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RESEARCH ARTICLE

Enhancing Geometry and Measurement Learning Experiences through Rigorous Problem Solving and Equitable Instruction

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Abstract

This paper details case study vignettes that focus on enhancing the teaching and learning of geometry and measurement in the elementary grades with attention to pedagogical practices for teaching through problem solving with rigor and centering equitable teaching practices. Rigor is a matter of equity and opportunity (Dana Center, 2019). Rigor matters for each and every student and yet research indicates historically disadvantaged and underserved groups have more of an opportunity gap when it comes to rigorous mathematics instruction (NCTM, 2020). Along with providing a conceptual framework that focuses on the importance of equitable instruction, our study unpacks ways teachers can leverage their deep understanding of geometry and measurement learning trajectories to amplify the mathematics through rigorous problems using multiple approaches including learning by doing, challenged-based and mathematical modeling instruction. Through these vignettes, we provide examples of tasks taught through rigorous problem solving approaches that support conceptual teaching and learning of geometry and measurement. Specifically, each of the three vignettes presented includes a task that was implemented in an elementary classroom and a vertically articulated task that engaged teachers in a professional learning workshop. By beginning with elementary tasks to more sophisticated concepts in higher grades, we demonstrate how vertically articulating a deeper understanding of the learning trajectory in geometric thinking can add to the rigor of the mathematics.

Keywords: learning by doing, equitable instruction, geometric thinking

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I. INTRODUCTION

One of the prevalent and misguided notion that persists in education, according to NCTM's recent publication, *Catalyzing Change in Early Childhood and Elementary Education* (NCTM, 2020) is "that children must 'master the basics' prior to engaging with complex problem solving—a belief leading to children frequently being removed from the grade-level curriculum and thus remaining behind grade-level despite intervention" (NCTM 2020, p. 28). This misguided belief leads to differential learning experiences and opportunity gaps particularly with marginalized populations. In fact, the survey on the National Center for Education Statistics (NCES, 2017), show that children who have regular opportunities to collaborate on challenging tasks, use varied solution approaches, and focus on sense making have higher mathematics achievement. Key recommendations from *Catalyzing Change* include implementing equitable mathematics instruction that nurture children's positive mathematical identities and developing deep mathematical understanding where "elementary school should build a strong foundation of deep mathematical understanding, emphasizing reasoning and sense making, and ensure the highest-quality mathematics education for each and every child" (NCTM, 2020, p. 123). Following these recommendations, we posit that rigorous mathematics learning opportunities through problem solving are central to equitable mathematics instruction.

Our paper proposes a conceptual framework to enhance pedagogical practices for teaching and learning of geometry and measurement in the grades K-12. We provide a conceptual framework that focuses on the importance of equitable instruction and how teachers can leverage their deep understanding of geometry and measurement learning trajectories to amplify mathematics through rigorous problem solving approaches. The overall framing and purpose is driven by a theoretical framework that supports this conceptual framework via three interconnected inquiry-based approaches. We also illustrate three classroom vignettes with rich tasks in the elementary grades that support conceptual teaching and learning of geometry and measurement and vertically articulate the learning trajectory with the connection to the rigor of the mathematics employed and equitable teaching practices.

II. RELATED LITERATURE

Our work builds on a theoretic framework that embraces the importance of sociocultural context that accounts for a student's learning. Embedding Problem Based Learning helps to recognize the important role of social contexts in enhancing students' cognition, understanding and competence. Vygotsky's Sociocultural Theory (Vygotsky, 1978) highlights the need for an integrative sociocultural approach for cognitive development and identity formation. In addition, social cultural theory accommodates authentic engagement, experimentation, social dimensions of learning, problem solving and active learning participation. In this work, we employ the sociocultural aspects of Problem Based Learning with a conceptual framework that integrates equitable instruction, deeper understanding and rigorous problem solving.

For our conceptual framework, we build on the notion that for equitable mathematics teaching we must provide students with learning experiences that provide rigorous mathematics learning through problem solving. This includes the well known Polya's (1957) method of problem solving as well as getting students to become problem posers by mathematizing real contextual problems using mathematical modeling. By using these rich approaches, teachers are better able to attend to students' assets and tap into students' funds of knowledge using local context and relevant problems. In addition, we place importance in teachers unpacking the mathematics learning trajectories that weave together research from cognitive development, instructional practice, and mathematical ideas so that they can be better equipped to notice the strength in children's mathematical thinking, which then can be used to identify and build instruction from children's knowledge bases (Clements & Sarama 2014; Aguirre et al., 2013; Suh and Seshaiyer, 2019). Equity scholars promote this asset based approach to mathematics teaching and learning by leveraging the varied cultural, linguistic, and mathematical strengths and experiences children bring to the classroom (Featherstone et al. 2011; Gutiérrez & Irving 2012; Turner et al., 2013).

To provide a background on how these three approaches complement one another, we detail prior literature that focus on the importance of equitable instruction and how teachers can leverage their deep understanding of geometry and measurement learning trajectories to amplify the mathematics through rigorous problems using multiple approaches including learning by doing, challenge-based and mathematical modeling instruction (see Figure 1).

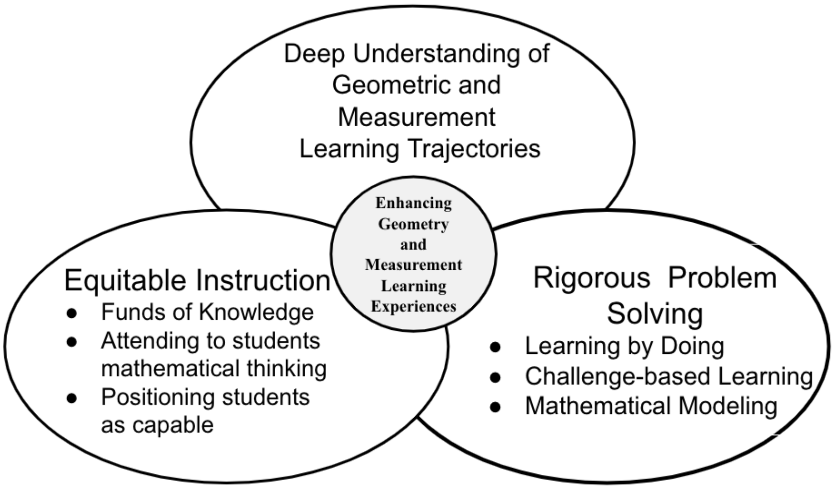

Figure 1. Conceptual Framework to Enhance Geometry and Measurement Learning Experiences

This work focuses on enhancing geometry and measurement learning experiences. For this, we integrated into this conceptual framework the van Hiele's theory that was structured around higher-level thinking using three important aspects including *existence* of levels, *properties* of the levels and then *movement* from one level to the next higher level (Van Heile, 1999; Haviger and Vojkůvková, 2014). These abstract levels of geometric thinking have been further broken down by other researchers for concrete examples that includes *visual* (identification of shapes through concrete examples), *descriptive* (Identification through properties), *relational* (identification of relationships through deduction); *formal* deduction (using logic) and *rigor* (using axiomatic approaches). Next, we describe the three structures around which our conceptual framework is built.

