KYUNGPOOK Math. J. 62(2022), 467-484
https://doi.org/10.5666/KMJ.2022.62.3.467
pISSN 1225-6951 eISSN 0454-8124

Some Approximation Results by Bivariate Bernstein-Kantorovich Type Operators on a Triangular Domain

Reşat Aslan* and Aydin Izgi
Department of Mathematics, Faculty of Sciences and Arts, Harran University, 63100 Şanlıurfa, Turkey
e-mail : resat63@hotmail.com *and* a_izgi@harran.edu.tr

Abstract. In this work, we define bivariate Bernstein-Kantorovich type operators on a triangular domain and obtain some approximation results for these operators. We start off by computing some moment estimates and prove a Korovkin type convergence theorem. Then, we estimate the rate of convergence using the partial and complete modulus of continuity, and derive a Voronovskaya-type asymptotic theorem. Further, we calculate the order of approximation with regard to the Peetre's K-functional and a Lipschitz type class. In addition, we construct the associated GBS type operators and compute the rate of approximation using the mixed modulus of continuity and class of the Lipschitz of Bögel continuous functions for these operators. Finally, we use the two operators to approximate example functions in order to compare their convergence.

1. Introduction

In [28], Weierstrass showed in his famous approximation theorem that any continuous function f on a compact set can be approximated uniformly by a polynomial sequence p_n. Since the proof of Weierstrass's approximation theorem is very long and complex, many authors have subsequently worked on the proof of this theorem, but the most elegant one was presented by Bernstein [6]. In 1930, Kantorovich [18] introduced approximations for Lebesgue integrable functions. In [19], Kingsley introduced and studied the Bernstein polynomials in the bivariate case of the class $C^{(k)}$. Stancu [26] obtained a new method for dealing with Bernstein operators for two variables. Very recently, Pop and Fărcaş [22] investigated several approximation properties of bivariate Kantorovich type operators. Kajla and Goyal [17] obtained direct results for the modified Bernstein–Kantorovich operators and

* Corresponding Author.
Received February 3, 2021; revised January 17, 2022; accepted January 24, 2022.
2020 Mathematics Subject Classification: 41A10, 41A25, 41A36.
Key words and phrases: Bernstein-Kantorovich operators, Modulus of continuity, Voronovskaya-type asymptotic theorem, Peetre's K-functional, GBS type operators.

considered the bivariate case of these operators. Moreover, Deshwal et al. [11] proposed and studied bivariate operators of Bernstein-Kantorovich type on a triangle. In [16], Kajla constructed generalized Bernstein-Kantorovich–type operators on a triangle and presented Voronovskaja-type and Grüss Voronovskaja-type asymptotic theorems; he estimated of the rate of approximation using Peetre's K-functional. In 2017, Goyal et al. [15] considered a bivariate extension of the Bernstein-Durrmeyer type operators on a triangle domain. Başcanbaz-Tunca et al. [6] considered bivariate Cheney-Sharma operators which preserve the Lipschitz condition. Agrawal et al. [1] discussed the deferred weighted A-statistical approximation and investigated the convergence estimates for the functions in a Bögel space by Bernstein-Kantorovich type operators on a triangle. Local and global approximation results in terms of modulus of continuity, Peetre's K-functional, second-order modulus of smoothness and statistical convergence for certain Bernstein-Kantorovich, bivariate Bernstein-Kantorovich and Bernstein-Stancu operators are studied in very recent papers [2, 20, 25].

Now, let $\nabla := \{(x,y) : -1 \le x, y \le 1, x+y \le 0\}$ be a triangular domain and let $C(\nabla)$ be the set of all real functions h that are continuous on ∇ and bounded on $\mathbb{R} \times \mathbb{R}$. The norm on $C(\nabla)$ is

$$\|h\| = \sup_{(x,y)\in\nabla} |h(x,y)|.$$

Inspired by the works mentioned above, we construct bivariate Bernstein-Kantorovich type operators for $(x,y) \in \nabla$ and $h : \nabla \to \mathbb{R}$ as follows:

$$R_n(h;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} \nu_{n,k,l}(x,y) \int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1} \int_{2\frac{l}{n+1}-1}^{2\frac{l+1}{n+1}-1} h(s,t)\,dsdt, \tag{1.1}$$

where $\nu_{n,k,l}(x,y) = \left(\frac{n+1}{2}\right)^2 \binom{n}{k}\binom{n-k}{l} \left(\frac{1+x}{2}\right)^k \left(\frac{1+y}{2}\right)^l \left(1 - \frac{1+x}{2} - \frac{1+y}{2}\right)^{n-k-l}$.

The structure of this work is planned as follows. In Section 2, we give some auxiliary results, such as computing moment estimates and proving a Korovkin's type approximation using Volkov's theorem [27]. In Section 3, we investigate the rate of approximation with regard to the partial and complete modulus of continuity and derive a Voronovskaya-type asymptotic theorem. In Section 4, we discuss the rate of convergence in terms of the Peetre's K-functional and Lipschitz type class. In Section 5, we construct the GBS type of newly defined operators and estimate the order of convergence in terms of the mixed modulus of smoothness and the Lipschitz class of Bögel-continuous function. Finally, we give a comparison of the convergence of the newly defined operators and their associated GBS operators with example computations.

2. Basic Results

Lemma 2.1. *Let $r_{u,v} : \nabla \to \mathbb{R}, r_{u,v}(s,t) = s^u t^v$. Then, for any $(x, y) \in \nabla$ and $0 \le u, v \le 4$, the operators given by (1.1) satisfy the following equalities:*

$$(i)\ R_n(r_{0,0}; x, y) = 1,$$

$$(ii)\ R_n(r_{1,0}; x, y) = x\left(1 - \frac{1}{n+1}\right),$$

$$(iii)\ R_n(r_{0,1}; x, y) = y\left(1 - \frac{1}{n+1}\right),$$

$$(iv)\ R_n(r_{1,1}; x, y) = xy\left(1 - \frac{3n+1}{(n+1)^2}\right) - (x+y+1)\left(\frac{n}{(n+1)^2}\right),$$

$$(v)\ R_n(r_{2,0}; x, y) = x^2\left(1 - \frac{3n+1}{(n+1)^2}\right) + \frac{n+\frac{1}{3}}{(n+1)^2},$$

$$(vi)\ R_n(r_{0,2}; x, y) = y^2\left(1 - \frac{3n+1}{(n+1)^2}\right) + \frac{n+\frac{1}{3}}{(n+1)^2},$$

