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An Alternative Perspective of Near-rings of Polynomials and Power series

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Abstract. Unlike for polynomial rings, the notion of multiplication for the near-ring of polynomials is the substitution operation. This leads to somewhat surprising results. Let S be an abelian left near-ring with identity. The relation $\sim$ on S defined by letting $a \sim b$ if and only if $ann_S(a) = ann_S(b)$, is an equivalence relation. The compressed zero-divisor graph $\Gamma_E(S)$ of S is the undirected graph whose vertices are the equivalence classes induced by $\sim$ on S other than $[0]_S$ and $[1]_S$, in which two distinct vertices $[a]_S$ and $[b]_S$ are adjacent if and only if $ab = 0$ or $ba = 0$. In this paper, we are interested in studying the compressed zero-divisor graphs of the zero-symmetric near-ring of polynomials $R_0[x]$ and the near-ring of the power series $R_0[[x]]$ over a commutative ring R. Also, we give a complete characterization of the diameter of these two graphs. It is natural to try to find the relationship between $\mathrm{diam}\big(\Gamma_E(R_0[x])\big)$ and $\mathrm{diam}\big(\Gamma_E(R_0[[x]])\big)$. As a corollary, it is shown that for a reduced ring R, $\mathrm{diam}\big(\Gamma_E(R)\big) \leq \mathrm{diam}\big(\Gamma_E(R_0[x])\big) \leq \mathrm{diam}\big(\Gamma_E(R_0[[x]])\big)$.

1. Introduction

Throughout this paper, all rings are associative rings with identity and all near-rings are abelian left near-rings with unity. Recall that a non-empty set S with two binary operations " $+$ " and " $\cdot$ " is an *abelian left near-ring* if $(S, +)$ forms an abelian group, $(S, \cdot)$ forms a semi-group, and $a \cdot (b + c) = a \cdot b + a \cdot c$ for each $a, b, c \in S$. Clearly, every ring is a near-ring. The *zero-symmetric part* of a near-ring S is the set of all elements $a \in S$ such that $0 \cdot a = 0$ and it is denoted

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by S_0. Moreover, a near-ring N is called *zero-symmetric* if $S = S_0$. Let S be a near-ring and $A \subseteq S$. Then $ann_S(A) = \ell.ann_S(A) \cup r.ann_S(A)$, where

$$\ell.ann_S(A) = \{s \in S \mid sa = 0 \text{ for each } a \in A\}$$

and $r.ann_S(A) = \{s \in S \mid as = 0 \text{ for each } a \in A\}$. Also, we write $Z_\ell(S)$, $Z_r(S)$ and $Z(S)$ for the set of all left zero-divisors of S, the set of all right zero-divisors and the set $Z_\ell(S) \cup Z_r(S)$, respectively. Moreover, we use $\langle A \rangle$ to denote the ideal generated by A. For basic definitions and comprehensive discussion on near-rings, we refer the reader to [21].

Let G be a graph. Recall that G is *connected* if there is a path between any two distinct vertices of G. Also, the *diameter* of G is

$$\operatorname{diam}(G) = \sup\big\{d(a,b)|a, b \text{ are vertices of } G\big\},$$

where $d(a,b)$ is the length of the shortest path from a to b. Moreover, the *girth* of G, $\operatorname{gr}(G)$, is the length of the shortest cycle of the graph, and $\operatorname{gr}(G) = \infty$ if G has no cycles.

The concept of a zero-divisor graph of a commutative ring R was introduced by Beck in [5]. However, he let all elements of R be vertices of the graph and was mainly interested in coloring. Inspired by his study, Anderson and Livingston [3], redefined and studied the (undirected) zero-divisor graph $\Gamma(R)$, whose vertices are the non-zero zero-divisors of a ring such that distinct vertices x and y are adjacent if and only if $xy = 0$. According to [3, Theorems 2.3 and 2.4], $\Gamma(R)$ is connected with $\operatorname{diam}(\Gamma(R)) \leq 3$, and $gr(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle. Redmond [22] extended the concept of the zero-divisor graph to noncommutative rings. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings (cf. [3, 15, 17, 18, 20, 22]).

In [8], the authors generalized this concept to a zero-symmetric near-ring S. They defined an undirected graph $\Gamma(S)$ with vertices in the set $Z^*(S) = Z(S) \setminus \{0\}$ and such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. Following [8, Theorem 2.2], the zero-divisor graph of zero-symmetric near-ring S is connected and $\operatorname{diam}\big(\Gamma(S)\big) \leq 3$.

For a ring or near-ring S, define $a \sim b$ if and only if $ann_S(a) = ann_S(b)$. As in [20], one can see that $\sim$ is an equivalence relation on S. For any $a \in S$, let $[a]_S = \{b \in S \mid a \sim b\}$ (for short we can use $[a]$ instead of $[a]_S$). For instance, it is clear that $[0]_S = \{0\}$ and $[1]_S = S \setminus Z(S)$, and that $[a]_S \subseteq Z(S) \setminus \{0\}$ for each $a \in S \setminus ([0]_S \cup [1]_S)$.

As in [23], $\Gamma_E(S)$ will denote the (undirected) graph, called the *compressed zero-divisor graph* of S, whose vertices are the elements of $S_E \setminus \{[0]_S, [1]_S\}$ such that distinct vertices $[a]_S$ and $[b]_S$ are adjacent if and only if $ab = 0$ or $ba = 0$. Note that if a and b are distinct adjacent vertices in $\Gamma(S)$, then $[a]_S$ and $[b]_S$ are adjacent in $\Gamma_E(S)$ if and only if $[a]_S \neq [b]_S$. Clearly, $\operatorname{diam}\big(\Gamma_E(S)\big) \leq \operatorname{diam}\big(\Gamma(S)\big)$. For a commutative ring R, Anderson and LaGrange [2], showed that $\operatorname{gr}\big(\Gamma_E(R)\big) = 3$ if

$\Gamma_E(R)$ contains a cycle, and determined the structure of $\Gamma_E(R)$ when it is a cyclic and the monoid R_E when $\Gamma_E(R)$ is a star graph.

Let R be a ring. Since $R[x]$ is an abelian near-ring under addition and substitution, it is natural to investigate the near-ring of polynomials $(R[x], +, \circ)$. The binary operation of substitution, denoted by " $\circ$ ", of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for polynomials $f = \sum_{i=0}^{m} a_i x^i$ and $g \in R[x]$,

$$g \circ f = \sum_{i=0}^{m} a_i g^i.$$

For example, $(a_0 + a_1 x) \circ x^2 = (a_0 + a_1 x)^2 = a_0^2 + (a_0 a_1 + a_1 a_0)x + a_1^2 x^2$. However, the operation $\circ$, left distributes but does not right distribute over addition. Thus $(R[x], +, \circ)$ forms a left near-ring but not a ring. We use $R[x]$ to denote the left near-ring $(R[x], +, \circ)$ with coefficients from R and

$$R_0[x] = \{f \in R[x] \mid f \text{ has zero constant term}\}$$

is the zero-symmetric left near-ring of polynomials with coefficients in R. Also, for each $f = \sum_{i=0}^{m} a_i x^i$ and $g = \sum_{j=0}^{n} b_j x^j \in R[x]$, we write

$$fg = \sum_{k=0}^{n+m} (\sum_{i+j=k} a_i b_j) x^k.$$

The aim of this paper is the study of the compressed zero-divisor graphs of zero-symmetric near-ring of polynomials $R_0[x]$ and near-ring of formal power series $R_0[[x]]$ over a commutative ring R. For a reduced ring R, we prove that $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = i$ if and only if $\operatorname{diam}\big(\Gamma_E(R[x])\big) = i$ for each $i = 1, 2, 3$. Moreover, we show that $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$ if and only if $|\Gamma_E(R)| \leq 2$, $Nil(R)^2 = 0$, $Z(R) = ann_R(a)$ for some $a \in R$, and $ann_R(c) = Nil(R)$ for each $c \in Z(R) \setminus Nil(R)$. Also, it is proved that $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$ if and only if $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$. In addition, we are interested in characterizing the diameter of graph $\Gamma_E(R_0[[x]])$. In fact, The diameter of the graphs $\Gamma_E(R[[x]])$ and $\Gamma_E(R_0[[x]])$ are the same when R is a reduced ring. Also, we try to relate $\operatorname{diam}\big(\Gamma_E(R)\big)$ to $\Gamma_E(R_0[[x]])$. As a corollary, it is shown that

$$\operatorname{diam}\big(\Gamma_E(R)\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[x])\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[[x]])\big),$$

where R is reduced. Moreover, we give a complete characterization for the possible diameters of $\Gamma_E(R_0[[x]])$, where R is a non-reduced Noetherian ring.