Equitable Math Teaching Practices

Bartell et al (2017) proposed nine research-based equitable mathematics teaching practices with a student-centered lens including drawing on students funds of knowledge, establishing classroom norms for participation, positioning students as capable, monitoring how students position each other, attending to race and culture, recognizing multiple forms of discourse and language as a resource, pressing for academic success, attending to students' mathematical thinking and supporting the development of productive disposition.

When thinking about fostering geometric thinking specifically, there are several "habits of mind" that rely on students' reasoning with relationships between geometric figures, generalizing geometric ideas, exploring and reflection (Driscoll et al., 2007). This requires teachers to implement tasks that allow for students to draw on their funds of knowledge to experiences that rely on spatial reasoning, like building, measuring, covering two dimensional shapes as well as building and filling three dimensional shapes.

Implementing problem solving tasks that promote reasoning "Engage students in tasks that provide multiple pathways for success and that require reasoning, problem solving, and modeling, thus enhancing each student's mathematical identity and sense of agency" (NCTM, 2018, p. 32). For example, in designing lessons that enhance geometry and measurement instruction, teachers can tap into *students' funds of knowledge* (Moll et al., 1992) by building on *community and cultural knowledge and practices* as Civil & Khan (2001) found when interviewing families engaged in gardening and created lessons that connected to measurement in one, two and three dimensions. In addition, teachers can tap into the robust knowledge of students, validate shared ideas and experiences, and connect instruction to students' experiences and interests (Aguirre et al., 2013; Bartell, 2011; Hedges, Cullen, & Jordan, 2011; Wager, 2014). We also connect the equitable teaching practice of *positioning students as capable* (Bartell et al., 2017) where the teachers assign competence (Cohen et al., 1999) and distribute power in the classroom by allowing students to provide meaningful input in developing collective knowledge. This teaching practice has the power to challenge and counteract societal stereotypes and inequities to which students and communities are subjected (Bartell, 2011; Gay, 2002; Ladson-Billings, 1995). Finally, we focus on *attending to students' mathematical thinking* to recognize, understand, and build from children's understanding of mathematics (Carpenter et al., 1999). This practice also requires responding to developmental needs to tailor instruction and differentiate, which is why we place importance on teachers' deep understanding of the learning trajectories (Clements and Sarama, 2004).

Deep Understanding of Learning Trajectories in Geometry and Measurement

Learning trajectory (LT) is detailed as the “description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning the topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence” (Battista, 2010, p. 61). LT researchers have reported on two types of LT, *hypothetical* and *actual*. Simon (1995) proposed that a "hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p 136). In contrast, descriptions of actual learning trajectories can be specified only during and after a student has progressed through such a learning path. Steffe (2004) described an actual LT as

> a model of [children's] initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. Such earning trajectory of children is then constructed during and after the experience in intensively interacting with children. (p. 131)

Researchers interested in studying the actual learning trajectory must attend to how instructional variation affects trajectories and how specific instructional sequence or curriculum may reveal a particular learning trajectory. Battista (2010) posed whether one constructs a prototypical hypothetical LT for a particular topic, how do the actual LT for individual students vary about this prototypical path? One might think of a prototypical trajectory as a "mean" of the actual student pathways, so the "standard deviation" of the distribution of actual trajectories is also relevant.

Learning Trajectory research on geometric thinking has been heavily influenced by van Hiele’s abstract levels which were primarily introduced for secondary education where geometry is often taught, it was also suggested that

> the levels are not age dependent in the sense of developmental stages of Piaget. An elementary grade or a high school student could be at level 1. In fact, some students and adults remain forever at level 1 and a significant number of adults never reach level 3. But age is certainly related to the amount and type of geometric experiences that we have. (van de Walle, 2004, p. 348)

Without such authentic geometry experiences, it is also known that early grades often create difficulty for students to deal with complex geometric thinking because of insufficient understanding of the concepts of space and shape (Elia et al., 2018). In order to use the van Hiele model of learning to describe how students reason in geometry with deeper understanding, it is also important to develop instructional interventions carefully to promote awareness of the theory and improved knowledge of geometry content. Implications on the research on the van Hiele model can help to provide insights for shaping curricula, teacher education, and classroom practice.

In addition, scholars have differentiated learning trajectory constructs for specific domains and the view of the actual learning trajectory path and what happens in a classroom.

When presenting a problem solving task, students will have a variety of student pathways and if we are to use the asset based lens, teachers need to be able to appreciate the deviation of the “prototypical trajectory” to acknowledge the multiple pathways. In addition, although LT research focuses on levels of sophistication, students' levels of thinking may fluctuate as they move from familiar content to unfamiliar content (Battista, 2007). In fact, Battista (2010) makes researchers consider that even if we develop an adequate definition for what it means to be "at" a level, the periods of time when students meet the strict requirement for being at levels may be short, with students spending much time "in transition." Learning trajectories then provides a “constructive itinerary” with levels that are compilations of empirical observations of the thinking of many students, and because students' learning backgrounds and mental processing differ, a particular student might not pass through every level for a topic; he or she might skip some levels or pass through them so quickly that the passage is difficult to detect. Even with the recognition of variability, LT research is valuable for teachers because they describe the plateaus that students achieve in their development of reasoning about a topic.

Rigorous Problem Solving

Over the past years, there has been a debate in the mathematics education community on the definition of rigor in mathematics. Some define it in multiple developmental stages including pre-rigor, rigor and post-rigor[1]. Other research has divided rigor into two categories, including one for content and another for instruction (Hull, Balka, & Harbin Miles, 2014). Students being able to use mathematical language to communicate effectively and describe their work with clarity and precision, being able to demonstrate what they have done works, being able to answer why and how they know are essential attributes of rigorous problem solving. Rigor is also a matter of equity and opportunity (Dana Center, 2019). While rigor matters for every student, it is particularly important for students who are more likely to encounter mathematics courses and instruction that are focused on procedural fluency and adherence to rules rather than the skills of reasoning, logical thinking, argumentation, and precision that lead to deeper engagement in mathematics. Also, many textbooks and traditional activities in elementary grades focus on recognizing shapes and defining their unique attributes. For example, in geometry, students might have lessons that focus on classifying and sorting shapes by characteristics. However, there is a danger that these lessons repeat over several grade levels and students miss opportunities to engage in inquiry based learning and discovery of geometry relationships through problem solving.