$$(vii)\ R_n(r_{3,0}; x, y) = x^3\left(1 - \frac{6n^2+n+1}{(n+1)^3}\right) + x\left(\frac{3n^2-n}{(n+1)^3}\right),$$

$$(viii)\ R_n(r_{0,3}; x, y) = y^3\left(1 - \frac{6n^2+n+1}{(n+1)^3}\right) + y\left(\frac{3n^2-n}{(n+1)^3}\right),$$

$$(ix)\ R_n(r_{4,0}; x, y) = x^4\left(1 - \frac{10n^3-5n^2+10n+1}{(n+1)^4}\right) + 6x^2\left(\frac{n^3+10n^2-5n}{(n+1)^4}\right) + \left(\frac{39n^2-36n+\frac{1}{5}}{(n+1)^4}\right),$$

$$(x)\ R_n(r_{0,4}; x, y) = y^4\left(1 - \frac{10n^3-5n^2+10n+1}{(n+1)^4}\right) + 6y^2\left(\frac{n^3+10n^2-5n}{(n+1)^4}\right) + \left(\frac{39n^2-36n+\frac{1}{5}}{(n+1)^4}\right).$$

Lemma 2.2. *Let $k_{u,v} : \nabla \to \mathbb{R}$, $k_{u,v} = (s-x)^u (t-y)^v$. Then, for any $(x, y) \in \nabla$ and $0 \le u, v \le 4$, we have the following central moments:*

$$(i)\ R_n(k_{0,0}; x, y) = 1,$$

$$(ii)\ R_n(k_{1,0}; x, y) = -\frac{x}{n+1},$$

$$(iii)\ R_n(k_{0,1};x,y) = -\frac{y}{n+1},$$

$$(iv)\ R_n(k_{1,1};x,y) = \frac{xy(1-n)}{(n+1)^2} - (x+y+1)\left(\frac{n}{(n+1)^2}\right),$$

$$(v)\ R_n(k_{2,0};x,y) = \frac{x^2(1-n)+n+\frac{1}{3}}{(n+1)^2},$$

$$(vi)\ R_n(k_{0,2};x,y) = \frac{y^2(1-n)+n+\frac{1}{3}}{(n+1)^2},$$

$$(vii)\ R_n(k_{4,0};x,y) = x^4\left(\frac{3n^2-20n+1}{(n+1)^4}\right) + 2x^2\left(\frac{33n^2-8n+1}{(n+1)^4}\right) + \left(\frac{39n^2-36n^2+\frac{1}{5}}{(n+1)^4}\right),$$

$$(viii)\ R_n(k_{0,4};x,y) = y^4\left(\frac{3n^2-20n+1}{(n+1)^4}\right) + 2y^2\left(\frac{33n^2-8n+1}{(n+1)^4}\right) + \left(\frac{39n^2-36n^2+\frac{1}{5}}{(n+1)^4}\right).$$

Lemma 2.3. *For the operators given by (1.1), we have the following relations:*

$$(i)\ \lim_{n\to\infty} nR_n((s-x);x,y) = -x,$$

$$(ii)\ \lim_{n\to\infty} nR_n((t-y);x,y) = -y,$$

$$(iii)\ \lim_{n\to\infty} nR_n((s-x)^2;x,y) = 1-x^2,$$

$$(iv)\ \lim_{n\to\infty} nR_n((t-y)^2;x,y) = 1-y^2,$$

$$(v)\ \lim_{n\to\infty} nR_n((s-x)(t-y);x,y) = -(xy+x+y+1),$$

$$(vi)\ \lim_{n\to\infty} n^2R_n((s-x)^4;x,y) = 3\left(x^4+22x^2+13\right),$$

$$(vii)\ \lim_{n\to\infty} n^2R_n((t-y)^4;x,y) = 3\left(y^4+22y^2+13\right).$$

Theorem 2.4. *If $h(x,y)\in C(\nabla)$, then operators given by (1.1) convergence uniformly to h on ∇ as $n\to\infty$.*

Proof. As a consequence of [27], we have to show operators given by (1.1) verifies that:

$$(i) \lim_{n\to\infty} \|R_n(r_{0,0}) - 1\|_{C(\nabla)} \to 0,$$
$$(ii) \lim_{n\to\infty} \|R_n(r_{1,0}) - x\|_{C(\nabla)} \to 0,$$
$$(iii) \lim_{n\to\infty} \|R_n(r_{0,1}) - y\|_{C(\nabla)} \to 0,$$
$$(iv) \lim_{n\to\infty} \left\|R_n(r_{2,0} + r_{0,2}) - (x^2 + y^2)\right\|_{C(\nabla)} \to 0.$$

From Lemma 2.1. (i), it is obvious that

$$(i) \lim_{n\to\infty} \|R_n(r_{0,0}) - 1\|_{C(\nabla)} \to 0.$$

In view of Lemma 2.1. $(ii) - (iii)$, we get

$$(ii) \lim_{n\to\infty} \|R_n(r_{1,0}) - x\|_{C(\nabla)} = \lim_{n\to\infty} \max_{-1\le x\le 1} |R_n(r_{1,0}) - x|$$
$$= \lim_{n\to\infty} \max_{-1\le x\le 1} \left|-\frac{x}{n+1}\right| \le \lim_{n\to\infty} \frac{1}{n+1} \to 0.$$

Similarly, we obtain

$$(iii) \lim_{n\to\infty} \|R_n(r_{0,1}) - y\|_{C(\nabla)} \to 0.$$

Also, from Lemma 2.1. (iv), we have

$$\begin{aligned}
&(iv) \lim_{n\to\infty} \left\|R_n(r_{2,0} + r_{0,2}) - (x^2 + y^2)\right\|_{C(\nabla)} \\
&= \lim_{n\to\infty} \max_{-1\le x,y\le 1} \left|R_n(r_{2,0} + r_{0,2}) - (x^2 + y^2)\right| \\
&= \lim_{n\to\infty} \max_{-1\le x,y\le 1} \left|(x^2 + y^2)\left(1 - \frac{3n+1}{(n+1)^2}\right) + \frac{2n + \frac{2}{3}}{(n+1)^2} - (x^2 + y^2)\right| \\
&\le \lim_{n\to\infty} \frac{8}{n+1} \to 0,
\end{aligned}$$

which gives the proof of the Theorem 2.4. □

3. Main Results

In this section, we will investigate the rate of approximation with regard to the partial and complete modulus of continuity and prove a Voronovskaya-type asymptotic theorem. Let the complete modulus of continuity for $h(x, y) \in C(\nabla)$ is given by

$$\begin{aligned}
\varpi(h, \gamma_1, \gamma_2) = \sup\{&|h(u, v) - h(x, y)| : (u, v), (x, y) \in \nabla \\
&|u - x| \le \gamma_1, |v - y| \le \gamma_2\}
\end{aligned}$$

and $\varpi(h,\gamma_1,\gamma_2)$ satisfy the following properties:

$$i)\ \varpi(h,\gamma_1,\gamma_2)\to 0,\ \gamma_1\to 0,\ \gamma_2\to 0$$

$$ii)\ |h(u,v)-h(x,y)|\le \varpi(h,\gamma_1,\gamma_2)\times\left(1+\frac{|u-x|}{\gamma_1}\right)\left(1+\frac{|v-y|}{\gamma_2}\right). \tag{3.1}$$