2. On the Diameter of the Compressed Zero-divisor Graph of $R_0[x]$

Let R be a commutative ring. Following [1, Theorem 2.7], we have

$$2 \leq \operatorname{diam}\big(\Gamma(R_0[x])\big) \leq 3.$$

Hence $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \leq 3$, since $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \leq \mathrm{diam}\big(\Gamma(R_0[x])\big)$.

Proposition 2.1. *Let R be a commutative ring with $Z(R) \neq 0$. Then* $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \geq 1$.

Proof. First suppose that R is a reduced ring and $0 \neq a \in Z(R)$. Thus $ab = 0$ for some non-zero $b \neq a$ of R. If $[ax] = [bx]$, then $ax \in ann_{R_0[x]}(ax)$, and so $a^2 = 0$, which is a contradiction. Hence $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \geq 1$. Now assume R is a non-reduced ring. Then there exists $0 \neq a \in R$ such that $a^2 = 0$. Thus $ax, ax + x^2 \in Z\big(R_0[x]\big)$. Also, $x^2 \in ann_{R_0[x]}(ax)$ but $x^2 \notin ann_{R_0[x]}(ax + x^2)$, which implies that $[ax] \neq [ax + x^2]$, and so $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \geq 1$. □

For any $f \in R[x]$, we denote by C_f the set of all coefficients of f. Also, the set of all non-zero coefficients of f is denoted by $C_f^* = C_f \setminus \{0\}$.

To characterize the diameter of $\Gamma_E(R_0[x])$, where R is a reduced ring, we need the following lemma.

Lemma 2.2. *Let R be a reduced ring. Then*

(1) [4, Lemma 1] For each $f, g \in R[x]$, $fg = 0$ if and only if $a_i b_j = 0$ for each $a_i \in C_f$ and $b_j \in C_g$.

(2) [7, Lemma 3.4] For each $f, g \in R_0[x]$, $f \circ g = 0$ if and only if $a_i b_j = 0$ for each $a_i \in C_f$ and $b_j \in C_g$.

Let R be a reduced ring and f, g be elements of the ring $R[x]$. Then $fg = 0$ if and only if $a_i b_j = 0$ for each $a_i \in C_f$ and $b_j \in C_g$, by Lemma 2.2. Hence $fx \circ gx = 0$, by Lemma 2.2. On the other hand, $Z(R_0[x]) \subseteq Z(R[x])$, by Lemma 2.2. Thus $d([f], [g]) = t$ in $\Gamma_E\big(R[x]\big)$, if and only if $d([fx], [gx]) = t$ in $\Gamma_E\big(R_0[x]\big)$. Therefore we can conclude the next result.

Proposition 2.3. *Let R be a reduced ring. Then*

(1) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 1$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 1$.

(2) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 2$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 2$.

(3) $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 3$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Corollary 2.4. *Let R be a reduced commutative ring. Then* $\mathrm{diam}\big(\Gamma(R_0[x])\big) = 3$ *if and only if* $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Proof. ($\Rightarrow$) Since $\mathrm{diam}\big(\Gamma(R_0[x])\big) = 3$, then we have $\mathrm{diam}\big(\Gamma(R[x])\big) = 3$, by [1, Proposition 2.10]. Thus $\mathrm{diam}\big(\Gamma_E(R[x])\big) = 3$, by [12, Theorem 3.3]. Hence $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 3$, by Proposition 2.3.

($\Leftarrow$) It is clear, since $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \leq \mathrm{diam}\big(\Gamma(R_0[x])\big) \leq 3$. □

Now, we investigate the diameter of $\Gamma_E(R_0[x])$, when R is not reduced. For this purpose, we bring the following lemmas which are used extensively in the sequel.

Lemma 2.5. ([1, Lemma 2.4]) *Let R be a commutative ring and $f = \sum_{i=1}^{n} a_i x^i$, $g = \sum_{j=1}^{m} b_j x^j$ be non-zero elements of $R_0[x]$ with $f \circ g = 0$. Then*

(1) $rf = 0$ for some non-zero $r \in R$.

(2) f is nilpotent or $sg = 0$ for some non-zero $s \in R$.

Lemma 2.6. ([1, Proposition 2.5]) *Let R be a non-reduced commutative ring. Then*

$$Z_r\big(R_0[x]\big) = Z_\ell\big(R_0[x]\big) \cup \{\sum_{i=1}^{n} a_i x^i \in R_0[x] \mid ann_R(a_1) \cap Nil(R) \neq 0 \text{ and } a_i \in R \text{ for each } i \geq 2\},$$

where $Z_\ell\big(R_0[x]\big) = \{f \in R_0[x] \mid rf = 0, \text{ for some } 0 \neq r \in R\}$.

Lemma 2.7. *Let R be a non-reduced commutative ring and for each $a, b \in Z(R)$, $ann_R\big(\{a, b\}\big) \cap Nil(R) \neq 0$. Then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) \leq 2$. *Also, if there exists $c \in Nil(R)$ such that $c^k = 0 \neq c^{k-1}$ for some $k \geq 3$, then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 2$.

Proof. By [1, Theorem 2.9], we have $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) \leq \operatorname{diam}\big(\Gamma(R_0[x])\big) = 2$.

Now assume that $c^k = 0$ but $c^{k-1} \neq 0$ for some $c \in Nil(R)$ and $k \geq 3$. Since $c^2x \circ x^{k-1} = 0$, then $x^{k-1} \in Z\big(R_0[x]\big)$. Also, $cx \circ x^{k-1} \neq 0 \neq x^{k-1} \circ cx$. Since $x^k \in ann_{R_0[x]}(cx)$ but $x^k \notin ann_{R_0[x]}(x^{k-1})$, then $[cx] \neq [x^{k-1}]$. It follows that $d(cx, x^{k-1}) \geq 2$, and thus $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 2$. □

Following [14], a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for each $a, b \in R$.

Remark 2.8. *Let R be a commutative ring. Then R is a semicommutative ring, and so $Nil\big(R[x]\big) = Nil(R)[x]$, by [16]. On the other hand, $Nil\big(R_0[x]\big) = Nil(R)_0[x]$, by [11, Corollary 2]. Therefore $Nil\big(R_0[x]\big) = Nil\big(R[x]\big)x$. We use this fact freely in the sequel.*

For any $f \in R_0[x]$, we use $\deg(f)$ to denote the degree of f.