To engage students in rigorous problem solving through van Heile’s student-thinking levels, a five-phase sequence of instructional practice introduced by van Heile may also be introduced in the classroom that includes a) An *inquiry* phase in which materials lead children to explore and discover certain structures; b) A *direct orientation* phase where tasks are presented in such a way that the characteristic structures appear gradually to the children; c) An *explicitation* phase where the teacher introduces

[1] https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/

terminology and encourages children to use it in their conversations and written work about geometry; d) a *free orientation* phase, the teacher presents tasks that can be completed in different ways and enables children to become more proficient with what they already know and; e) an *integration* phase where children are given opportunities to pull together what they have learned, perhaps by creating their own activities. In order to achieve rigorous problem solving, these student-thinking levels take the students from recognizing basic geometry concepts holistically through visual presentation of concepts without regard to properties to students performing analysis in terms of identifying components and relationships. Then the expectation is for students to think abstractly and be able to logically relate previously discovered properties or rules. Once they achieve a higher level of intellectual maturity where they can make their own deduction and the role of geometric properties, they have learnt to solve problems in geometry rigorously.

III. METHODS

Next, we share three inquiry approaches including learning by doing, challenge-based learning and mathematical modeling to problem solving that can provide varied and rich opportunities for students to engage in making sense of and making conjectures using problem solving. Within each of these approaches we present examples of teaching vignettes that illustrate inquiry-based instruction where we move from several measurement and geometric learning trajectories. We make an effort in all of these lesson vignettes of geometric thinking to move beyond Level 1 and 2 of Van Hiele's level of visualization and analysis to rigorous problem solving. We also try to make a connection between the vignette of geometric thinking and the inquiry approach described next.

Approach #1: Learning by Doing. One powerful way to engage students in learning geometric concepts in early grades is through learning by doing. The emphasis on "doing" seems to imply that a different way of geometric learning occurs when the hands, or the senses, are engaged. While literature on embodied cognition theory (Foglia & Wilson, 2013) has helped to make connections between "doing" and learning of geometry, there is still a need to connect the learning of geometric concepts through doing deeper knowledge. More often, the need for learning by doing experience is presented as overcoming the emphasis on knowing "that" (specific geometric concepts and principles) versus knowing "how" (when, where and why those geometric concepts and principles are applied to understand real-world problems). Play is an important part of learning by doing- as stated by van Heile (1999), "For many children geometry begins with play. Rich and stimulating instruction in geometry can be provided through playful activities with mosaics, such as pattern blocks or design tiles, with puzzles like tangrams" (p. 310). While there is some evidence between the relationship between geometric thinking and kinesthetic thinking and their implications for education (SellarÈs & Toussaint, 2003), there is still a lack of research in understanding the pedagogical value of the interplay between kinesthetic learning and geometric thinking.

Approach #2: Challenge-based Learning. Challenge-based learning is a strategy for enhancing learning while solving real-world challenges that combines features from problem-based learning, project-based learning and inquiry-based learning applied to the problems that are authentic and more open-ended. In particular, participants engage in activities that help them to identify big ideas, make meaningful assumptions, ask good questions, gain in-depth subject area knowledge and develop 21st-century skills. While there has been a growing literature on problem solving frameworks ever since the introduction of Polya's *How to Solve It* (Polya, 1957), research suggests that the greatest difficulty in the problem-solving process is in the identification of an appropriate mathematical model, which requires contextual knowledge of the real-world situation as well as creativity. One framework to engage students in employing a challenge-based learning approach to interpret a real-world context and solving an associate problem using the *FERMI problem solving framework* (Peter-Koop, 2004, Ärlebäck & Bergsten, 2013). Such approaches provide opportunities for students to learn from mistakes, explore multiple pathways for solving problems, reevaluate and revise knowledge and strategies as they work in teams.

Approach #3: Mathematical Modeling. Over the years, there have been continuous efforts to find an effective strategy for teaching spatial concepts to young children. One of the approaches proposed for presenting geometry content is to do it through mathematical modeling combined with a proficiency in communication, creativity, critical thinking, and collaboration to solve big, complex and non-routine real-world problems (Suh, Matson and Seshaiyer, 2017; Seshaiyer and Suh, 2019). This modeling process may involve a storytelling context which could be based on cognitive research that documents the benefits of a story framework for retention of material (Mishra, 2003), as well as the motivational benefits of embedding mathematics content within meaningful context (Cordova & Lepper, 1996). Along with that the need to employ data collection, interpretation, visualization, analysis and prediction is very important to extract and discover the meaningful mathematical context. Along with this learning from storytelling, mathematical modeling (COMAP & SIAM, 2019) is aligned well with van Heile's sequence of instruction as it is an *inquiry-based approach* to problem solving starting with problem posing, making assumptions and identifying variables, doing the math, analyzing and assessing the solution to interpret the model with in the real world context.

In this work, we argue that learning and applying in a real-situation is helpful for students to understand and develop new knowledge and rigor to geometric reasoning. Specifically, we propose that using inquiry-based approaches such as learning by doing, challenge-based learning and mathematical modeling (COMAP & SIAM, 2019) to stimulate geometric thinking for students in early grades. While the van Hiele theory for higher-order student-thinking and instructional phases has helped to grow a large body of work using, evaluating and modifying the theory itself as well as deepening and expanding research in the learning and teaching of geometry, we believe such practical frameworks can help to make the needed connections among theory, research and the practice of teaching and student's thinking and learning.

IV. RESULTS

Context of Our Study

Each of the vignettes we present in this work stems from a larger project on mathematical modeling in the elementary grades. The three lesson vignettes came from lessons co-designed with the teachers and the two authors. The teachers were engaged in a year-long professional development project consisting of six face to face workshops and ongoing Lesson Study to plan for three total lessons during the year. Two of the vignettes called the Bee Hotel and the Lantern project come from Ms. Sim's second and third lesson implementation cycles. Ms. Sims was a mathematics coach working in a second grade classroom during the year to support a novice teacher and had known the students like her own. The Canstruction Sphinx lesson vignette comes from a 6th grade teacher who was also part of this mathematical modeling project. She co-planned all of her three mathematical modeling lessons with another 5th grade teacher. This lesson was the second lesson implementation of the three required lesson implementations. Mathematical modeling was a novel approach to these teachers but they were familiar with problem based learning (PBL) and hands on learning. We tried to leverage their existing knowledge of PBL and hands-on learning to introduce them to challenge based tasks and mathematical modeling.