Also, according to x and y integers, the partial modulus of continuity is defined by

$$\omega_1(h,\delta)=\sup\left\{|h(x_1,y)-h(x_2,y)| : y\in[-1,1], |x_1-x_2|\le\delta,\delta>0\right\},$$
$$\omega_2(h,\delta)=\sup\left\{|h(x,y_1)-h(x,y_2)| : x\in[-1,1], |y_1-y_2|\le\delta,\delta>0\right\}.$$

Further, we denote with $C^2(\nabla)$ the set of all functions of h such that $\frac{\partial_j h}{\partial x_j}, \frac{\partial_j h}{\partial y_j}$ $(j=1,2)$ belong to $C(\nabla)$.

Theorem 3.1. *Let $h\in C(\nabla)$. For the operators given by (1.1), the following inequality holds:*

$$|R_n(h;x,y)-h(x,y)|\le 4\varpi(h,\frac{2}{\sqrt{n+1}},\frac{2}{\sqrt{n+1}}).$$

Proof. In view of (3.1), we get

$$\begin{aligned}|h(s,t)-h(x,y)| &\le \varpi(h,|s-x|,|t-y|)\\ &\le \varpi(h,\gamma_1,\gamma_2)\left(1+\frac{|s-x|}{\gamma_1}\right)\left(1+\frac{|t-y|}{\gamma_2}\right).\end{aligned}$$

Operating $R_n(.;x,y)$ to the above inequality, one has

$$\begin{aligned}|R_n(h;x,y)-h(x,y)| &= |R_n(h(s,t);x,y)-R_n(h(x,y);x,y)|\\ &= |R_n(h(s,t)-h(x,y));x,y)|\\ &\le R_n(|h(s,t)-h(x,y)|;x,y)\\ &\le \varpi(h,\gamma_1,\gamma_2)\left(1+\frac{1}{\gamma_1}R_n(|s-x|;x,y)\right)\\ &\times\left(1+\frac{1}{\gamma_2}R_n(|t-y|;x,y)\right).\end{aligned}$$

Utilizing the Cauchy-Schwarz inequality and by Lemma 2.2. $(v)-(vi)$, we obtain

$$\begin{aligned}&|R_n(h;x,y)-h(x,y)|\\ &\le \varpi(h,\gamma_1,\gamma_2)\left(1+\frac{1}{\gamma_1}\sqrt{R_n((s-x)^2;x,y)}\right)\left(1+\frac{1}{\gamma_2}\sqrt{R_n((t-y)^2;x,y)}\right)\\ &\le \varpi(h,\gamma_1,\gamma_2)\left(1+\frac{1}{\gamma_1}\sqrt{\frac{4}{n+1}}\right)\left(1+\frac{1}{\gamma_2}\sqrt{\frac{4}{n+1}}\right).\end{aligned}$$

Choosing $\gamma_1 = \gamma_2 = \frac{2}{\sqrt{n+1}}$, which gives the proof of Theorem 3.1.. □

Theorem 3.2. *Let $h \in C(\nabla)$. For the operators given by (1.1), we get the following inequality:*

$$|R_n(h;x,y) - h(x,y)| \leq 2\left(\omega_1(h, \frac{2}{\sqrt{n+1}}) + \omega_2(h, \frac{2}{\sqrt{n+1}})\right).$$

Proof. Applying the Cauchy-Schwarz inequality and by the definition of partial modulus of continuity, hence the desired result can be obtained easily. □

Theorem 3.3. (Voronovskaya-type theorem) *Let $h \in C^2(\nabla)$. For the operators given by (1.1), we have the following relation:*

$$\begin{aligned}\lim_{n\to\infty} n(R_n(h(s,t);x,y) - h(x,y)) = (-x)h'_x(x,y) + (-y)h'_y(x,y)\\ + (-xy - x - y - 1)h''_{xy}(x,y)\\ + (\frac{1-x^2}{2})h''_{xx}(x,y) + (\frac{1-y^2}{2})h''_{yy}(x,y)\end{aligned}$$

uniformly on ∇.

Proof. For arbitrary $(x,y) \in \nabla$ and using Taylor's formula, we may write

$$\begin{aligned}h(s,t) &= h(x,y) + h'_x(x,y)(s-x) + h'_y(x,y)(t-y)\\ &+ \frac{1}{2}\left\{h''_{xx}(x,y)(s-x)^2 + 2h''_{xy}(x,y)(s-x)(t-y) + h''_{yy}(x,y)(t-y)^2\right\}\\ &+ \theta_1(s,t;x,y)(s-x)^2 + \theta_2(s,t;x,y)(t-y)^2 \qquad (3.2)\end{aligned}$$

for $(s,t) \in \nabla$ where $\theta_1(s,t;x,y), \theta_2(s,t;x,y) \in C(\nabla)$ and $\theta_1(s,t;x,y) \to 0$, $\theta_2(s,t;x,y) \to 0$, as $(s,t) \to (x,y)$.