Theorem 2.9. *Let R be a non-reduced commutative ring. Then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$ *if and only if $|\Gamma_E(R)| \leq 2$, $Nil(R)^2 = 0$, $Z(R) = ann_R(a)$ for some $a \in R$, and $ann_R(c) = Nil(R)$ for each $c \in Z(R) \setminus Nil(R)$.*

Proof. ($\Rightarrow$) Let $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$. Since R is a non-reduced ring, there exists $0 \neq a \in R$ such that $a^2 = 0$. Let $b \in Z(R)$. If $[ax] = [bx]$, then $ax \in ann_{R_0[x]}(bx)$, since $a^2 = 0$. Thus $ax \circ bx = 0$, and so $ab = 0$. Also, if $[ax] \neq [bx]$, then $ax \circ bx = 0$, by hypothesis. Hence $ab = 0$. Therefore $Z(R) = ann_R(a)$. It follows that for each $b \in Nil(R)$, $b^2 = 0$, by Lemma 2.7. Now assume that b, c are distinct elements of $Nil(R)$. If $[bx] = [cx]$, then $cx \in ann_{R_0[x]}(bx)$, and so $bc = 0$. If $[bx] \neq [cx]$, then $0 = bcx = bx \circ cx$, by assumption. Hence $Nil(R)^2 = 0$.

Now suppose that $c \in Z(R) \setminus Nil(R)$ and $d \in ann_R(c)$. Thus $[x^2] = [cx]$, since $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$. Hence $dx \in ann_{R_0[x]}(cx) = ann_{R_0[x]}(x^2)$, which implies that $d^2 = 0$, and so $ann_R(c) \subseteq Nil(R)$. Also, by a similar way as used above, we have

$Z(R) = ann_R(b)$ for each $b \in Nil(R)$, since $b^2 = 0$. Hence $Nil(R) \subseteq ann_R(c)$. Therefore $Nil(R) = ann_R(c)$.

Let $c \in Z(R)$. If c is nilpotent, then $ann_R(c) = Z(R)$, and if $c \notin Nil(R)$, then $ann_R(c) = Nil(R)$. Hence there exist at most two different vertices $[a]_R$ and $[b]_R$ in $\Gamma_E(R)$, where $a \in Nil(R)$ and $b \notin Nil(R)$. This shows that $|\Gamma_E(R)| \leq 2$.

($\Leftarrow$) We claim that for each $c \in Nil(R)$, $ann_{R_0[x]}(cx) = Z\big(R_0[x]\big)$ and $ann_R(c) = Z(R)$. Since $Nil(R)^2 = 0$ and $c \in Nil(R)$, then $Nil(R) \subseteq ann_R(c)$. Now assume $d \in Z(R) \setminus Nil(R)$. Hence $ann_R(d) = Nil(R)$, and thus $cd = 0$. It means that $ann_R(c) = Z(R)$. Now suppose that $g = \sum_{j=1}^{m} b_j x^j \in Z\big(R_0[x]\big)$. Thus $cx \circ g = 0$, since $Nil(R)^2 = 0$ and $b_1 \in Z(R)$, by Lemma 2.6. Hence $ann_{R_0[x]}(cx) = Z\big(R_0[x]\big)$. On the other hand, since R is non-reduced, $x^2 \in Z\big(R_0[x]\big)$. Also, $x^2 \in ann_{R_0[x]}(cx)$ but $x^2 \notin ann_{R_0[x]}(x^2)$. Hence we have at least two vertices $[cx]$ and $[x^2]$ in $\Gamma_E(R_0[x])$. Clearly, $r.ann_{R_0[x]}(x^2) = 0$. On the other hand, if $g \in \ell.ann_{R_0[x]}(x^2)$, then $g^2 = 0$, and so $g \in Nil(R)_0[x]$. Since $Nil(R)^2 = 0$, then $Nil(R)_0[x] \subseteq \ell.ann_{R_0[x]}(x^2)$, and thus

$$ann_{R_0[x]}(x^2) = Nil(R)_0[x] = Nil\big(R_0[x]\big).$$

Now let f be a non-zero element of $Z\big(R_0[x]\big)$. We can write $f = f_1 + f_2 + f_3$ such that $C^*_{f_1} \subseteq Nil(R)$, $C^*_{f_2} \subseteq Z(R) \setminus Nil(R)$, and $C^*_{f_3} \subseteq R \setminus Z(R)$. We consider the following cases:

Case 1. Let $f = f_1 = \sum_{i=1}^{n} a_i x^i$ and $g = \sum_{j=1}^{m} b_j x^j \in Z\big(R_0[x]\big)$. Since $C^*_{f_1} \subseteq Nil(R)$, then $ann_R(a_i) = Z(R)$ for each $1 \leq i \leq n$. Also, by Lemma 2.6, $b_1 \in Z(R)$. Hence $f \circ g = 0$, since $Nil(R)^2 = 0$. Therefore $ann_{R_0[x]}(f) = Z\big(R_0[x]\big)$, and so $[f] = [f_1] = [cx]$.

Case 2. Let $f = f_2 = \sum_{i=1}^{n} a_i x^i$. Then $ann_R(a_i) = Nil(R)$ for each $1 \leq i \leq n$. Suppose that $g = \sum_{j=1}^{m} b_j x^j \in r.ann_{R_0[x]}(f)$. It means that

$$f \circ g = b_1 f + b_2 f^2 + \cdots + b_m f^m = 0.$$

Thus $b_m a_n^m = 0$, since it is the leading coefficient of $f \circ g = 0$. Also, from $a_n \notin Nil(R)$ yields $a_n^m \notin Nil(R)$, and so $b_m \in ann_{R_0[x]}(a_n^m) = Nil(R)$. Hence $b_m \in Nil(R)$, which implies that $b_m f = 0$, since $ann_R(a_i) = Nil(R)$. Thus $f \circ g = b_1 f + b_2 f^2 + \cdots + b_{m-1} f^{m-1} = 0$. Continuing this process, we see that $b_j \in Nil(R)$ for each $1 \leq j \leq m-1$. Hence g is a nilpotent element of $R_0[x]$, and so $r.ann_{R_0[x]}(f) \subseteq Nil(R)_0[x]$. Now assume that $g \in \ell.ann_{R_0[x]}(f)$. Thus $g \circ f = a_1 g + a_2 g^2 + \cdots + a_n g^n = 0$, and so $a_n b_m^n = 0$. This shows that $b_m^n \in ann_R(a_n) = Nil(R)$, which implies that $b_m \in Nil(R)$. Hence

$$g \circ f = a_1 g_1 + a_2 g_1^2 + \cdots + a_n g_1^n = 0,$$

where $g_1 = \sum_{j=1}^{m-1} b_j x^j$. By repeating this argument, we can conclude that $b_j \in Nil(R)$ for each $1 \leq j \leq m-1$. Therefore $\ell.ann_{R_0[x]}(f) \subseteq Nil(R)_0[x]$. Since $ann_R(a_i) = Nil(R)$ for each $1 \leq i \leq n$, then $g \circ f = 0 = f \circ g$ for each $g \in Nil(R)_0[x]$. Hence $ann_{R_0[x]}(f) = Nil(R)_0[x] = Nil\big(R_0[x]\big)$. Therefore $[f] = [f_2] = [x^2]$.

Case 3. Let $f = f_3 = \sum_{i=1}^{n} a_i x^i$. Then $a_1 = 0$, by Lemma 2.6. Since $rf \neq 0$ for each $0 \neq r \in R$, then $r.ann_{R_0[x]}(f) = 0$, by Lemma 2.5. Also, if $g \in \ell.ann_{R_0[x]}(f)$, then g is nilpotent, by Lemma 2.5. Since $Nil(R)^2 = 0$, then

$$h \circ f = a_2 h^2 + \cdots + a_n h^n = 0$$

for each $h \in Nil(R)_0[x]$. Therefore

$$ann_{R_0[x]}(f) = \ell.ann_{R_0[x]}(f) = Nil(R)_0[x] = Nil\big(R_0[x]\big).$$

Hence $[f] = [f_3] = [x^2]$.