We used Lesson Study (Lewis, 2002; Suh and Seshaiyer, 2015, 2016) as our professional development model to study, plan, teach and reflect on each of the lessons. The Study phase involved teachers working through a rich task that was vertically articulated so that they would see how an elementary task would develop over time to important geometric theorems, postulates and big ideas. This also provided opportunities to introduce rigor through problem solving and an articulation of geometry and measurement concepts. Each vignette showcases a classroom task where the lesson was planned and taught in the elementary grades as well as a related task at different levels of rigor that teachers in a workshop engaged in to deepen their understanding of the learning trajectory of geometric and measurement concepts.

Vignettes and Connections to Learning Trajectories of Geometric Thinking

Vignette #1: Geometric Thinking with Area using Learning by Doing. The discussion of the first task in this section includes observations from a second grade classroom where a geometric task to establish the area of a given shape was presented. The task evolved from discussions with second graders around the importance of bees for the environment and therefore the need for creating a bee-hotel to provide long term accommodation for bees to lay eggs and emerge as fully-grown adults (Schmitt, Demary and Wilson-Rich, 2021). The shape of the hotels were cylindrical coffee cans that the students related the cross-section of top traced onto paper as small, medium and large *circles* (See Figure 2).

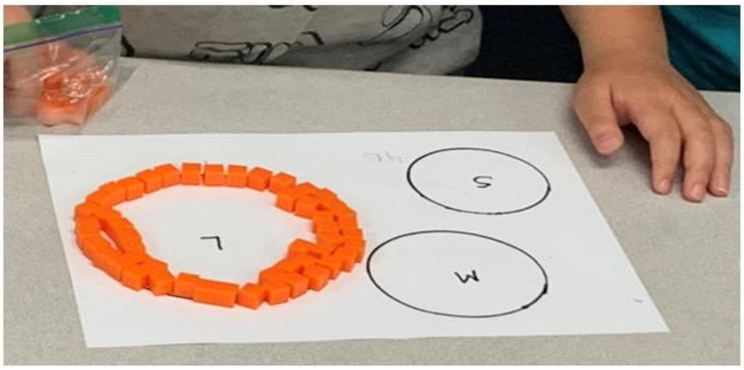

Figure 2. Estimating area of circle using base-ten unit blocks

There has been a lot of research around misconceptions and errors in elementary grades on the lack of comprehension of the fundamental concepts of spatial measurement and their relationships and the procedures and formulas used for measuring length, area, and volume (Tan Sisman & Aksu, 2016). However, the second grade students had no problem understanding "area" means to "fill" covering a two-dimensional space with equal-size units without gaps (Sarama & Clements, 2009). As they built towards the long-term unit project activity which was to build such a bee-hotel and learn different STEAM concepts, Ms. Sims, the second grade teacher also motivated the students to first think of the concept of "*area*" of each of the circles motivated by the holes on the cross-sectional surface of the bee-hotel. To help the students visualize this better through *learning by doing*, the teacher provided packets of base-ten unit blocks for them to cover the circular area with these blocks. Without any prompting most student groups working as teams went on to "fill up" the circle completely with several base-ten unit blocks and then started to count the total number (see Figure 3). Following that each team was asked to report the number of cubes that fit the circle. Six groups of students with 4 students in each group reported numbers ranging between 117 cubes to 125 cubes. While all students engaged in this "area" identification process through counting, they demonstrated several different approaches. These included filling the cubes around the circumference and going through concentric circles and counting the number of cubes in each of those circles and adding. Another group just filled up the circle completely with the cubes and just counted all of them loud. Each of these groups realized that it took time to count.

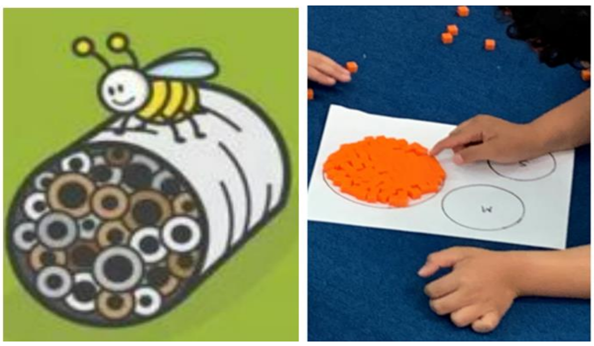

Figure 3. A conceptualization of "area" though learning by doing

Ms. Sims then focused on one particular group that did the counting differently and asked the students in the group to explain their thinking. The students in this group were intuitively able to discover that it is faster to count if they grouped 10 unit-cubes and pulled them out, which helped them to work towards an approximate answer quickly which was in the same range as the other groups (see Figure 4).

Figure 4. Understanding "area" as a sum though learning by doing

In this vignette, Ms. Sims attends to students' mathematical thinking as an equitable teaching practice. By eliciting students' thinking and making use of the strategy of grouping by tens to make the counting more efficient sends a positive message about students' mathematical identities. Ms. Sims made her student thinking public, and then chose to elevate a student to a more prominent position in the discussion by identifying his or her idea as worth exploring, to cultivate a positive mathematical identity.

While the first task with the bee-hotel may be an elementary exercise and appropriate for engaging second graders in geometric thinking, it demonstrated some of the foundational aspects from geometry that is taught when introducing the concepts of area later such as the "***Area Addition Postulate***" which states that if *a figure (e.g. circle) is composed of two or more parts (e.g. base-10 unit block) that do not overlap each other, then the area of the figure is the sum of the areas of the parts.*

A teacher could continue to use the same context with bees which often tend to build flat honeycombs from just three shapes: squares, triangles or hexagons. Specifically, the students could then be given the opportunity to compare how these shapes are related in terms of the number of base-10 cubes and even evolve through stages in Van Hiele's theories from *visualization* (understanding size and shape), *analysis* (counting the base-10 cubes), *abstraction* (group-counting), *deduction* (making connections between the counts for various shapes), and *rigor* (moving from standard to non-standard shapes), respectively. An example of rigor for middle school students could be to expand on the bee-hotel activity to a more challenging activity where the students are asked to calculate the area of non-standards shapes that arise in real-world applications. This is presented next.

Connecting Vignette #1 to Trajectory of Area Thinking through Learning by Doing. Consider the following second task that was presented in a workshop for middle and secondary teachers of finding the area of a bacterial colony (of unknown shape) inside a petri-dish (See Figure 5).