Operating $R_n(.;x,y)$ on both sides of (3.2), thus

$$\begin{aligned}R_n(h(s,t);x,y)) &= h(x,y) + h'_x(x,y)R_n((s-x);x,y) + h'_y(x,y)R_n((t-y);x,y)\\ &+ \frac{1}{2}\left\{h''_{xx}(x,y)R_n((s-x)^2;x,y) + h''_{yy}(x,y)R_n((t-y)^2;x,y\right.\\ &\left.+ 2h''_{xy}(x,y)R_n((s-x)(t-y);x,y)\right\}\\ &+ R_n(\theta_1(s,t;x,y)(s-x)^2 + \theta_2(s,t;x,y)(t-y)^2;x,y). \qquad (3.3)\end{aligned}$$

Implementing the Cauchy-Schwarz inequality in the last part of (3.3), then

$$\begin{aligned}&\left|R_n(\theta_1(s,t;x,y)(s-x)^2 + \theta_2(s,t;x,y)(t-y)^2;x,y)\right|\\ &\leq \sqrt{R_n(\theta_1^2(s,t;x,y)(s-x)^4;x,y)} + \sqrt{R_n(\theta_2^2(s,t;x,y)(t-y)^4;x,y)}\\ &\leq \left\{\sqrt{R_n(\theta_1^2(s,t;x,y);x,y)}\sqrt{R_n((s-x)^4;x,y)}\right.\\ &\left.+ \sqrt{R_n(\theta_2^2(s,t;x,y);x,y)}\sqrt{R_n((t-y)^4;x,y)}\right\}.\end{aligned}$$

Because of $\theta_1(s,t;x,y) \to 0$ and $\theta_2(s,t;x,y) \to 0$, as $(s,t) \to (x,y)$ and also by Lemma 2.3. $(vi) - (vii)$, obviously we get

$$\lim_{n\to\infty} n\left[R_n(\theta_1(s,t;x,y)(s-x)^2 + \theta_2(s,t;x,y)(t-y)^2;x,y)\right] = 0 \tag{3.4}$$

uniformly on $(x,y) \in \nabla$.

Considering to Lemma 2.2. and (3.4), the required result is obtained as:

$$\begin{aligned}\lim_{n\to\infty} n(R_n(h(s,t);x,y) - h(x,y)) &= (-x)h'_x(x,y) + (-y)h'_y(x,y) \\ &+ (-xy-x-y-1)h''_{xy}(x,y) \\ &+ (\frac{1-x^2}{2})h''_{xx}(x,y) + (\frac{1-y^2}{2})h''_{yy}(x,y).\end{aligned}$$

□

4. Local Approximation

In this section, we will estimate the rate of convergence in terms of Peetre's K-functional and a Lipschitz-type function. The norm on $C^2(\nabla)$ and Peetre's K-functional are given, respectively as follows:

$$\|h\|_{C^2(\nabla)} = \|h\|_{C(\nabla)} + \sum_{j=1}^{2}\left(\left\|\frac{\partial_j h}{\partial x_j}\right\|_{C(\nabla)} + \left\|\frac{\partial_j h}{\partial y_j}\right\|_{C(\nabla)}\right),$$

$$K_2(h,\zeta) = \inf\left\{\|h-g\|_{C(\nabla)} + \zeta\,\|g\|_{C^2(\nabla)} : g \in C^2(\nabla)\right\} \quad (\zeta > 0).$$

Also, for an absolute constant $D > 0$ such that

$$K_2(h,\zeta) \le D\overset{*}{\omega}_2(h,\sqrt{\zeta}), \tag{4.1}$$

where $\overset{*}{\omega}_2(h,\sqrt{\zeta})$ denote the second order of modulus of continuity. (See: [3, 29]).

Further, for $h \in C(\nabla)$, $(x,y),(t,s) \in \nabla$ and $\beta,\gamma \in (0,1]$, the Lipschitz-type class for bivariate case is assigned as

$$Lip_\alpha(\beta,\gamma) = \left\{h \in C(\nabla) : |h(t,s) - h(x,y)|\,; x,y \le \alpha\ \ |t-x|^\beta\,|s-y|^\gamma\right\}. \tag{4.2}$$

Theorem 4.1. *Let $h \in Lip_\alpha(\beta,\gamma)$. Then, for each $(x,y) \in \nabla$ the following inequality verifies that*

$$|\ R_n(h;x,y) - h(x,y)| \le \alpha\ \Pi_{n_1}(x)^{\frac{\beta}{2}}\Pi_{n_2}(y)^{\frac{\gamma}{2}}$$

where $\Pi_{n_1}(x) = R_n(k_{2,0};x,y)$ and $\Pi_{n_2}(y) = R_n(k_{0,2};x,y)$.

Proof. In view of (4.2) and using the definition of (1.1), then

$$\begin{aligned} | R_n(h;x,y)-h(x,y)| &\leq R_n\left(|h(t,s)-h(x,y)|\,;x,y\right) \\ &\leq \alpha R_n\left(|t-x|^{\beta}|s-y|^{\gamma};x,y\right) \\ &= \alpha R_n(|t-x|^{\beta};x,y) R_n\left(|s-y|^{\gamma};x,y\right). \end{aligned}$$

Utilizing the Hölder's inequality with $(p_1,q_1)=\left(\frac{2}{\beta},\frac{2}{2-\beta}\right)$ and $(p_2,q_2)=\left(\frac{2}{\gamma},\frac{2}{2-\gamma}\right)$, we get

$$\begin{aligned} | R_n(h;x,y)-h(x,y)| \leq \alpha\Bigg(& R_n\left((t-x)^2;x,y\right)^{\frac{\beta}{2}} R_n\left(r_{0,0};x,y\right)^{\frac{2-\beta}{2}} \\ & \times R_n\left((s-y)^2;x,y\right)^{\frac{\gamma}{2}} R_n\left(r_{0,0};x,y\right)^{\frac{2-\gamma}{2}}\Bigg). \end{aligned}$$

From Lemma 2.1. (i), which leads to the required result as

$$| R_n(h;x,y)-h(x,y)| \leq \alpha\ \Pi_{n_1}(x)^{\frac{\beta}{2}}\Pi_{n_2}(y)^{\frac{\gamma}{2}}.$$

□

Theorem 4.2. *Let $h\in C^1(\nabla)$. Then, we obtain the following inequality*

$$| R_n(h;x,y)-h(x,y)| \leq \|h_x\|\sqrt{\Pi_{n_1}(x)}+\|h_y\|\sqrt{\Pi_{n_2}(y)}$$

where $\Pi_{n_1}(x)$ and $\Pi_{n_2}(y)$ are same as in Theorem 4.1..

