \sup \{ \langle B'(u), v \rangle ; \|v\|_{1, G(\cdot)} \leq 1 \}
\leq \sup \left| \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \right|
\leq 2 \|g(x, |\nabla u|)|_{G(\cdot)}\|v\|_{1, G(\cdot)}.
$$

Since $u_i \to u$ in $W^{1, G(\cdot)}_0(\Omega)$, then by the compact embedding Theorem 2.13, we have $u_i \to u$ in $L^{G(\cdot)}(\Omega)$. So, using the reverse dominate convergence theorem, Lemma 2.9, there are a subsequence $\{u_{i_j}\}_j$ and a function $h \in L^{G(\cdot)}(\Omega)$ such that $u_{i_j} \to u$ for a.e. in $\Omega$ and $|u_{i_j}| \leq h$.

As the function $t \to g(x, t)$ is continuous and increasing, we have

$$
\frac{g(x, |u_{i_j}|)}{|u_{i_j}|} u_{i_j} \to \frac{g(x, |u|)}{|u|} u \text{ for a.e. in } \Omega \text{ and } \frac{g(x, |u_{i_j}|)}{|u_{i_j}|} u_{i_j} \leq \frac{g(x, |h|)}{|h|} h.
$$

Note that $\frac{g(x, |h|)}{|h|} h \in L^{G(\cdot)}(\Omega)$. Indeed, by the Young equality and the condition $(SC)$, we have

$$
\int_{\Omega} G^*(x, |\frac{g(x, |h|)}{|h|} h|) \, dx = \int_{\Omega} G^*(x, g(x, |h|)) \, dx
\leq g^0 \int_{\Omega} G(x, |h|) \, dx < \infty.
$$

Then, by the dominate convergence theorem, Lemma 2.8, we have

$$
\frac{g(x, |u_{i_j}|)}{|u_{i_j}|} u_{i_j} \to \frac{g(x, |u|)}{|u|} u \text{ in } L^{G(\cdot)}(\Omega).
$$

Hence, by inequality (3.8), we have

$$
\int_{\Omega} \frac{g(x, |u_{i_j}|)}{|u_{i_j}|} u_{i_j} v \, dx \to \int_{\Omega} \frac{g(x, |u|)}{|u|} u v \, dx.
$$

Since the weak limit is independent of the choice of the subsequence, it follows that

$$
\int_{\Omega} \frac{g(x, |u_{i_j}|)}{|u_{i_j}|} u_{i_j} v \, dx \to \int_{\Omega} \frac{g(x, |u|)}{|u|} u v \, dx.
$$

Therefore $A'(u_i) \to A(u)$ in $W^{1, G(\cdot)}_0(\Omega)^*$. 

Inspired by the proof of Theorem 1.7 of Lieberman [10], we have:

**Lemma 3.5.** Let $B$ be defined in (3.5). Then $B'$ satisfies $(LS_3)$.

**Proof.** Using the Hölder inequality, we have

$$
||B'|| \left( W_{\alpha}^{1, G(\cdot)}(\Omega) \right)^* = \sup \{ \langle B'(u), v \rangle ; \|v\|_{1, G(\cdot)} \leq 1 \}
\leq \sup \left| \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \right|
\leq 2 \|g(x, |\nabla u|)|_{G(\cdot)}\|v\|_{1, G(\cdot)}.
$$
Using the equivalent between Luxembourg norm and Orlicz norm, and the Young inequality, we have
\[ ||g(x, |\nabla u|)||_{G^*} \leq ||g(x, |\nabla u|)||^0_{G^*} \leq 1 + \int_\Omega G^*(x, g(x, |\nabla u|)) \, dx. \]

Then, by the Young equality and the condition \((SC)\), we obtain
\[ ||g(x, |\nabla u|)||_{G^*} \leq g^0 \int_\Omega G(x, |\nabla u|) \, dx. \]

So, by the inequality (3.9), we get
\[ ||B'||_{W^1_{0, G^*}(\Omega)} \leq C \left( g^0 \int_\Omega G(x, |\nabla u|) \, dx + 1 \right) ||u||_{1, G(\cdot)}. \]

Hence \(B'\) is bounded. Moreover, a similar argument to the one we used to prove \((LS_2)\), we get the continuity of \(B'\).