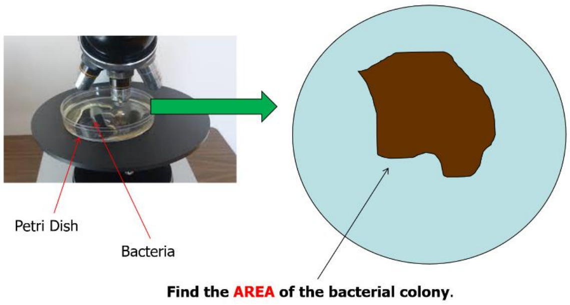

Figure 5. Illustration of a bacterial colony in a petri-dish

As additional information, they were given the radius of the circle that represented the petri-dish, which helped them to find the exact area of the petri-dish by using the area of a circle formula. To help them with their geometric thinking, graph paper of various grid sizes were also provided at their tables. Without any prompting, the teachers were asked to place the graph paper with the bigger grid size on top of the picture (see Figure 6).

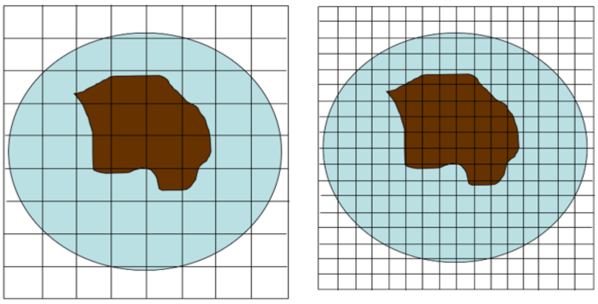

Figure 6. Computing area using graph paper with different grid sizes

Following that, they were guided to perform some computations such as counting the number of squares (or parts of) that covered the non-standard shape. This consisted of finding an approximate number of squares for the non-standard shape and the circle (representing the petri-dish). Following this guided inquiry-based learning part, the teachers had the discovery moment when they realized that they can use the respective counts they obtained with the area of the petri-dish that they have and use proportional reasoning to estimate the area of the non-standard shape. As a continuation, the teachers were asked to repeat the activity using the graph paper with a finer grid (see Figure 6). When asked what they observed, they immediately noted that the answer they obtained through geometric thinking and proportional reasoning argument this time would be much closer to the exact area of the non-standard shape. Without prompting, they also had figured out that more finer grid papers can yield much better estimates of the area. Needless to say the teachers were excited with their new mathematical discovery of geometric learning and even checked if their educated answer was less than the area of the circle calculated using the radius provided and using the formula for area of a circle, which was an upper-bound. This task also provided the opportunity to introduce connections between geometric thinking for areas of non-standard shapes as fractions of the whole (e.g. of the circle).

Vignette #2: Geometric Thinking with Volume using Challenge-based Learning. This classroom vignette of a Challenge-based learning task called canstruction[2] is from a lesson study with a fifth and sixth grade teacher, Ms. Green, who focused on a charitable fundraiser to fill a donation for a local food pantry. The activity started with Ms. Green showed a picture of a 3D solid (sphinx) built out of cans (Figure 7) and asked: *How many cans do we need to create the geometry of the sphinx and is it reasonable for us to do this design?*

Figure 7. Geometric canstruction

This provided the students with the opportunity to pose their own questions including "How much space would it take up?", "What is the height or width of the design?", "How many cans will we need?", "Does it have to be a certain size or shape?". Following that the students identified questions to get started as well as discussed steps for this process. Some of these are identified in Table 1.

[2] https://www.canstruction.org/

Table 1. Problem posed by students

Questions to get started	Steps to this process
● How far back do each row in the pyramid go? ● How tall is the structure? ● How many cans are in each section of the structure? ● How big are the cans? ● How many layers is this structure? ● Where are the cans going to come from?	● How am I going to design the building to have structural integrity? ● Where are you going to put this? Inside the building or outside? ● What color cans? ● How much will this cost? ● Will the cans all be the same size?

The students then worked in small groups to outline details on the design approach. By keeping this challenge open, students had multiple opportunities to become what the Barrett et al. (Children Measurement, n. d) LT calls a "3-D Array Structurer", where one understands the rectangular prism volume formula and shows the coordination of multiplicative and additive thinking flexibly. That is, they show indications of knowing how to coordinate the three dimensions, multiplicatively iterating cubes in a row, column, and/or layer to determine volume (see more on https://www.childrensmeasurement.org/volume.html). In addition, decomposing shapes to find the volume of irregular shapes like the one with the Canstruction Sphinx provides opportunity for critical thinking and problem solving.

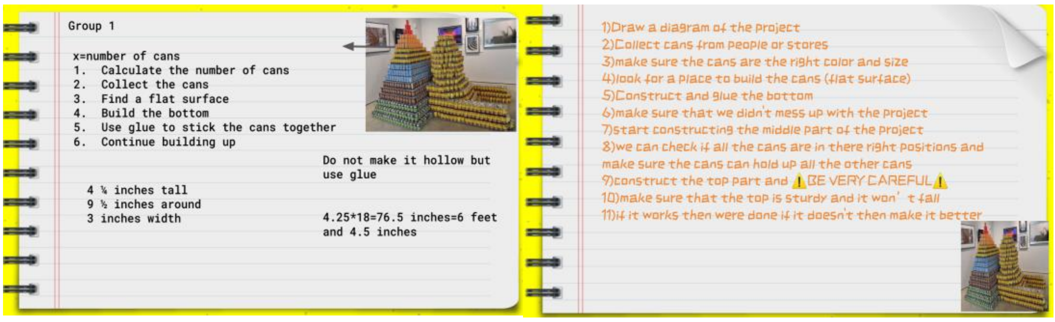

Figure 8. Student work to identify strategies to predict the number of cans

The work of two of the groups are shown in Figure 8 which shows the diverse ways of student thinking. Both groups engaged in algorithmic thinking writing out a set of instructions to accomplish the task.

Group 1 students further worked on the design to identify that a total of 3606 cans are in the pyramid. Group 2 used a different approach to yield 3758 cans which was not too far from Group 1's prediction. Both these groups' work is shown in Figure 9 that shows the different approaches each group took but landed in answers that were in the same ballpark.

Ms. Green *positions students as being capable* as an equitable teaching practice by providing them with the opportunity to become problem posers. By asking students what questions they have about the Canstruction project, she positions students and their groups to explore an important mathematical pathway. Each group shared some different questions to consider about the canstruction project like knowing how much space it would take up as a display, how tall the structure would be and how they might have to consider ceiling height, how many cans it would actually take to complete the design and what that would means in terms of the number of cans each family in the school might need to donate for them to reach their charitable goal. By sharing the problem, Ms. Green shares power in the classroom and positions them as capable mathematicians who can provide meaningful input in making decisions about their classroom project.