Proof. For a given fixed point $(x,y)\in\nabla$, we may write

$$h(u,v)-h(x,y)=\int_x^u h_t(t,v)dt+\int_y^v h_s(x,s)ds.$$

Operating $R_n(.;x,y)$ to the both sides of above equality, then

$$| R_n(h;x,y)-h(x,y)| \leq R_n(\int_x^u h_t(t,v)dt;x,y)+R_n(\int_y^v h_s(x,s)ds;x,y).$$

Since

$$\left|\int_x^u h_t(t,v)dt\right| \leq \|h_x\|\,|u-x| \ \text{ and } \ \left|\int_y^v h_s(x,s)ds\right| \leq \|h_y\|\,|v-y|\,,$$

hence,

$$| R_n(h;x,y)-h(x,y)| \leq \|h_x\|\,R_n(|u-x|\,;x,y)+\|h_y\|\,R_n(|v-y|\,;x,y).$$

Applying the Cauchy-Schwarz inequality to the above inequality, one has

$$\begin{aligned}| R_n(h;x,y)-h(x,y)| &\leq \|h_x\|\, R_n((u-x)^2;x,y)^{\frac{1}{2}}R_n(r_{0,0};x,y)^{\frac{1}{2}}\\ &+\|h_y\|\, R_n((v-y)^2;x,y)^{\frac{1}{2}}R_n(r_{0,0};x,y)^{\frac{1}{2}}\\ &\leq \|h_x\|\sqrt{\Pi_{n_1}(x)}+\|h_y\|\,\Pi_{n_2}(y),\end{aligned}$$

which gives the proof of Theorem 4.2. □

Theorem 4.3. *Let $g\in C(\nabla)$. Then, for the operators defined by (1.1) we have the following inequality*

$$\begin{aligned}|R_n(g;x,y)-g(x,y)| \quad &\leq N\left\{\overset{*}{\omega}_2(g;\frac{\sqrt{A_n(x,y)}}{2}+\min\{1,A_n\}\,\|g\|_{C(\nabla)}\right\}\\ &+\varpi(g;\chi_n(x,y)),\end{aligned}$$

where $\chi_n(x,y)=\frac{\sqrt{x^2+y^2}}{n+1}$, $A_n(x,y)=\left(\Pi_{n_1}(x)+\Pi_{n_2}(y)+\chi_n^2(x,y)\right)$ *and* $N>0$ *is a constant.*

Proof. By first, we define the following auxiliary operators

$$\overset{*}{R}_n(g;x,y)=R_n(g;x,y)-g(x\frac{n}{n+1},y\frac{n}{n+1})+g\left(x,y\right). \tag{4.3}$$

From Lemma 2.1., it is clear that

$$\overset{*}{R}_n((s-x);x,y)=0,\ \overset{*}{R}_n((t-y);x,y)=0.$$

For $h\in C^2(\nabla)$, $(s,t)\in\nabla$, applying Taylor's formula, then

$$\begin{aligned}h(s,t)-h(x,y)&=h(s,y)-h(x,y)+h(s,t)-h(s,y)\\ &=\frac{\partial h(x,y)}{\partial x}(s-x)+\int_x^s(s-u)\frac{\partial^2 h(u,y)}{\partial^2 u}du\\ &+\frac{\partial h(x,y)}{\partial y}(t-y)+\int_y^t(t-v)\frac{\partial^2 h(x,v)}{\partial^2 v}dv.\end{aligned} \tag{4.4}$$

Operating $R_n(.;x,y)$ to (4.4), hence

$$\begin{aligned}
&\overset{*}{R}_n(h;x,y)-h(x,y)\\
&=\overset{*}{R}_n\left(\int_x^s (s-u)\frac{\partial^2 h(u,y)}{\partial^2 u}du;x,y\right)+\overset{*}{R}_n\left(\int_y^t (t-v)\frac{\partial^2 h(x,v)}{\partial^2 v}dv;x,y\right)\\
&=R_n\left(\int_x^s (s-u)\frac{\partial^2 h(u,y)}{\partial^2 u}du;x,y\right)-\int_x^{x\frac{n}{n+1}}\left(x\frac{n}{n+1}-u\right)\frac{\partial^2 h(u,y)}{\partial^2 u}du\\
&+R_n\left(\int_y^t (t-v)\frac{\partial^2 h(x,v)}{\partial^2 v}dv;x,y\right)-\int_y^{y\frac{n}{n+1}}\left(y\frac{n}{n+1}-v\right)\frac{\partial^2 h(x,v)}{\partial^2 v}dv.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&|R_n(h;x,y)-h(x,y)|\\
&\le R_n\left(\left|\int_x^s |s-u|\left|\frac{\partial^2 h(u,y)}{\partial^2 u}\right|du\right|;x,y\right)+\left|\int_x^{x\frac{n}{n+1}}\left|x\frac{n}{n+1}-u\right|\left|\frac{\partial^2 h(u,y)}{\partial^2 u}\right|du\right|\\
&+R_n\left(\left|\int_y^t |t-v|\left|\frac{\partial^2 h(x,v)}{\partial^2 v}\right|dv\right|;x,y\right)+\left|\int_y^{y\frac{n}{n+1}}\left|y\frac{n}{n+1}-v\right|\left|\frac{\partial^2 h(x,v)}{\partial^2 v}\right|dv\right|\\
&\le\left\{R_n((s-x)^2;x,y)+(x\frac{n}{n+1}-x)^2+R_n((t-y)^2;x,y)+(y\frac{n}{n+1}-y)^2\right\}\|h\|_{C^2(\nabla)}.
\end{aligned}$$

Choosing $\chi_n(x,y)=\frac{\sqrt{x^2+y^2}}{n+1}$, $A_n(x,y)=\left(\Pi_{n_1}(x)+\Pi_{n_2}(y)+\chi_n^2(x,y)\right)$, we get

$$|R_n(h;x,y)-h(x,y)|\le A_n(x,y)\|h\|_{C^2(\nabla)}. \tag{4.5}$$

Taking Lemma 2.2. into account, thus

$$\left|\overset{*}{R}_n(g;x,y)\right|\le|R_n(g;x,y)|+\left|g(x\frac{n}{n+1},y\frac{n}{n+1})\right|+|g(x,y)|\le 3\|g\|_{C(\nabla)}. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$\begin{aligned}
|R_n(g;x,y)-g(x,y)| &\leq \left|\overset{*}{R}_n(g-h;x,y)\right| + \left|\overset{*}{R}_n(h;x,y)-h(x,y)\right| \\
&+ |h(x,y)-g(x,y| + \left|g(x\frac{n}{n+1}, y\frac{n}{n+1}) - g(x,y)\right| \\
&\leq 4\|g-h\| + |R_n(h;x,y)-h(x,y)| \\
&+ \left|g(x\frac{n}{n+1}, y\frac{n}{n+1}) - g(x,y)\right| \\
&\leq \left(4\|g-h\| + A_n(x,y)\|h\|_{C^2(\nabla)}\right) + \varpi(g;\chi_n(x,y)). \qquad (4.7)
\end{aligned}$$

Taking the infimum over all $h \in C^2(\nabla)$ on the right hand side of (4.7), we arrive

$$|R_n(g;x,y)-g(x,y)| \leq 4K_2(g;\frac{A_n(x,y)}{4}) + \varpi(g;\chi_n(x,y)).$$

Applying (4.1) to the above inequality, it gives the required result as

$$\begin{aligned}
|R_n(g;x,y)-g(x,y)| \quad &\leq N\left\{\overset{*}{\omega}_2(g;\frac{\sqrt{A_n(x,y)}}{2} + \min\{1,A_n\}\|g\|_{C(\nabla)}\right\} \\
&+\varpi(g;\chi_n(x,y)).
\end{aligned}$$

□

5. GBS(Generalized Boolean Sum) Type Operators

The concept of the B-continuous and B-differentiable functions were firstly used by Bögel [8, 9]. Dobrescu and Matei [12] considered the GBS(Generalized Boolean Sum) type of the bivariate Bernstein operators. Next, using of the B-continuous functions by the GBS operators, which is related to a quantitative variant of the Korovkin's type theorem, was firstly improved by Badea [4, 3]. Pop and Fărcas [21] obtained some approximation of B-continuous and B-differentiable functions by GBS type of Bernstein bivariate operators. We refer also some papers of various linear positive operators, which are related to the GBS operators, ([23, 24, 5, 13]).