Case 4. Let $f = f_1 + f_2$, where $0 \neq f_1 = \sum_{i=1}^{n} a_i x^i$ and $0 \neq f_2 = \sum_{s=1}^{t} c_s x^s$. Suppose that $g \in \ell.ann_{R_0[x]}(f)$. Then $rg = 0$ for some $0 \neq r \in R$, by Lemma 2.5. Thus $C_g^* \subseteq Z(R)$. Since $ann_R(a_i) = Z(R)$ for each $a_i \in C_{f_1}^*$, we have $g \circ f = c_1 g + c_2 g^2 + \cdots + c_t g^t = g \circ f_2 = 0$, which implies that $g \in \ell.ann_{R_0[x]}(f_2)$. Thus $\ell.ann_{R_0[x]}(f) \subseteq Nil\big(R_0[x]\big)$, by Case 2. Now, assume

$$g = \sum_{j=1}^{m} b_j x^j \in r.ann_{R_0[x]}(f).$$

Since f is not nilpotent, then $C_g^* \subseteq Z(R)$, by Lemma 2.5. Hence

$$0 = f \circ g = b_1 f + b_2 f^2 + \cdots + b_m f^m = b_1 f_2 + b_2 f_2^2 + \cdots + b_m f_2^m = f_2 \circ g,$$

which implies that $g \in r.ann_{R_0[x]}(f_2)$, and so $g \in Nil\big(R_0[x]\big)$, by Case 2. Since $ann_R(a_i) = Z(R)$ for each $a_i \in C_{f_1}^*$ and $ann_R(c_s) = Nil(R)$ for each $c_s \in C_{f_2}^*$, then $\ell.ann_{R_0[x]}(f) = r.ann_{R_0[x]}(f) = Nil\big(R_0[x]\big)$. Hence $ann_{R_0[x]}(f) = Nil\big(R_0[x]\big)$. Therefore $[f] = [f_1 + f_2] = [x^2]$.

Case 5. Let $f = f_1 + f_3$, where $0 \neq f_1 = \sum_{i=1}^{n} a_i x^i$ and $0 \neq f_3 = \sum_{s=1}^{t} c_s x^s$. Then $a_1 + c_1$ is the coefficient of x in f. By Lemma 2.6, we have $a_1 + c_1 \in Z(R)$. Thus $c_1 = 0$, since $a_1 \in Z(R)$ and $Z(R) = ann_R(a)$ for some $a \in R$. Hence $\deg(f_3) \geq 2$. Similar to Case 3, we can conclude that $r.ann_{R_0[x]}(f) = 0$. On the other hand, if $g \circ f = 0$ for some $g \in R_0[x]$, then g is nilpotent, by Lemma 2.5. Hence $\ell.ann_{R_0[x]}(f) \subseteq Nil\big(R_0[x]\big)$. Since $Nil(R)^2 = 0$ and $ann_R(a_i) = Z(R)$ for each $a_i \in C_{f_1}^*$, then $g \circ f = 0$ for each $g \in Nil\big(R_0[x]\big)$. Therefore we have $ann_{R_0[x]}(f) = \ell.ann_{R_0[x]}(f) = Nil\big(R_0[x]\big)$. Thus $[f] = [f_1 + f_3] = [x^2]$.

Case 6. Let $f = f_2 + f_3$, where $f_i \neq 0$ for each $i \in \{2, 3\}$. Since $\deg(f_3) \geq 2$ and $ann_R(a_i) = Nil(R)$ for each $a_i \in C_{f_2}^*$, then by a similar way as used in Case 5 one can show that $ann_{R_0[x]}(f) = \ell.ann_{R_0[x]}(f) = Nil\big(R_0[x]\big)$. Hence $[f] = [x^2]$.

Case 7. Let $f = f_1 + f_2 + f_3$, where $f_i \neq 0$ for each $i \in \{1, 2, 3\}$. Since $C_{f_3}^* \subseteq R \setminus Z(R)$, then $r.ann_{R_0[x]}(f) = 0$ and $\ell.ann_{R_0[x]}(f) \subseteq Nil\big(R_0[x]\big)$, by Lemma 2.5. Hence $ann_{R_0[x]}(f) = Nil\big(R_0[x]\big)$, and so $[f] = [f_1 + f_2 + f_3] = [x^2]$.

Therefore $|\Gamma_E(R_0[x])| = 2$, and thus $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$. □

Corollary 2.10. *Let R be a non-reduced commutative ring with $Z(R) \neq 0$. If $Z(R)^2 = 0$, then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$.

From [1, Theorems 2.7 and 2.9], we immediately deduce the following result.

Proposition 2.11. *Let R be a non-reduced commutative ring. Then there exist $a, b \in Z(R)$ with $ann_R\big(\{a,b\}\big) \cap Nil(R) = 0$ if and only if* $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$

Lemma 2.12. *Let R be a commutative ring and $a, b \in R$. If*

$$ann_R\big(\{a,b\}\big) \cap Nil(R) = 0,$$

then $ann_R\big(\{a^k, b^s\}\big) \cap Nil(R) = 0$ for each positive integer k, s with $a^k \neq 0 \neq b^s$.

Proof. Let $a^k \neq 0$ for some positive integer k. On the contrary, assume that and $0 \neq t \in ann_R\big(\{a^k, b\}\big) \cap Nil(R)$. Then $ta^k = 0 = tb$. Hence there exists $1 \leq r \leq k-1$ such that $ta^r \neq 0$ but $ta^{r+1} = 0$. Thus $ta^r \in ann_R\big(\{a,b\}\big) \cap Nil(R)$, which is a contradiction. Now suppose $b^s \neq 0$ for some positive integer s. Put $a' = a^k \neq 0$. Hence $ann_R\big(\{a', b\}\big) \cap Nil(R) = 0$, and so by a similar way as used above, $ann_R\big(\{a', b^s\}\big) \cap Nil(R) = 0$, as desired. □

Theorem 2.13. *Let R be a non-reduced commutative ring. Then* $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Proof. ($\Rightarrow$) Let $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$. Then there exist $a, b \in Z(R)$, such that $ann_R\big(\{a,b\}\big) \cap Nil(R) = 0$, by Proposition 2.11. Notice that if a or $b \in Nil(R)$ and $ab = 0$, then $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$, which is a contradiction. Hence we consider the following cases:

Case 1. Let $a, b \notin Nil(R)$. Since $ann_R\big(\{a,b\}\big) \cap Nil(R) = 0$, then either there exists $c \in Nil(R)$ such that $ca = 0$ but $cb \neq 0$ or for each $c \in Nil(R)$, $ca \neq 0 \neq cb$.

First assume $ca = 0$ but $cb \neq 0$ for some $c \in Nil(R)$. There exists a positive integer k such that $c^k = 0$. Hence $ax + x^k, bx \in Z\big(R_0[x]\big)$. Since

$$cx \in ann_{R_0[x]}(ax + x^k)$$

but $cx \notin ann_{R_0[x]}(bx)$, then $[ax+x^k] \neq [bx]$. Also, $bx \circ (ax+x^k) \neq 0 \neq (ax+x^k) \circ bx$. Since for each $0 \neq r \in R$, $r(ax+x^k) \neq 0$, then

$$ann_{R_0[x]}(ax + x^k) = \ell.ann_{R_0[x]}(ax + x^k) \subseteq Nil(R_0[x]),$$

by Lemma 2.5. Suppose that $g = \sum_{i=s}^{n} c_i x^i \in ann_{R_0[x]}(ax + x^k) \cap ann_{R_0[x]}(bx)$ and $c_s \neq 0$. Then $g \circ (ax + x^k) = 0$ and either $g \circ bx = 0$ or $bx \circ g = 0$. Hence $c_i \in Nil(R)$ for each i and $ac_s = 0$. If $g \circ bx = 0$, then $bc_s = 0$, which implies that $c_s \in ann_R\big(\{a,b\}\big) \cap Nil(R)$, a contradiction. If $0 = bx \circ g = c_s b^s x^s + \cdots + c_n b^n x^n$, then $c_s b^s = 0$. Since $b \notin Nil(R)$, then $b^s \neq 0$. Hence $c_s \in ann_R\big(\{a, b^s\}\big) \cap Nil(R)$, which is a contradiction by Lemma 2.12. Thus bx and $ax + x^k$ have not common non-zero annihilator, and so $d([ax+x^k], [bx]) \geq 3$. Therefore $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Now assume for each $c' \in Nil(R)$, $c'a \neq 0 \neq c'b$. Since R is not reduced, there exists $c \in R$ such that $c^2 = 0$. Thus $cb \neq 0$ and $cbx + x^2 \in Z\big(R_0[x]\big)$. Hence $[cbx + x^2] \neq [ax]$, since $cx \in ann_{R_0[x]}(cbx + x^2) \setminus ann_{R_0[x]}(ax)$. Obviously, $(cbx + x^2) \circ ax \neq 0 \neq ax \circ (cbx + x^2)$. By Lemma 2.5, we have

$$ann_{R_0[x]}(cbx + x^2) = \ell.ann_{R_0[x]}(cbx + x^2) \subseteq Nil(R_0[x]).$$