Complete the proof of Lemma 3.5, that is, prove that \(B\) satisfies condition \((S_0)\). Let \(\{u_i\}\) be a sequence in \(W^1_{0, G(\cdot)}(\Omega)\) such that
\[ u_i \rightharpoonup u, \quad B'(u_i) \rightharpoonup v \quad \text{and} \quad \langle B'(u_i), u_i \rangle \to \langle v, u \rangle \]
for some \(v \in W^1_{0, G(\cdot)}(\Omega)^*\) and \(u \in W^1_{0, G(\cdot)}(\Omega)\). Then, we have
\[ \langle B'(u_i) - B'(u), u_i - u \rangle \]
On the other hand, from the condition \((SC)\) and Cauchy-Schwarz inequality, we have for \(\theta_t = tu + (1 - t)u_i, \ t \in (0, 1)\)
\[ \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \]
\[ = \left( \int_0^1 \frac{\partial}{\partial t} \left( \frac{g(x, |\nabla \theta_t|)}{|\nabla \theta_t|} \nabla \theta_t \right) \, dt \right) \cdot (\nabla u - \nabla u_i) \]
\[ = |\nabla u - \nabla u_i|^2 \int_0^1 \frac{g(x, |\nabla \theta_t|)}{|\nabla \theta_t|} \, dt \]
\[ + \int_0^1 g(x, |\nabla \theta_t|) \left( \frac{|\nabla \theta_t| g'(x, |\nabla \theta_t|) - 1}{g(x, |\nabla \theta_t|)} \right) \frac{(|\nabla \theta_t| \cdot (\nabla u - \nabla u_i))^2}{|\nabla \theta_t|^2} \, dt \]
\[ \geq \min(1, g_0) |\nabla u - \nabla u_i|^2 \int_0^1 \frac{g(x, |\nabla \theta_t|)}{|\nabla \theta_t|} \, dt. \]

Which implies
\[ \langle B'(u_i) - B'(u), u_i - u \rangle \geq \min(1, g_0) \int_\Omega \int_0^1 \frac{g(x, |\nabla \theta_t|)}{|\nabla \theta_t|} |\nabla u - \nabla u_i|^2 \, dt \, dx. \]
Now we write \( S_1 = \{ x \in \Omega, |\nabla u - \nabla u_i| \leq 2|\nabla u| \} \) and \( S_2 = \{ x \in \Omega, |\nabla u - \nabla u_i| > 2|\nabla u| \} \). Then \( S_1 \cup S_2 = \Omega \) and

\[
\begin{align*}
\frac{1}{2}|\nabla u| \leq |\nabla \theta| \leq 3|\nabla u| & \quad \text{on } S_1 \text{ for } t \geq \frac{3}{4}, \\
\frac{1}{4}|\nabla u - \nabla u_i| \leq |\nabla \theta| \leq 3|\nabla u - \nabla u_i| & \quad \text{on } S_2 \text{ for } t \leq \frac{1}{4}.
\end{align*}
\]

Therefore

\[
(3.11) \quad \langle B'(u_i) - B'(u), u_i - u \rangle \geq C \left( \int_{S_1} g(x, |\nabla u|) |\nabla u - \nabla u_i|^2 \, dx + \int_{S_2} G(x, |\nabla u - \nabla u_i|) \, dx \right).
\]

Hence

\[
(3.12) \quad \int_{S_2} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \langle B'(u_i) - B'(u), u_i - u \rangle.
\]

To estimate the integrals over \( S_1 \), using the condition \((SC), t \to g(x, t)\) is a nondecreasing function and the Hölder inequality in \( L^2(S_1) \), we have

\[
\int_{S_1} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \left( \int_{S_1} g(x, |\nabla u|) |\nabla u - \nabla u_i| \, dx \right)^\frac{1}{2} \left( \int_{S_1} G(x, |\nabla u|) \, dx \right)^\frac{1}{2}.
\]