Let a function $h:\nabla \to \mathbb{R}$. For any $(x,y),(t_0,s_0) \in \nabla$, the mixed difference of the function h is given by

$$\phi_{(x,y)}h\left[t_0,s_0;x,y\right] = h(x,y) - h(x,s_0) - h(t_0,y) + h(t_0,s_0). \qquad (5.1)$$

A function $h:\nabla \to \mathbb{R}$ is called Bögel-continuous (B-continuous) at $(t_0,s_0) \in \nabla$, if

$$\lim_{(x,y)\to(t_0,s_0)} \phi_{(x,y)}h\left[t_0,s_0;x,y\right] = 0.$$

A function $h : \nabla \to \mathbb{R}$ is called Bögel-differentiable (B-differentiable) at $(t_0, s_0) \in \nabla$, if

$$\lim_{(x,y)\to(t_0,s_0)} \frac{\phi_{(x,y)} h\left[t_0, s_0; x, y\right]}{(x-t_0)(y-s_0)} < \infty. \tag{5.2}$$

Note that, by $C_b(\nabla)$ and $D_b(\nabla)$, we denote the sets of all B-differentiable and B-continuous functions on ∇, respectively. Also, $C(\nabla) \subset C_b(\nabla)$, see details in [10].

The mixed modulus of smoothness for $h \in C_b(\nabla)$ is given by

$$\omega_{mixed}\left(h; \delta_1, \delta_2\right) := \sup\left\{\left|\phi_{(x,y)} h\left[t_0, s_0; x, y\right]\right| : |t_0 - x| < \delta_1, |s_0 - y| < \delta_2\right\} \tag{5.3}$$

where $(x, y), (t_0, s_0) \in \nabla$ and $\delta_1, \delta_2 \in \mathbb{R}^+$. Also for all $\lambda_1, \lambda_2 \geq 0$, the following inequality holds

$$\omega_{mixed}\left(h; \lambda_1\delta_1, \lambda_2\delta_2\right) \leq (1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}\left(h; \delta_1, \delta_2\right). \tag{5.4}$$

In view of (5.4), one has

$$\begin{aligned} \left|\phi_{(x,y)} h\left[t_0, s_0; x, y\right]\right| &\leq \omega_{mixed}\left(h; |t_0 - x|, |s_0 - y|\right) \\ &\leq \left(1 + \frac{|t_0 - x|}{\delta_1}\right)\left(1 + \frac{|s_0 - y|}{\delta_2}\right)\omega_{mixed}\left(h; \delta_1, \delta_2\right). \end{aligned} \tag{5.5}$$

Some details on ω_{mixed} can be found [14].

Let $h \in C_b(\nabla)$, the Lipschitz class $Lip_\alpha(\beta, \gamma)$ with $\alpha > 0$,$(t_0, s_0), (x, y) \in \nabla$ and $\beta, \gamma \in (0, 1]$ is defined by

$$Lip_\alpha(\beta,\gamma) = \left\{h \in C_b(\nabla) : \left|\phi_{(x,y)} h\left[t_0, s_0; x, y\right]\right| \leq \alpha\ |t_0 - x|^\beta\,|s_0 - y|^\gamma\right\}.$$

Now, for $h \in C_b(\nabla)$ and $(t_0, s_0), (x, y) \in \nabla$, we define the associated GBS type of operators (1.1) as follows:

(5.6)

$$P_n(h; x, y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} \nu_{n,k,l}(x,y) \int\limits_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}\int\limits_{2\frac{l}{n+1}-1}^{2\frac{l+1}{n+1}-1} \left[h(x, s_0) + h(t_0, y) - h(t_0, s_0)\right] ds_0 dt_0,$$

where $\nu_{n,k,l}(x,y) = \left(\frac{n+1}{2}\right)^2 \binom{n}{k}\binom{n-k}{l}\left(\frac{1+x}{2}\right)^k\left(\frac{1+y}{2}\right)^l\left(1 - \frac{1+x}{2} - \frac{1+y}{2}\right)^{n-k-l}$.

Theorem 5.1. *For all $h \in C_b(\nabla)$ and $(x, y) \in \nabla$, the operators given by (5.6) verify that*

$$\left|\ P_n(h; x, y) - h(x, y)\right| \leq 4\ \omega_{mixed}\left(h; \frac{2}{\sqrt{n+1}}, \frac{2}{\sqrt{n+1}}\right).$$

Proof. From (5.1), it is clear

$$h(x,y)-\phi_{(x,y)}h\left[t_0,s_0;x,y\right]=h(x,s_0)+h(t_0,y)-h(t_0,s_0).$$

Operating $R_n(.;x,y)$ and using the definition of (5.6)

$$P_n(h;x,y)-h(x,y)=-R_n(\phi_{(x,y)}h\left[t_0,s_0;x,y\right];x,y).$$

Utilizing the Cauchy-Schwarz inequality to the above equation and in view of (5.5), thus

$$\begin{aligned}|P_n(h;x,y)-h(x,y)| &\le R_n(\left|\phi_{(x,y)}h\left[t_0,s_0;x,y\right]\right|;x,y)\\ &\le \Big(R_n(r_{0,0};x,y)+\delta_1^{-1}\sqrt{R_n\left((t_0-x)^2;x,y\right)}\\ &+\delta_2^{-1}\sqrt{R_n\left((s_0-y)^2;x,y\right)}\\ &+\frac{1}{\delta_1\delta_2}\sqrt{R_n\left((t_0-x)^2;x,y\right)R_n\left((s_0-y)^2;x,y\right)}\Big)\,\omega_{mixed}\left(h;\delta_1,\delta_2\right).\end{aligned}$$