Let $g = \sum_{i=s}^{n} c_i x^i \in ann_{R_0[x]}(cbx + x^2) \cap ann_{R_0[x]}(ax)$ and $c_s \neq 0$. Hence either $g \circ ax = 0$ or $ax \circ g = 0$. If $g \circ ax = 0$, then $ac_s = 0$, which is a contradiction. If $ax \circ g = 0$, then $c_s a^s = 0$. Since $a^s \neq 0$, there exists $1 \leq t \leq s-1$ such that $c_s a^t \neq 0$ but $c_s a^{t+1} = 0$. Hence $c_s a^t \in ann_R(a) \cap Nil(R)$, which is a contradiction. Therefore $d([cbx + x^2], [ax]) \geq 3$, and so $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Case 2. Let $a \in Nil(R)$, $b \notin Nil(R)$ and $ab \neq 0$. Hence there exists a positive integer k such that $a^k = 0$ but $a^{k-1} \neq 0$. Thus $a^{k-1}x + x^k, bx \in Z\big(R_0[x]\big)$. Since $ax \in ann_{R_0[x]}(a^{k-1}x + x^k) \setminus ann_{R_0[x]}(bx)$, then $[bx] \neq [a^{k-1}x + x^k]$. Moreover, $bx \circ (a^{k-1}x + x^k) \neq 0 \neq (a^{k-1}x + x^k) \circ bx$. Let

$$g \in ann_{R_0[x]}(a^{k-1}x + x^k) \cap ann_{R_0[x]}(bx).$$

Hence $g = \sum_{i=s}^{n} c_i x^i$ with $c_s \neq 0$ is nilpotent, since

$$ann_{R_0[x]}(a^{k-1}x + x^k) = \ell.ann_{R_0[x]}(a^{k-1}x + x^k) \subseteq Nil(R_0[x]).$$

From $g \circ (a^{k-1}x + x^k) = 0$ yields $a^{k-1}c_s = 0$. On the other hand, if $g \circ bx = 0$, then $bc_s = 0$. Therefore $0 \neq c_s \in ann_R\big(\{a^{k-1}, b\}\big) \cap Nil(R)$, which is a contradiction by Lemma 2.12. Now assume that $bx \circ g = 0$. Then $c_s b^s = 0$. Since $b \notin Nil(R)$, then $b^s \neq 0$. Thus $0 \neq c_s \in ann_R\big(\{a^{k-1}, b^s\}\big) \cap Nil(R)$, which is a contradiction by Lemma 2.12. Hence $d([a^{k-1}x + x^k], [bx]) \geq 3$, and so the result follows.

Case 3. Let $a, b \in Nil(R)$ and $ab \neq 0$. Then there exist positive integers t, k such that $a^k = b^t = 0$ but $a^{k-1} \neq 0 \neq b^{t-1}$. Therefore

$$a^{k-1}x + x^k, b^{t-1}x + x^t \in Z\big(R_0[x]\big).$$

Notice that $(a^{k-1}x + x^k) \circ (b^{t-1}x + x^t) \neq 0 \neq (b^{t-1}x + x^t) \circ (a^{k-1}x + x^k)$. Moreover,

$$ann_{R_0[x]}(a^{k-1}x + x^k) = \ell.ann_{R_0[x]}(a^{k-1}x + x^k) \subseteq Nil(R_0[x])$$

and $ann_{R_0[x]}(b^{t-1}x + x^t) = \ell.ann_{R_0[x]}(b^{t-1}x + x^t)$. Also, if $ax \in ann_{R_0[x]}(b^{t-1}x + x^t)$, then $ax \circ (b^{t-1}x + x^t) = 0$, and so $a \in ann_R\big(\{a^{k-1}, b^{t-1}\}\big) \cap Nil(R)$, which is a contradiction by Lemma 2.12. Hence

$$ax \in ann_{R_0[x]}(a^{k-1}x + x^k) \setminus ann_{R_0[x]}(b^{t-1}x + x^t),$$

and so $[a^{k-1}x + x^k] \neq [b^{t-1}x + x^t]$. Let

$$g = \textstyle\sum_{i=s}^{n} c_i x^i \in ann_{R_0[x]}(a^{k-1}x + x^k) \cap ann_{R_0[x]}(b^{t-1}x + x^t),\ c_s \neq 0.$$

Hence $g \circ (a^{k-1}x + x^k) = 0 = g \circ (b^{t-1}x + x^t)$. Therefore

$$0 \neq c_s \in ann_R\big(\{a^{k-1}, b^{t-1}\}\big) \cap Nil(R),$$

which is a contradiction by Lemma 2.12. Hence $d([a^{k-1}x + x^k], [b^{t-1}x + x^t]) \geq 3$, as wanted.

($\Leftarrow$) Let $\text{diam}\big(\Gamma_E(R_0[x])\big) = 3$. Since $\text{diam}\big(\Gamma_E(R_0[x])\big) \leq \text{diam}\big(\Gamma(R_0[x])\big) \leq 3$, then the result follows. □

By using Theorems 2.9 and 2.13, we can determine when $\text{diam}\big(\Gamma_E(R_0[x])\big) = 2$.

Theorem 2.14. *Let R be a non-reduced commutative ring with $Z(R) \neq 0$. Then $\text{diam}\big(\Gamma_E(R_0[x])\big) = 2$ if and only if $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$ and one of the following conditions holds:*

(1) $\mid \Gamma_E(R) \mid \geq 3$.

(2) $Z(R) \neq ann_R(c)$ for each $c \in R$.

(3) $Nil(R)^2 \neq 0$.

(4) There exists $0 \neq c \in Z(R) \setminus Nil(R)$ such that $ann_R(c) \neq Nil(R)$.

Proof. ($\Rightarrow$) By Theorem 2.13, we have $\text{diam}\big(\Gamma(R_0[x])\big) = 2$. It follows that $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$, by [1, Theorem 2.9], Since $\text{diam}\big(\Gamma_E(R_0[x])\big) = 2$, then the result follows from Theorem 2.9.

($\Leftarrow$) Since $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$, we have $\text{diam}\big(\Gamma(R_0[x])\big) = 2$, by [1, Theorem 2.9]. Hence $\text{diam}\big(\Gamma_E(R_0[x])\big) \in \{1, 2\}$, since $\text{diam}\big(\Gamma_E(R_0[x])\big) \leq \text{diam}\big(\Gamma(R_0[x])\big)$. On the other hand, if one of the conditions $(1) - (4)$ holds, then $\text{diam}\big(\Gamma_E(R_0[x])\big) \neq 1$, by Theorem 2.9, and so the result follows. □

3. On the Diameter of the Compressed Zero-divisor Graph of $R_0[[x]]$

We denote the collection of all power series with positive orders using the operations of addition and substitution by $R_0[[x]]$, unless specifically indicated otherwise (i.e., $R_0[[x]]$ denotes $(R_0[[x]], +, \circ)$). Observe that the system $(R_0[[x]], +, \circ)$ is a zero-symmetric left near-ring. For any $f \in R_0[[x]]$, we denote by C_f the set of all coefficients of f. Also, the set of all non-zero coefficients of f is denoted by $C_f^* = C_f \setminus \{0\}$.