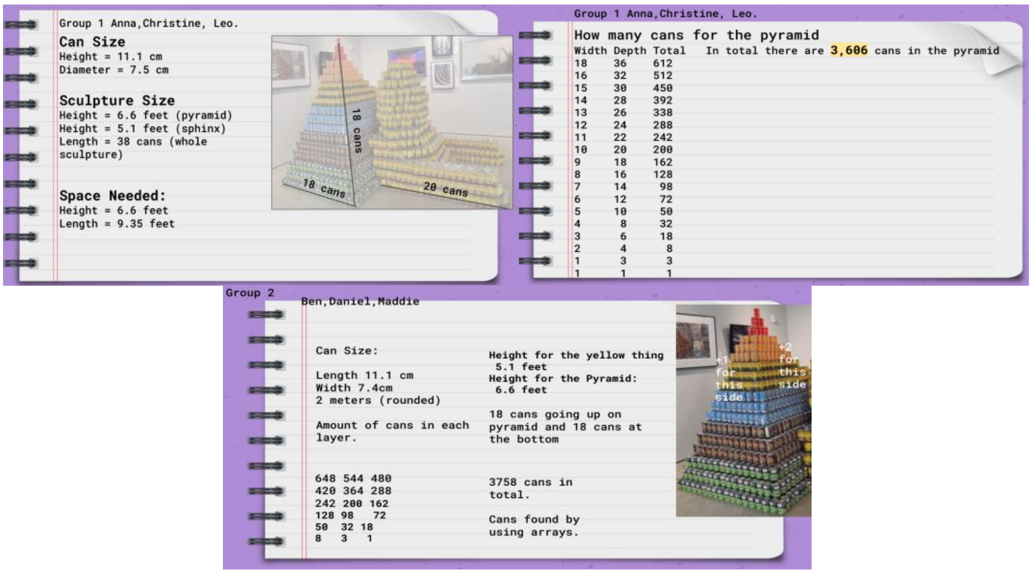

Figure 9. Comparison of the work of two groups to predict the number of cans

Connecting Vignette #2 to the Trajectory of Volume through Challenge Based Learning. Challenge-based learning provides students to gain knowledge, develop cognitive and metacognitive strategies and satisfaction of increased strategic competence (Piaget et al., 2013). It provides opportunities for students to use and adapt strategies to attain goals, basing their choices on personal preferences. To apply Challenge-based learning together with geometry instruction, teachers need to learn to encourage students to participate in a process that enables them to become aware of the ways in which they think, learn and problem-solve.

Figure 10. An illustration of the problem of filling popcorn into a room

The discussion of the task in the previous vignette relates to another professional development workshop that the authors led with primary grade teachers where the authors asked the following question to motivate challenge-based learning along with a picture (see Figure 7): "*How many popcorn kernels will it take to fill the room you are sitting in? Also, what mathematical competency are we trying to build through this task?"*

Enrico Fermi (1901-1954), who won the Nobel Prize for Physics in 1938 for his work on nuclear processes, was also well known for creating open real-world problems such as this task that could only be solved by giving a reasonable estimate. Some examples of Fermi problems include, "*How many piano tuners are there in Chicago*?" or "*How many people in the world are talking on their cell phones at this instant*?" These problems are not only very real-world but also give the problem solver a challenge and an indication that the solution will involve big numbers and a lot of data. The key in solving these problems involve the need for some additional information and then the type of assumptions that the problem solver would need to make to solve the problem. The Fermi problems also serve as "model-eliciting tasks', because the required modeling process necessitates multiple modeling cycles with multiple ways of thinking about givens, goals, and solution paths.

To develop deep understanding around volume, students need concrete experiences involving packing and building with cubes as well as reasoning with partially filled containers to move to more abstract thinking. The teacher participants in the workshop had absolutely no trouble identifying the geometric concept this activity tried to help build which is *determining volume*. They also recognized that this problem involved big data as they realized the number of popcorn kernels is a lot but they had to come up with a creative way to determine this number. The teachers were allowed to use any approach they felt would help get a good estimate. Next, we present an approach that the teachers discussed and reflected on in their solution strategy, which is illustrated in Figure 11.

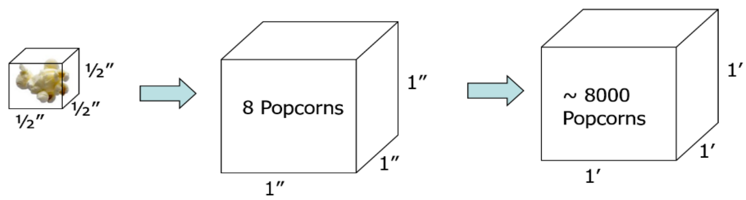

Figure 11. Steps in the Challenge-based learning and FERMI problem-solving process

As the teachers quoted, "*we note that assuming every popcorn kernel can fit into a half-inch by half-inch by half-inch cube, one can ask about the number of kernels that can fit into a one-inch by one-inch by one-inch cube. A quick visual analysis indicates a one-inch by one-inch by one-inch cube contains eight half-inch by half-inch by half-inch cubes and hence contains eight popcorn kernels. Approximating 1-feet as 10-inches instead of 12-inches (as a good geometric estimate), a 1-foot by 1-foot by 1-foot cube will contain 8000 popcorn kernels. Now that we have a measure for a 1-foot by 1-foot by 1-foot cube, all we have to do is to find an estimate of the length, width and height of the room in feet.*

That will tell us how many 1-foot by 1-foot by 1-foot cubes can fit into that room with those dimensions."

Note the steps in the geometric estimation that make the problem-solving efficient. For example, assuming 1-feet to be about 10-inches makes the multiplication in the computation easier. Ultimately, one is able to provide an educated guess for the answer to the number of popcorn kernels but also gets a good grasp of the notion of capacity or volume. The FERMI approach presented was also very rewarding and provided the opportunity to help students to learn simple geometric facts because they represented knowledge in a variety of ways. However, this task may not have been cognitively complex enough that allowed the student to pose and solve real problems, and use their knowledge to create artifacts. The challenge should also have the potential to frustrate students (and teachers) and to send them searching for alternative paths. Next, we describe such a task enacted within a classroom, that provides more mathematical rigor using a challenge-based learning approach.

Vignette #3: Geometric Thinking using Mathematical Modeling. In the final classroom vignette, we visit Ms. Sim's second grade classroom again where she introduced a mathematical modeling activity that is centered around lanterns that were traditionally used to decorate inside homes or on the streets of the town during Eid, the muslim festival of Ramadan. These lanterns also come in an array of colors and shapes and have been said to symbolize hope as they light the way through darkness. The specific modeling *problem posed* for the students was *to determine how many lanterns they needed to make to light up their front hallway*.

The students started with the *inquiry* phase (van Hiele - phase 1) where they engage in a "we know" and "we need to know" activity by asking questions to elicit decisions such as "what is the length of the hallway?", "Will there be lights on both sides of the hallway?", "Will the lights be staggered or parallel?" This phase allowed students to *consider the variables* like the length of the hallway as well as the spacing between lanterns which allowed them to *make assumptions* and decisions to constrain the problem. Following Ms. Sims provided a *direct orientation* (van Hiele - phase 2) with the structure of the hallway as shown as in Figure 12. And the students were given the options to arrange the lanterns in this hallway setup.