Hence, using the inequality (3.11), we have

\[
(3.13) \quad \int_{S_1} G(|\nabla u - \nabla u_i|) \, dx \leq C \left( \langle B'(u_i) - B'(u), u_i - u \rangle \right)^\frac{1}{2} \left( \int_{S_1} G(x, |\nabla u|) \, dx \right)^\frac{1}{2}.
\]

Collecting the inequalities (3.11), (3.12), (3.13), we have

\[
\int_{\Omega} G(|\nabla u - \nabla u_i|) \, dx \leq C \left( \langle B'(u_i) - B'(u), u_i - u \rangle \right)^\frac{1}{2} \left( \int_{\Omega} G(|\nabla u|) + G(x, |u|) \, dx \right)^\frac{1}{2}.
\]
\[ + \langle B'(u_i) - B'(u), u_i - u \rangle \].

Therefore, by the inequality (3.10), Theorem 2.12 and Lemma 2.7, we have \( u_i \to u \) in \( W^1_{0,G}(\Omega) \). □

**Lemma 3.6.** Let \( B \) be defined in (3.5). Then \( B \) and \( B' \) satisfies \((LS_4)\).

**Proof.** Let \( u \neq 0 \), by the inequalities (2.1) and (2.2), for all \( t \in \mathbb{R}^+ \) we have

\[
B(tu) = \int_\Omega G(x,|\nabla tu|) \, dx \leq \max(t^{g_0+1},t^{g_0+1}) \int_\Omega G(x,|\nabla u|) \, dx.
\]

Then \( \lim_{t \to +\infty} B(tu) = +\infty \).

Next, by the condition \((SC)\) and Proposition 2.2.7 in [8], there exists \( \tilde{G}(\cdot) \in \Phi(\Omega) \) with \( \tilde{G}(\cdot) \approx G(\cdot) \) which is a strictly increasing. Then, we have

\[
\langle B'(u), u \rangle = \int_\Omega g(x,|\nabla u|)|\nabla u| \, dx \\
\geq (g_0 + 1) \int_\Omega G(x,|\nabla u|) \, dx \\
\geq C \int_\Omega \tilde{G}(x,|\nabla u|) \, dx \\
\geq 0.
\]

Note that, the last inequality result from the fact if \( u \in W^1_{0,G}(\cdot) \) and \( \nabla u = 0 \) then \( u = 0 \).

So, if \( u \in S_B \), then, by the previous inequality, we have \( \langle B'(u), u \rangle \geq g_0 + 1 \).

Therefore \( B \) and \( B' \) satisfies the hypothesis \((LS_4)\). □

**Theorem 3.7.** Let \( G(\cdot) \in \Phi(\mathbb{R}^N) \) satisfy \((SC)\), \((A_0)\) and \((A_1)\). Let \( A \) and \( B \) be the two functionals defined in (3.4), (3.5). Then there exists a nondecreasing sequence of nonnegative eigenvalues \( \{\lambda_i\} \) of (3.2) such that \( \lambda_i \to \infty \) as \( i \to \infty \).

**Proof.** By Lemmas 3.3-3.6 combined with Theorem 3.1, there exists a nonnegative nonincreasing sequence \( \{\mu_i\} \) such that \( \mu_i \to 0 \) as \( i \to \infty \) and each \( \mu_i \) is an eigenvalue of \( A'(u) = \mu B'(u) \) which means, for every \( v \in W^1_{0,G}(\cdot) \), we have

\[
\int_\Omega g(x,|u|)|u|v \, dx = \mu \int_\Omega g(x,|\nabla u|)\nabla u \cdot \nabla v \, dx.
\]

Which is equivalent

\[
\int_\Omega g(x,|\nabla u|)\nabla u \cdot \nabla v \, dx = \frac{1}{\mu} \int_\Omega g(x,|u|)|u|v \, dx.
\]

Then \( \lambda_i := \frac{1}{\mu_i} \to \infty \) as \( i \to \infty \) is a nondecreasing sequence of nonnegative eigenvalues of (3.2). □
References


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