In this section, we characterize the diameter of the compressed zero-divisor graph of the near-ring $R_0[[x]]$.

Lemma 3.1. *Let R be a reduced ring. Then*

(1) [13, Proposition 2.3] For each $f, g \in R[[x]]$, $fg = 0$ if and only if $a_i b_j = 0$ for each $a_i \in C_f$ and $b_j \in C_g$.

(2) [6, Lemma 3.3] For each $f, g \in R_0[[x]]$, $f \circ g = 0$ if and only if $a_i b_j = 0$ for each $a_i \in C_f$ and $b_j \in C_g$.

By using Lemma 3.1 and a similar argument as used in the proof of Proposition 2.3, we can conclude the following nice fact.

Proposition 3.2. *Let R be a reduced ring. Then*

(1) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 1$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 1$.

(2) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 2$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 2$.

(3) $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$.

Let R be a commutative ring. For polynomials, McCoy's Theorem [19, Theorem 2] states that a polynomial $f \in R[x]$ is a zero-divisor if and only if there is a non-zero element $r \in R$ such that $rf = 0$. Based on this theorem, a ring R is said to be *McCoy ring* if each finitely generated ideal contained in $Z(R)$ has a non-zero annihilator [9].

Corollary 3.3. *Let R be a reduced commutative ring. Then* $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$ *if and only if* $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$.

Proof. ($\Rightarrow$) Let $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$. Then $\operatorname{diam}\big(\Gamma(R[[x]])\big) = 3$, by Lemma 3.1. Thus by [17, Theorem 4.9], one of the following cases occurs:

Case 1. R is a McCoy ring with $Z(R)$ an ideal but there exist countably generated ideals I and J with non-zero annihilators such that $I + J$ does not have a non-zero annihilator. Since $Z(R)$ is an ideal, then R has more than two minimal primes. Therefore $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$, by [12, Theorem 4.3].

Case 2. $Z(R)$ is an ideal and each two generated ideal contained in $Z(R)$ has a non-zero annihilator but R is not a McCoy ring. Then R has more than two minimal primes and there exists $K = \langle a_1, \ldots, a_n\rangle \subseteq Z(R)$ with $ann_R(K) = 0$, since R is not McCoy. Hence $n \geq 3$. Therefore one can easily show that there exist finitely generated ideals I and J with non-zero annihilators such that $I + J$ does not have a non-zero annihilator. Hence $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$, by [12, Theorem 4.3].

Case 3. R has more than two minimal primes and there is a pair of zero-divisors a and b such that $\langle a\rangle + \langle b\rangle = \langle a, b\rangle$ does not have a non-zero annihilator. Then $\operatorname{diam}\big(\Gamma_E(R[[x]])\big) = 3$, by [12, Theorem 4.3].

Therefore $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$, by Proposition 3.2.

The backward direction is clear. □

Corollary 3.4. *Let R be a reduced commutative ring. If* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$*, then* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$.

Proof. Let $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$. Then $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$, by Corollary 2.4. Thus $\operatorname{diam}\big(\Gamma(R[x])\big) = 3$, by [1, Proposition 2.10], and so $\operatorname{diam}\big(\Gamma(R[[x]])\big) = 3$, by [17, Theorem 4.9]. Hence $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$, by Lemma 3.1. Therefore the result follows from Corollary 3.3. □

Proposition 3.5. *Let R be a reduced commutative ring. Then*

$$\operatorname{diam}\big(\Gamma_E(R)\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[x])\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[[x]])\big).$$

Proof. Clearly, if $\operatorname{diam}\big(\Gamma_E(R)\big) = 0$, then we have $\operatorname{diam}\big(\Gamma_E(R)\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[x])\big)$. Also, $\operatorname{diam}\big(\Gamma_E(R)\big) = 1$ if and only if $\operatorname{diam}\big(\Gamma_E(R[x])\big) = 1$ if and only if $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$, by [12, Theorem 3.3] and Proposition 2.3. Therefore if

$\operatorname{diam}(\Gamma_E(R)) = 2$, then $\operatorname{diam}(\Gamma_E(R_0[x])) \geq 2$. Finally, if $\operatorname{diam}(\Gamma_E(R)) = 3$, then $\operatorname{diam}(\Gamma_E(R[x])) = 3$, by [12, Theorem 4.4]. Hence $\operatorname{diam}(\Gamma_E(R_0[x])) = 3$, by Proposition 2.3.

Obviously, $\operatorname{diam}(\Gamma_E(R_0[x])) \leq \operatorname{diam}(\Gamma_E(R_0[[x]]))$, if $\operatorname{diam}(\Gamma_E(R_0[x])) = 1$. Now assume that $\operatorname{diam}(\Gamma_E(R_0[x])) = 2$. Then there exist $f, g \in Z(R_0[x])$ with $d([f]_{R_0[x]}, [g]_{R_0[x]}) = 2$. On the contrary, suppose that $\operatorname{diam}(\Gamma_E(R_0[[x]])) = 1$. Since $d([f]_{R_0[x]}, [g]_{R_0[x]}) = 2$, we have $f \circ g \neq 0$. Therefore $[f]_{R_0[[x]]} = [g]_{R_0[[x]]}$, which implies that $[f]_{R_0[x]} = R_0[x] \cap [f]_{R_0[[x]]} = R_0[x] \cap [g]_{R_0[[x]]} = [g]_{R_0[x]}$, a contradiction. Hence $\operatorname{diam}(\Gamma_E(R_0[x])) \leq \operatorname{diam}(\Gamma_E(R_0[[x]]))$, by Corollary 3.4. □

The following lemmas play an important role in proving Theorem 3.10.

Lemma 3.6. ([10, Corollary 1]) *Let R be a commutative Noetherian ring. Then $Nil(R[[x]]) = Nil(R)[[x]]$.*

For each $f \in R_0[x]$ and positive integer n, we write

$$f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

Lemma 3.7. *Let R be a commutative Noetherian ring. Then*

$$Nil(R_0[[x]]) = Nil(R)_0[[x]].$$

Proof. First, Suppose that $f = \sum_{r=1}^{\infty} a_r x^r \in Nil(R_0[[x]])$. Then there exists a positive integer n such that $f^{(n)} = 0$. We show that for each $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in C_f$, we have $a_{i_1} a_{i_2} \cdots a_{i_n} \in Nil(R)$, which implies that $a_r \in Nil(R)$ for each $a_r \in C_f$, as wanted. We use induction on n. Assume that $n = 2$ and $\overline{R} = R/Nil(R)$. Since $0 = f \circ f \in Nil(R)_0[[x]]$, then $\overline{f} \circ \overline{f} = \overline{0}$ in $\overline{R}_0[[x]]$. By Lemma 3.1, we have $\overline{a}_i \overline{a}_j = \overline{0}$ for each $\overline{a}_i, \overline{a}_j \in C_{\overline{f}}$, since $\overline{R}$ is a reduced ring. Thus $a_i a_j \in Nil(R)$ for each i, j. Now suppose that $n > 2$. Let $g = f^{(n-1)}$. Thus $f \circ g \in Nil(R)_0[[x]]$. By a similar argument as used above, we have $a_r a_g \in Nil(R)$, where $a_g \in C_g$ and $a_r \in C_f$. Therefore for each $a_{i_1} \in C_f$,

$$g \circ a_{i_1} x = f^{(n-1)} \circ a_{i_1} x = f^{(n-2)} \circ (f \circ a_{i_1} x) = f^{(n-2)} \circ (a_{i_1} f) \in Nil(R)_0[[x]].$$

By induction, we have $a_{i_2} a_{i_3} \cdots a_{i_1} a_{i_n} \in Nil(R)$, where $a_{i_j} \in C_f$ for each j and the coefficients of $a_{i_1} f$ are $a_{i_1} a_{i_n}$. Therefore $a_r \in Nil(R)$ for each $a_r \in C_f$.