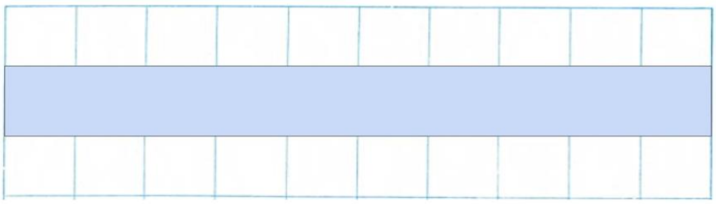

Figure 12. A pictorial representation of the hallway provided by teachers

As students were *building their model*, they engaged in creative geometric design that included a variety of symmetric and asymmetric patterns and arrangements (See Figure 13). In this *explicitation* phase (van Hiele - phase 3), Ms. Sims was able to support with terminology for the various patterns as shown in Figure 13 and helped engage the students in more conversations. Following that Ms. Sims provided the opportunity for the students to engage in a *free orientation* phase (van Hiele - phase 4) where the students were able to build their various patterns and continued to discuss the advantages and disadvantages of these various patterns. Finally, the students were able to make connections by *integrating* (van Hiele - phase 5) information from the various patterns which helped to *analyze and interpret* the “best” lantern arrangement based on a rubric they created.

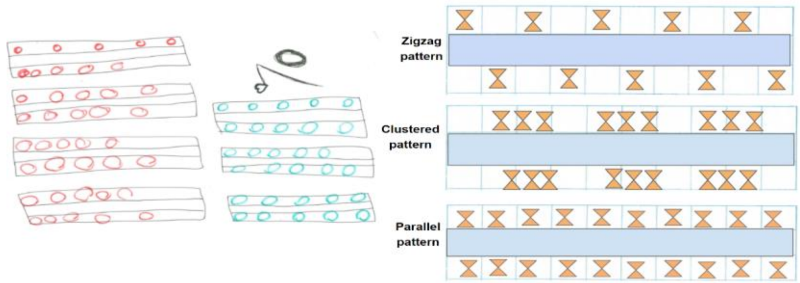

Figure 13. Eid lantern decoration along the school hallway

By having the students go through these phases the teacher also practiced various instructional roles including planning tasks, directing children’s attention to geometric qualities of shapes, introducing terminology and engaging children in discussions using these terms, and encouraging explanations and problem-solving approaches that make use of children’s descriptive thinking about shapes. As they cycled through mathematical modeling via these five phases with materials to build the lanterns enabled children to build a rich background in visual and descriptive thinking that involves various shapes and their properties, specifically providing them an opportunity to move through all the van Heile levels from visualization, analysis, informal deduction, deduction to rigor.

In this classroom task, the teacher also provided an opportunity for students to take ownership and convince (*prove*) themselves and others on why their lantern patterns were the “best” in the *analysis* phase of the mathematical modeling. The teacher leveraged students' funds of knowledge by connecting to festivals and celebrations that students celebrate and the decoration that they put up for these holidays. By connecting to students' lived experiences and their prior knowledge around decorating, she provided students equitable access to the descriptive model of figuring how many lanterns needed for a long hallway using visuals to count by tens, grouping numbers, and writing number sentences to make sense of extending the pattern in the hallway.

Connecting Vignette #3 Spatial Reasoning Trajectory through Mathematical Modeling. This final example presented in this work was presented at a professional learning workshop and includes how curriculum materials should integrate mathematical modeling approaches to support conceptual teaching and learning of geometry and measurement. The following task is an example of employing real-world data to build a geometric model and discover new knowledge and is from a professional development workshop given by the authors to elementary, middle and high school teachers.

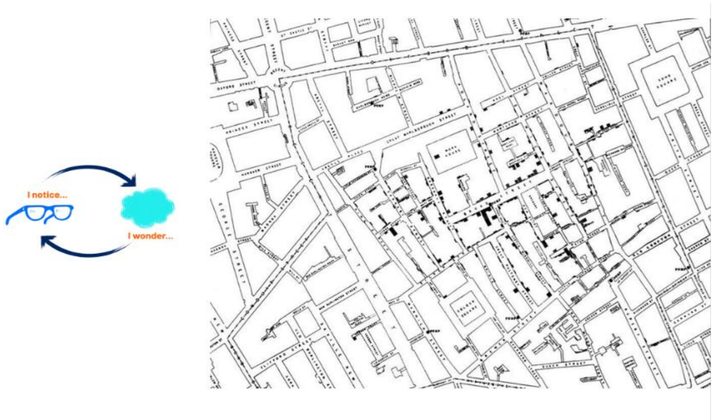

Figure 14. A Notice and Wonder routine applied to a map

Starting with a typical *Notice and Wonder* visual routine (van Heile - Level 1), the participants were asked to display the problem scenario such as Figure 14 to the classroom and asks the students "what do they notice" and ask them to form their responses as "I notice this", "I notice that" and so on. Typical responses were: “I notice a map”, “I notice streets”, “I notice black dots”. After a lot of noticing’s were shared, the participants were asked, “What are you wondering?” Typical wondering included “I wonder what is this a map of?”, “I wonder what the black dots are?” to “I wonder why the black dots are crowded in certain parts of the map?” Once the participants were hooked into the geometric visual, they were now eager to learn about what came next. Figure 15 shows the outline of the entire instructional routine that participants were shown to enhance their own pedagogical strategies.

The person’s picture was shown next and introduced as John Snow (often referred to as *father of epidemiology*) who helped to map out a cholera outbreak in London in September, 1854. In fact, the black dots were the number of deaths due to cholera in those locations that Snow helped to identify through data he collected by going door to door and doing analysis (van Hiele - Level 2). Now that the participants had an idea of the person in history and what they are looking at, the next thing was to recollect some of what happened in the story. From his basic analysis, John Snow informally deduced from the maps and interviews that the reason for the deaths was that the water that the people who died were exposed to was contaminated (van Hiele - Level 3). However, the doctors of the day blamed the regular outbreaks of disease in the city on the stench and believed that the disease was spread through Miasma or "toxic air". While Snow alerted the health authorities about his theory in 1849, it was largely ignored in part because of squeamishness about the fecal-oral route of transmission that was involved. Note that this part of the story now gives opportunity to students that are interested in Science to research about bacteria, viruses and other water-borne diseases.