Using Lemma 2.1. (i), Lemma 2.2. $(v)-(vi)$ and (5.3), we get

$$\begin{aligned}|P_n(h;x,y)-h(x,y)| \le \Bigg(1+\delta_1^{-1}\frac{2}{\sqrt{n+1}}+\delta_2^{-1}\frac{2}{\sqrt{n+1}}\\ +\delta_1^{-1}\delta_2^{-1}\sqrt{\frac{2}{n+1}\frac{2}{n+1}}\Bigg)\,\omega_{mixed}\left(h;\delta_1,\delta_2\right).\end{aligned}$$

Taking $\delta_1=\delta_2=\frac{2}{\sqrt{n+1}}$, hence we get

$$|\;P_n(h;x,y)-h(x,y)|\le 4\;\omega_{mixed}\left(h;\frac{2}{\sqrt{n+1}},\frac{2}{\sqrt{n+1}}\right).$$

□

Theorem 5.2. *Let $h\in Lip_\alpha(\beta,\gamma)$. Then, operators (5.6) satisfy the following inequality:*

$$|\;P_n(h;x,y)-h(x,y)|\le\alpha\;\Pi_{n_1}(x)^{\frac{\beta}{2}}\Pi_{n_2}(y)^{\frac{\gamma}{2}}.$$

Proof. In view of (4.2) and by (5.6), we get

$$\begin{aligned}|\;P_n(h;x,y)-h(x,y)| &\le R_n\left(\left|\phi_{(x,y)}h\left[t_0,s_0;x,y\right]\right|;x,y\right)\\ &\le \alpha R_n\left(\left|t_0-x\right|^\beta\left|s_0-y\right|^\gamma;x,y\right)\\ &=\alpha R_n\left(\left|t_0-x\right|^\beta;x,y\right)R_n\left(\left|s_0-y\right|^\gamma;x,y\right).\end{aligned}$$

Applying the Hölder's inequality with $(p_1, q_1) = \left(\frac{2}{\beta}, \frac{2}{2-\beta}\right)$, $(p_2, q_2) = \left(\frac{2}{\gamma}, \frac{2}{2-\gamma}\right)$ to the above inequality, thus

$$| \; P_n(h; x, y) - h(x, y)| \leq \alpha \Bigg(R_n \left((t_0 - x)^2; x, y\right)^{\frac{\beta}{2}} R_n \left(r_{0,0}; x, y\right)^{\frac{2-\beta}{2}} \times R_n \left((s_0 - y)^2; x, y\right)^{\frac{\gamma}{2}} R_n \left(r_{0,0}; x, y\right)^{\frac{2-\gamma}{2}} \Bigg).$$

Considering to Lemma 2.1. (i) and Lemma 2.2. $(v) - (vi)$, then

$$| \; P_n(h; x, y) - h(x, y)| \leq \alpha \; \Pi_{n_1}(x)^{\frac{\beta}{2}} \Pi_{n_2}(y)^{\frac{\gamma}{2}},$$

which completes the proof. □

6. Graphics and Error Estimation Tables

In this section, we present graphs (made with Maple) and error estimation tables comparing of the convergence of the operators $R_n(h; x, y)$ and $P_n(h; x, y)$, from (1.1) and (5.6), to some example functions $h(x, y)$.

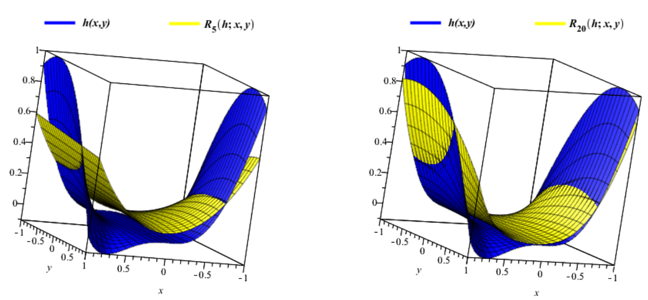

Figure 1: Approximations $R_n(h; x, y)$ of $h(x, y) = x^3 \sin(x - y)$.

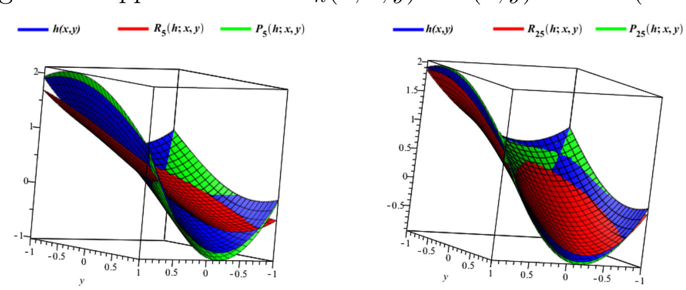

Figure 2: Approximations $R_n(h; x, y)$ and $P_n(h; x, y)$ of $h(x, y) = x^2 - \sin(y - x)$.

Example 6.1. In Figure 1, we show the function $h(x, y) = x^3 \sin(x - y)$ in blue, and the approximations $R_n(h; x, y)$ from 1.1, for $n = 5$ and 20, in yellow. In Table 1, we give the error of the same approximation operators for $n = 5, 25$ and 125. It is clear from Table 1 that, as the value of n increases, the absolute error between the operators and the function $h(x, y)$ decreases.

Example 6.2. In Figure 2, we show the function $h(x, y) = x^2 - \sin(y - x)$ in blue; the operator $R_n(h; x, y)$ from (1.1), for $n = 5$ and 25 is shown in red and its associated GBS operators from (5.6) are shown in green. In addition, in Table 2 we show the error of the operator approximations when $n = 250$. It is evident from the table that GBS type operator gives a better approximation.