Now assume that $f \in Nil(R)_0[[x]]$. Since R is Noetherian, there exists a positive integer k such that $Nil(R)^k = 0$. It follows that $C_f^k = 0$. Since for each $n \geq 1$, the coefficient of x^n in $f^{(k)}$ is a sum of such elements $a_{i_1} a_{i_2} \cdots a_{i_l}$, where $a_{i_j} \in C_f$ and $l \geq k$, then we have $f^{(k)} = 0$. Hence $f \in Nil(R_0[[x]])$. □

Lemma 3.8. *Let R be a commutative ring. If $f = \sum_{i=1}^{\infty} a_i x^i$ is a zero-divisor of $R_0[[x]]$, then $a_1 \in Z(R)$.*

Proof. Let $a_1 \neq 0$. Since $f \in Z(R_0[[x]])$, then there exists $g = \sum_{i=1}^{\infty} b_i x^i \in R_0[[x]]$

such that $f \circ g = 0$ or $g \circ f = 0$. Let b_k be the first non-zero coefficient of g. Assume that $f \circ g = 0$. Then $b_k a_1^k = 0$. Hence there exists $1 \leq t \leq k-1$ such that $b_k a_1^t \neq 0$ but $b_k a_1^{t+1} = 0$, which implies that $a_1 \in Z(R)$. On the other hand, if $g \circ f = 0$, then $a_1 b_k = 0$, and so the result follows. □

Lemma 3.9. *Let R be a Noetherian commutative ring and $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j$ be non-zero elements of the near-ring $R_0[[x]]$. If $f \circ g = 0$, then*

(1) $rf = 0$ for some non-zero $r \in R$.

(2) f is nilpotent or $sg = 0$ for some non-zero $s \in R$.

Proof. (1) Let b_k be the first non-zero coefficient of g. Since $f \circ g = 0$, we have $b_k f^k + b_{k+1} f^{k+1} + \cdots = 0$. Hence $\big(b_k + b_{k+1} f + \cdots\big) f^k = 0$. If $f^k = 0$, then there exists $1 \leq t \leq k-1$ such that $f^t \neq 0 = f^{t+1}$. Therefore $rf = 0$ for some $0 \neq r \in R$, by McCoy's Theorem. Thus assume that $f^k \neq 0$. Since $0 \neq b_k + b_{k+1} f + \cdots$, then the result follows by McCoy's Theorem.

(2) Notice that $\langle C_g \rangle = \langle b_1, \ldots, b_n \rangle$ for some $n \geq 1$, since R is Noetherian. Suppose that f is not nilpotent. Thus there exists $a = a_i$ such that $a \notin Nil(R)$, by Lemma 3.6. Let $\overline{R} = R/Nil(R)$. Since $f \circ g = 0$, then $\overline{f} \circ \overline{g} = \overline{0}$ in the near-ring $\overline{R}_0[[x]]$. Since $\overline{R}$ is a reduced ring, it follows that $\overline{a}_i \overline{b}_j = \overline{0}$, by Lemma 3.1. Since R is Noetherian, then $Nil(R)$ is nilpotent, and so $Nil(R)^k = 0$ for some positive integer k. Thus $a^k b_j^k = 0$ for each $j \geq 1$. Hence there exist integers $0 \leq t_j \leq k$ such that $a^k b_j^{t_j} \neq 0$ but $a^k b_j^{t_j+1} = 0$ for each $j \geq 1$. Therefore there exist integers $0 \leq s_j \leq t_j$ such that $a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} \neq 0$ but $a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} b_j = 0$ for each $1 \leq j \leq n$. Let $s = a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n}$. Thus $sg = 0$, since $\langle C_g \rangle = \langle b_1, \ldots, b_n \rangle$. □

Theorem 3.10. *Let R be a non-reduced commutative ring. Then*

(1) If R is Noetherian and $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$, then $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 1$.

(2) If $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 1$, then $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$.

Proof. (1) Let $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 1$. Then $|\Gamma_E(R)| \leq 2$, $Z(R) = ann_R(a)$ for some $a \in R$, $Nil(R)^2 = 0$, and $ann_R(c) = Nil(R)$ for each $c \in Z(R) \setminus Nil(R)$, by Theorem 2.9. As shown in the proof of Theorem 2.9, for each $c \in Nil(R)$, $ann_R(c) = Z(R)$. Assume $c \in Nil(R)$ and $g = \sum_{j=1}^{\infty} b_j x^j \in Z\big(R_0[[x]]\big)$. Since $Nil(R)^2 = 0$, then $cx \circ g = 0$, by Lemma 3.8. Thus $ann_{R_0[[x]]}(cx) = Z\big(R_0[[x]]\big)$. It is clear that $r.ann_{R_0[[x]]}(x^2) = 0$. Also, we have $\ell.ann_{R_0[[x]]}(x^2) \subseteq Nil(R)_0[[x]]$, by Lemmas 3.7 and 3.9. Hence $ann_{R_0[[x]]}(x^2) = \ell.ann_{R_0[[x]]}(x^2) = Nil\big(R_0[[x]]\big)$, since $Nil(R)^2 = 0$ and $Nil\big(R_0[[x]]\big) = Nil(R)_0[[x]]$. Notice that $[cx] \neq [x^2]$, since $x^2 \in ann_{R_0[[x]]}(cx) \setminus ann_{R_0[[x]]}(x^2)$. Now suppose that f be a non-zero element of $Z\big(R_0[[x]]\big)$. We can write $f = f_1 + f_2 + f_3$ such that $C^*_{f_1} \subseteq Nil(R)$, $C^*_{f_2} \subseteq Z(R) \setminus Nil(R)$, and $C^*_{f_3} \subseteq R \setminus Z(R)$.

Assume $f = f_1 = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j \in Z\big(R_0[[x]]\big)$. Hence we have $ann_R(a_i) = Z(R)$ for each $a_i \in C^*_f$, since $C^*_f \subseteq Nil(R)$. Thus $f \circ g = 0$, since

$Nil(R)^2 = 0$ and $b_1 \in Z(R)$, by Lemma 3.8. Therefore $ann_{R_0[[x]]}(f) = Z\big(R_0[[x]]\big)$, which implies that $[f] = [cx]$.

Suppose that $f = f_2 = \sum_{i=q}^{\infty} a_i x^i$ and $a_q \neq 0$. Since $C_f^* \subseteq Z(R) \setminus Nil(R)$, then $ann_R(a_i) = Nil(R)$ for each $a_i \in C_f^*$. Hence for each $g \in Nil(R)_0[[x]]$, $f \circ g = 0$ and $g \circ f = 0$. Let $g = \sum_{j=1}^{\infty} b_j x^j \in r.ann_{R_0[[x]]}(f)$. Thus $f \circ g = \sum_{j=1}^{\infty} b_j f^j = 0$. Assume that b_t is the first non-zero coefficient of g. Then $b_t \in ann_R(a_q^t) = Nil(R)$, since $b_t a_q^t = 0$ and $a_q^t \notin Nil(R)$. Hence $b_t f = 0$, and so $f \circ g = \sum_{j=t+1}^{\infty} b_j f^j = 0$. By repeating this argument, one can deduce that $b_j \in Nil(R)$ for each $b_j \in C_g^*$. Thus $g \in Nil(R)_0[[x]]$, and so $r.ann_{R_0[[x]]}(f) = Nil(R)_0[[x]]$.