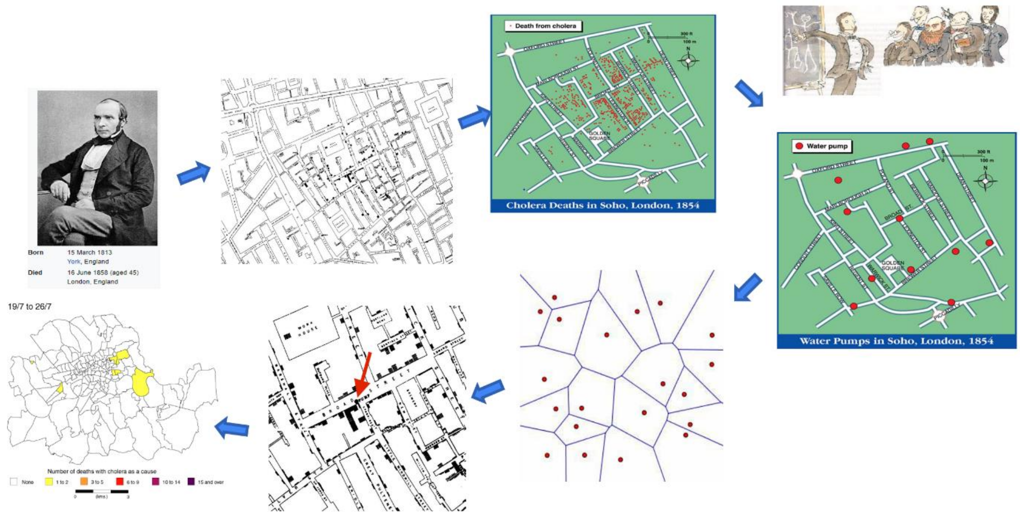

Figure 15. Example of a data-driven inquiry and geometric approach to teaching and learning

What the participants learn after this is how Snow transitions in his work to a formal deduction (van Hiele - Level 4) when he created an ingenious geometric map that dramatically showed the geographic spread of deaths in the outbreak. Each bar on the graph represented a death at that address, showing as many as 18 people dying in particular households. This geometric representation of the data showed that most of the deaths were tightly clustered in a specific area, crowded around the water pump at 40 Broad Street (now Broadwick Street) in Soho. Snow's research had led him to believe the Broad Street pump was the source of the disease, and this data backed up that theory.

His task was then to convince everyone with rigor (van Hiele - Level 5) that while there were other pumps nearby, it was most likely that this particular pump was the source. His next brilliant step was to represent the time it took to travel to the Broad Street pump on his map and to calculate who was most likely to use each water pump in the area. Snow drew a curve on the map that marked the points where the Broad Street pump was at equal walking distance from neighboring water pumps. For this discovery, it was remarkable that in 1854, John Snow used the simple geometric concept of a *perpendicular bisector* which is a line segment that intersects another line segment at a right angle and it divides that other line into two equal parts at its midpoint. More importantly he also used one of the most important theorems taught in geometry is the perpendicular bisector theorem states that any point on the perpendicular bisector is "*equidistant from both the endpoints of the line segment on which it is drawn*." With this rigorous and convincing mathematical analysis of the cholera outbreak in Soho using simple mathematical modeling and geometric visualization, he was able to convince the authorities that the disease was transmitted through water. Snow's mathematical evidence that cholera was water-borne is one of the founding moments of epidemiology and the use of geometry to understand disease, one of the greatest advances in medicine that has saved millions of lives. Therefore, integrating such pivotal stories into the curriculum that reinforces the importance of geometry is important to make students become active STEM learners and doers.

V. DISCUSSION

In this work, we presented a conceptual framework supporting the enhancement of geometry and measurement instruction through specific learning trajectories in teaching geometry and connections to equitable teaching practices. Three specific vignettes were also illustrated that integrated educational approaches such as Learning by doing, Challenge-based learning and Mathematical modeling along with tasks that vertically articulated across grade levels with rigor (see Table 2).

Table 2. Linking the domains in the conceptual framework

Inquiry-based Approaches with Vignettes	Learning Trajectory with Rigor of Math	Connection to Equitable Teaching Practices
• **Approach**: Learning by doing • **Task**: "Discover area of standard shapes for "Bee Hotels" to non-standard shapes for "Bacteria in a Petri Dish"	Area, Surface Area, Area Addition Postulate, Calculating Areas of Non-standard Shapes	Attend to students' mathematical thinking to build on their strength and advances their learning
• **Approach**: Challenge-based Learning • **Task**: "Introduction to Fermi Problems through Popcorn to Canstructions"	Estimation, Measurement, Surface Area, Volume	Position students as capable by honoring multiple approaches
• **Approach**: Mathematical Modeling • **Task**: "Light the Path to Medical Mysteries from the Path"	Surface Area, Lines and Angles, Parallel, Perpendicular Bisector	Funds of Knowledge and connecting to lived experiences

The work presented also reflected on how these tasks helped to illustrate van Hiele's theory on levels of student thinking in geometry (visual, analysis, abstraction, deduction and rigor) and the five-phase sequence of geometric instruction (inquiry, direct orientation, explicitation, free orientation, integration) at elementary grades. The study also gave us an opportunity to observe the effects of acquisition of the van Heile levels in geometry and students' attitudes towards geometry. Integrating such educational theories and approaches should be used in teaching in order to help students equitably to overcome their difficulties in mathematics.

By connecting these inquiry-based teaching approaches of learning by doing, challenge-based tasks, and mathematical modeling, we showcase children as "doers of mathematics" (NCTM, 2020, p. 70). We approach equitable instruction by centering the inquiry process of problem posing, exploring, conjecturing and problem solving and highlight students' funds of knowledge, position them as capable and honor their diverse strategies to build up mathematical strengths of children so they continuously see themselves as doers and sense makers of mathematics. In our three vignettes, we illustrate how teachers position students as young mathematicians, who use visualization, analysis, conjectures and justification to explore spatial reasoning as a humanizing experience where “people, natural objects, human-made objects and structures exist somewhere in space, and the interaction of people and things must be understood in terms of locations, distances, directions, shapes and patterns” (NRC, 2006, p. 5).

As educators commit to engaging in equitable instruction, we recommend a focus on inquiry-based instruction that develop deep and well-connected understandings of spatial reasoning that can be vertically articulated to concepts that build on one another. This ensures students' understanding of big ideas to build a strong foundation for geometric and measurement concepts. Couple this deep understanding with equity practices by positioning students as capable learners and allowing diverse learners to have time to share their thinking and elevating their status to form positive mathematics identities in the elementary classrooms. Finally, we offer a professional development model and focus that deepens teachers' knowledge of spatial reasoning through vertically articulating rich tasks and focusing on analysis of student thinking to develop confidence in mathematics thinkers, doers, and sense makers.

Acknowledgments

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