(x, y)	$n = 5$	$n = 25$	$n = 125$
$(0.095, -0.095)$	0.080565	0.010006	0.001126
$(0.05, -0.05)$	0.077840	0.008023	0.000554
$(0.035, -0.035)$	0.077289	0.007625	0.000472
$(-0.025, 0.025)$	0.077027	0.007437	0.000424
$(-0.01, -0.01)$	0.076414	0.007196	0.000363
$(0.015, -0.075)$	0.075718	0.007179	0.000389
$(-0.03, -0.06)$	0.075131	0.007040	0.000364
$(-0.04, -0.09)$	0.074803	0.006933	0.000361
$(-0.02, -0.08)$	0.074716	0.006894	0.000333
$(-0.01, -0.099)$	0.074421	0.006830	0.000325

Table 1: Error of approximation operators $R_n(h; x, y)$ to the function $h(x, y) = x^3 \sin(x - y)$ for $n = 5, 25, 125$

(x, y)	$\lvert R_{250}(h; x, y) - h(x, y)\rvert$	$\lvert P_{250}(h; x, y) - h(x, y)\rvert$
$(0.01, -0.01)$	0.00373539672	0.00007801713
$(-0.02, 0.02)$	0.00444218352	0.00015595725
$(0.03, -0.04)$	0.00313838356	0.00026983045
$(-0.03, -0.05)$	0.00373219945	0.00007189246
$(0.05, -0.05)$	0.00276456558	0.00038854967
$(0.06, -0.06)$	0.00251728506	0.00046541592
$(-0.07, -0.05)$	0.00414246641	0.00006892313
$(-0.07, -0.07)$	0.00391506974	0.00000000012
$(0.08, -0.08)$	0.00202380312	0.00061769752
$(0.09, -0.09)$	0.00177541194	0.00069296224

Table 2: Error of approximation operators $R_n(h; x, y)$ and $P_n(h; x, y)$ to the function $h(x, y) = x^2 - \sin(y - x)$ for $n = 250$

References

[1] P. N. Agrawal, A. M. Acu, R. Chaugan and T. Garg, *Approximation of Bögel Continuous functions and deferred weighted A-statistical convergence by Bernstein-Kantorovich type operators on a triangle*, J. Math. Inequal., **15(4)**(2021), 1695–1711.

[2] K. J. Ansari, F. Özger and Z. Ödemiş Özger, *Numerical and theoretical approximation results for Schurer–Stancu operators with shape parameter* λ, Comp. Appl. Math., 41, 181(2022).

[3] C. Badea, *K-functionals and moduli of smoothness of functions defined on compact metric spaces*, Comput. Math. with Appl., **30**(1995), 23–31.

[4] C. Badea, I. Badea and H. H. Gonska, *Notes on degree of approximation on b-continuous and b-differentiable functions*, Approx. Theory Appl., **4**(1988), 95–108.

[5] D. Bărbosu and O. T. Pop, *A note on the gbs Bernstein's approximation formula*, An. Univ. Craiova Ser. Mat. Inform., **35**(2008), 1–6.

[6] G. Başcanbaz-Tunca, A. Erençin and H. İnce-İlarslan, *Bivariate Cheney-Sharma operators on simplex*, Hacet. J. Math. Stat., **47(4)**(2018), 793.804.

[7] S. N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Comp. Comm. Soc. Mat. Charkow Ser., **13(1)**(1912), 1–2.

[8] K. Bögel, *Über mehrdimensionale differentiation von funktionen mehrerer reeller veränderlichen*, J. für die Reine und Angew. Math., **170**(1934), 197–217.

[9] K. Bögel, *Über mehrdimensionale differentiation, integration und beschränkte variation*, J. für die Reine und Angew. Math., **173**(1935), 5–30.

[10] K. Bögel, *Über die mehrdimensionale differentiation*, Jahresber. Deutsch. Math. Verein., **65**(1962), 45–71.

[11] S. Deshwal, N. Ispir and P. N. Agrawal, *Bivariate operators of Bernstein-Kantorovich type on a triangle*, Appl. Math. Inf. Sci., **11(2)**(2017), 423–432.

[12] E. Dobrescu and I. Matei, *The approximation by Bernstein type polynomials of bidimensional continuous functions*, An. Univ. Timi¸soara Ser. Sti. Mat.-Fiz., **4**(1966), 85–90.

[13] B. R. Draganov, *Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated boolean sums*, J. Approx. Theory, **200**(2015), 92–135.

[14] S. G. Gal, *Shape-preserving approximation by real and complex polynomials*, Springer Science and Business Media(2010).

[15] M. Goyal, A. Kajla, P. N. Agrawal and S. Araci, *Approximation by bivariate Bernstein-Durrmeyer operators on a triangle*, Appl. Math. Inf. Sci., **11**(2017), 693–702.

[16] A. Kajla, *Generalized Bernstein-Kantorovich–type operators on a triangle*, Math. Methods Appl. Sci., **42(12)**(2019), 4365–4377.

[17] A. Kajla and M. Goyal, *Modified Bernstein–Kantorovich operators for functions of one and two variables*, Rend. Circ. Mat. Palermo(2), **67(2)**(2018), 379–395.

[18] L. V. Kantorovich, *Sur certain développements suivant les polynômes de la forme de s. Bernstein*, I, II, CR Acad. URSS, **563**(1930), 568.

[19] E. H. Kingsley, *Bernstein polynomials for functions of two variables of class* $C^{(k)}$, Proc. Amer. Math. Soc., **2(1)**(1951), 64–71.

[20] M. Heshamuddin, N. Rao, B. P. Lamichhane, A. Kiliçman and M. Ayman-Mursaleen, *On One- and Two-Dimensional* α*–Stancu–Schurer–Kantorovich Operators and Their Approximation Properties*. Mathematics, 10(18), 3227 (2022).

[21] O. T. Pop and M. D. Fărcas, *Approximation of b-continuous and b-differentiable functions by GBS operators of Bernstein bivariate polynomials*, J. Inequal. Pure Appl. Math., **7(9)**(2006).

[22] O. T. Pop and M. D. Fărcas, *About the bivariate operators of Kantorovich type*, Acta Math. Univ. Comenian., **78(1)**(2009), 43–52.

[23] R. Ruchi, B. Baxhaku and P. N. Agrawal, *GBS operators of bivariate Bernstein-Durrmeyer type on a triangle*, Math. Methods Appl. Sci., **41(7)**(2018), 2673-2683.

[24] M. Sidharth, N. Ispir and P. N. Agrawal, *GBS operators of Bernstein–Schurer–Kantorovich type based on q-integers*, Appl. Math. Comput., **269**(2015), 558–568.

[25] H. Srivastava, K. J. Ansari, F. Özger and Z. Ödemiş Özger, *A link between approximation theory and summability methods via four-dimensional infinite matrices*. Mathematics 9:1895 (2021).

[26] D. D. Stancu, *A method for obtaining polynomials of Bernstein type of two variables*, Am. Math. Mon., **70(3)**(1963), 260–264.

[27] V. I. Volkov, *On the convergence of sequences of linear positive operators in the space of continuous functions of two variables*, Dokl. Akad. Nauk SSSR, **151(1)**(1957), 17–19.

[28] K. Weierstrass, *Über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen veränderlichen*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, **2**(1885), 633–639.

[29] D. X. Zhou, *On smoothness characterized by Bernstein type operators*, J. Approx. Theory, **81(3)**(1995), 303–315.