Now suppose that $g = \sum_{j=t}^{\infty} b_j x^j \in \ell.ann_{R_0[[x]]}(f)$, where $b_t \neq 0$. Therefore

$$g \circ f = \textstyle\sum_{i=q}^{\infty} a_i g^i = 0,$$

which implies that $a_q b_t^q = 0$. Hence $b_t^q \in ann_R(a_q) = Nil(R)$, and so $b_t \in Nil(R)$. Then $b_t a_i = 0$ for each $a_i \in C_f^*$, and thus $g \circ f = \sum_{i=q}^{\infty} a_i g_1^i = 0$, where $g_1 = \sum_{j=t+1}^{\infty} b_j x^j$. Continuing this process one can show that $b_j \in Nil(R)$ for each $b_j \in C_g^*$, and so $\ell.ann_{R_0[[x]]}(f) \subseteq Nil(R)_0[[x]]$. Hence $ann_{R_0[[x]]}(f) = Nil(R)_0[[x]]$. Therefore $[f] = [f_2] = [x^2]$.

If $f = f_3$ or $f = f_1 + f_2$ ($f_1 \neq 0 \neq f_2$) or $f = f_1 + f_3$ ($f_1 \neq 0 \neq f_3$) or $f = f_2 + f_3$ ($f_2 \neq 0 \neq f_3$) or $f = f_1 + f_2 + f_3$ (each f_i be non-zero), then by using Lemmas 3.7, 3.9 and a similar argument as used in the proof of Theorem 2.9, one can show that $[f] = [x^2] = Nil(R)_0[[x]]$. Hence $|\Gamma_E(R_0[[x]])| = 2$, and thus $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 1$.

(2) It is clear. □

Proposition 3.11. *Let R be a non-reduced commutative ring. Then*

(1) If $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$, then $ann_R\big(\{a,b\}\big) \cap Nil(R) = 0$ for some $a, b \in Z(R)$.

(2) Let R be a Noetherian ring. If $ann_R\big(\{a,b\}\big) \cap Nil(R) = 0$ for some $a, b \in Z(R)$, then $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$.

Proof. (1) Since $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$, then there exist $f, g \in R_0[[x]]$ such that $d(f,g) = 3$. Let a_t and b_q be the first non-zero coefficients of f and g, respectively. On the contrary, suppose that $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$. By Lemma 3.8, we have $a_t, b_q \in Z(R)$. Hence there exists $c \in Nil(R)$ such that $ca_t = b_q c = 0$. Let $c^r = 0 \neq c^{r-1}$ for some positive integer r. Therefore $f - c^{r-1}x - g$ is a path in $\Gamma(R_0[[x]])$, which is a contradiction.

(2) Since R is non-reduced, there exists $c \in R$ such that $c^2 = 0$. It follows that $x^2, x^3 \in Z(R_0[[x]])$ and $x^2 \circ x^3 \neq 0 \neq x^3 \circ x^2$. Thus $d(x^2, x^3) \geq 2$, and so $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) \geq 2$. On the contrary, suppose that $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) \neq 3$. Therefore $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 2$, by [8, Theorem 2.2]. Let $a, b \in Z(R)$. We show that $ax + x^2, bx + x^2 \in Z(R_0[[x]])$. If $a^{k-1} \neq 0 = a^k$ for some positive integer k, then $a^{k-1}x \circ (ax + x^2) = 0$. Thus assume that $a \notin Nil(R)$. Since $ax, x^2 \in Z(R_0[[x]])$ and

$ax \circ x^2 \neq 0 \neq x^2 \circ ax$, then there exists a non-zero nilpotent element $f = \sum_{i=r}^{\infty} c_i x^i$ with $c_r \neq 0$ such that $ax - f - x^2$ is a path. If $f \circ ax = 0$, then $ac_r = 0$. By Lemma 3.6, we have $c_r^{k-1} \neq 0 = c_r^k$ for some positive integer k. Therefore $c_r^{k-1} x \circ (ax + x^2) = 0$. If $ax \circ f = 0$, then $c_r a^r = 0$. Hence there exists $1 \leq t \leq r-1$ such that $c_r a^t \neq 0 = c_r a^{t+1}$, and so $c_r a^t x \circ (ax + x^2) = 0$. Similarly, we have $bx + x^2 \in Z(R_0[[x]])$. Since $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 2$ and

$$(ax + x^2) \circ (bx + x^2) \neq 0 \neq (bx + x^2) \circ (ax + x^2),$$

then $g \circ (ax+x^2) = 0 = g \circ (bx+x^2)$ for some non-zero nilpotent element g, by Lemma 3.9. Let s be the first non-zero coefficient of g. Therefore $s \in ann_R(\{a, b\}) \cap Nil(R)$, which is a contradiction. □

Corollary 3.12. *Let R be a non-reduced commutative ring. Then*

(1) If R is Noetherian and $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$.

(2) If $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma(R_0[x])\big) = 3$.

Proof. It follows from Propositions 2.11 and 3.11. □

Theorem 3.13. *Let R be a non-reduced commutative ring. Then*

(1) If R is Noetherian and $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$.

(2) If $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma(R_0[[x]])\big) = 3$.

Proof. (1) By using Lemmas 3.7, 3.9, Proposition 3.11 and a similar argument as used in the proof of Theorem 2.13, one can prove it.

(2) It is clear. □

Corollary 3.14. *Let R be a non-reduced commutative ring. Then*

(1) If R is Noetherian and $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$.

(2) If $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 3$, *then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 3$.

Proof. It follows from Theorems 2.13, 3.13 and Corollary 3.12. □

Proposition 3.15. *Let R be a non-reduced commutative ring. Then*

(1) If $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 2$, *then* $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 2$.

(2) If R is Noetherian and $\operatorname{diam}\big(\Gamma_E(R_0[[x]])\big) = 2$, *then* $\operatorname{diam}\big(\Gamma_E(R_0[x])\big) = 2$

Proof. This follows from Theorem 3.10 and Corollary 3.14. □

Proposition 3.16. *Let R be a non-reduced Noetherian commutative ring. If $Z(R) \neq ann_R(a)$ for each $a \in R$, then*

$$\operatorname{diam}\big(\Gamma_E(R)\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[x])\big) \leq \operatorname{diam}\big(\Gamma_E(R_0[[x]])\big).$$

Proof. Clearly, $\mathrm{diam}\big(\Gamma_E(R)\big) \leq \mathrm{diam}\big(\Gamma_E(R_0[x])\big)$, if $\mathrm{diam}\big(\Gamma_E(R)\big) \in \{0,1\}$. Hence suppose that $\mathrm{diam}\big(\Gamma_E(R)\big) = 2$. Then $\mid \Gamma_E(R)\big) \mid\geq 3$, which implies that $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \geq 2$, by Theorem 2.9.

Now assume that $\mathrm{diam}\big(\Gamma_E(R)\big) = 3$. Notice that $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \geq 2$, by Theorem 2.9. On the contrary, suppose that $\mathrm{diam}\big(\Gamma_E(R_0[x])\big) = 2$. Thus $Z(R)$ is an ideal and each pair of zero-divisors has a non-zero annihilator, by Theorem 2.14. Since $Z(R) \neq ann_R(a)$ for every $a \in Z(R)$, then $\mathrm{diam}\big(\Gamma_E(R)\big) = 2$, by [12, Theorem 2.3], which is a contradiction. Hence $\mathrm{diam}\big(\Gamma_E(R)\big) \leq \mathrm{diam}\big(\Gamma_E(R_0[x])\big)$.

Also, by Corollary 3.14 and Proposition 3.15, we have

$$\mathrm{diam}\big(\Gamma_E(R_0[x])\big) \leq \mathrm{diam}\big(\Gamma_E(R_0[[x]])\big).$$

□

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