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On Representable Rings and Modules

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Abstract. In this paper, we determine the structure of rings that have secondary representation (called representable rings) and give some characterizations of these rings. Also, we characterize Artinian rings in terms of representable rings. Then we introduce completely representable modules, modules every non-zero submodule of which have secondary representation, and give some necessary and sufficient conditions for a module to be completely representable. Finally, we define strongly representable modules and give some conditions under which representable module is strongly representable.

1. Introduction

Throughout this paper, R will denote a commutative ring with a non-zero identity and every module will be unitary. Given an R-module M, we shall denote the annihilator of M (in R) by $Ann_R(M)$ or $Ann(M)$. We shall follow Macdonald's terminology in [18] concerning secondary representation. Thus, an R-module N is *secondary* if $N \neq 0$ and for each $r \in R$, either $rN = N$ or there exists some positive integer n, such that $r^n N = 0$. If N is a secondary module then, $Ann(N)$ is a primary ideal and hence $P = \sqrt{Ann(N)}$ is prime and we say that N is P*-secondary.* A *secondary representation* of M is an expression for M as a sum $M = N_1 + N_2 + \cdots + N_t$ of finitely many secondary submodules of M, such that N_i is P_i-secondary for $i = 1, \ldots, t$. If such a representation exists, we shall say M is *representable.* Such a representation is said to be *minimal* if $P_1, \ldots, P_t$ are all different and none of the summands N_i are redundant. Every representable module has a minimal secondary representation. As for the primary decomposition of submodules, we have two uniqueness theorems for secondary representation of

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modules. The first uniqueness theorem (see [18, 2.2]) says that if $M = N_1 + \cdots + N_t$, with N_i being P_i-secondary, is a minimal secondary representation of M, then the set $\{P_1, ..., P_t\}$ is independent of the choice of minimal secondary representation of M. This set is called the *set of attached prime ideals of* M and is denoted by $Att(M)$. Every Artinian module and every injective module over a Noetherian ring is representable. These and other propositions about representable modules can be found in [5, 18, 22].

An R-module M is said to be *Laskerian* if every proper submodule of M is an intersection of a finite number of primary submodules, i.e. has a primary decomposition. A ring R is *Laskerian* if it is Laskerian as an R-module over itself.

We denote the set of all prime ideals and the set of all maximal ideals of a ring R by $Spec(R)$ and $Max(R)$, respectively. The *Jacobson radical* $J(R)$ of a ring R is defined to be the intersection of all the maximal ideals of R. The set of all nilpotent elements of R is called the *nilradical* of R and denoted by $N(R)$. The Krull dimension of R is denoted by $dim(R)$. If I is a proper ideal of R, then $\sqrt{I}$ and $Min(I)$ denote the radical ideal of I and the set of prime ideals of R minimal over I, respectively.

A topological space X is said to be irreducible if $X \neq \emptyset$, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$. The maximal irreducible subsets of X are called irreducible components of X. A topological space X is said to be *Noetherian* if the ascending chain condition holds for open subsets of X. If $X = Spec(R)$ with the Zariski topology, then X is Noetherian if and only if R satisfies the ascending chain condition for radical ideals.

In Section 2, we investigate representable rings and show that these rings and Artinian rings have similar properties in some cases. We then show that representable rings are strictly between Artinian and semiperfect rings. Therefore, we determine the structure of these rings and characterize them. Next we characterize Artinian rings in terms of representable rings.

In Section 3, we consider modules for which all non-zero submodules are representable and called these modules completely representable. Then we give some necessary and sufficient conditions for a module to be completely representable.

In Section 4, we define strongly representable modules as modules that can be written as a direct sum of finitely many secondary submodules and then give some conditions such that a representable module is strongly representable.

2. Representable Rings

A ring R is *representable* if it is representable as a module over itself. In this section, we first show that representable rings are strictly between Artinian and semiperfect rings. Then we determine the structure of these rings. Finally we give some characterization of these rings and characterize Artinian rings in terms of representable rings. By [18, 5.2], any Artinian ring is representable. But the converse is not true.

Example 2.1. The ring $R = k[X_1, X_2, \dots]/(X_1, X_2, \dots)^2$, where k is a field, is representable (in fact it is $m = (x_1, x_2, \dots)$-secondary, where x_i denotes the equivalence class of X_i in R) but it is not Artinian, because it is not Noetherian.

Representable rings have similar properties to Artinian rings. Artinian rings are Noetherian. We show that representable rings are Laskerian.

Proposition 2.2. *Let R be a representable ring. We have the following statements.*

(i) *If R is a domain, then R is a field.*

(ii) $dim(R) = 0$.

(iii) $J(R) = N(R)$, *and hence* $J(R)$ *is a nil ideal.*

(iv) *R is Laskerian.*

(v) $Spec(R)$ *is Noetherian.*

Proof. (i) Let R be a representable domain and $R = I_1 + I_2 + \cdots + I_t$ be a minimal secondary representation of R, where I_i is P_i-secondary ($i = 1, \dots, t$). Let $r \in P_i$. Then there exists $n \geqslant 1$ such that $r^n I_i = 0$. But $I_i \neq 0$, hence $r = 0$. Thus, $P_i = 0$ for all i, and hence $R = I_1$. So, R is 0-secondary and hence is a field.
(ii) Let P be a prime ideal of R. Then R/P is a representable R-module and also a representable domain. Hence by part (i), R/P is a field. Thus, P is a maximal ideal.
(iii) It follows by part (ii).
(iv) Let I be a proper ideal of a representable ring R. Then R/I is a representable R-module. Let $R/I = J_1 + \cdots + J_t$ be a minimal secondary representation of R/I. Then $I = Ann(R/I) = \bigcap_{i=1}^{t} Ann(J_i)$ is a primary decomposition for I.
(v) It follows from (iv) and [10, Theorem 4]. □

Every representable ring has finitely many prime ideals. Indeed, prime ideals are maximal, and we have the following.

Proposition 2.3. *Let R be a Laskerian ring. If $dim(R) = 0$, then it has finitely many maximal ideals. In particular, every representable ring has finitely many maximal ideals.*

Proof. Since R is Laskerian, it follows from [10, Theorem 4] that R has Noetherian spectrum. Hence by [20, Theorem 3.A.16], the $Spec(R)$ has finitely many irreducible components and by [20, Corollary 3.A.14], these components are of the forms $V(P) = \{Q \in Spec(R) \mid P \subseteq Q\}$, where P is a minimal prime ideal of R. Since $dim(R) = 0$, $Min(R) = Max(R)$. Thus, $V(P) = \{P\}$, for every $P \in Min(R)$. Therefore, $Spec(R)$ is finite. The last statement follows from Proposition 2.2 (ii), (iv). □

In the following, we show that every representable ring is semiperfect. By [27, 42.6], a ring R is *semiperfect* if and only if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$.

Proposition 2.4. *Every representable ring is semiperfect.*

Proof. Let R be a representable ring and let $Max(R) = \{m_1, m_2, \ldots, m_t\} = Spec(R)$. By the Chinese Remainder Theorem, $R/J(R) \cong \bigoplus_{i=1}^{t} R/m_i$. Hence $R/J(R)$ is semisimple. By Proposition 2.2 (iii), $J(R)$ is a nil ideal. Thus by [27, 42.7], idempotents lift modulo $J(R)$. Therefore, R is a semiperfect ring. □

The converse of Proposition 2.4, is not true in general. For example, the ring of formal power series $R = k[[X]]$ where k is a field, is semiperfect (because it is local) but it is not representable (because $dim(R) \neq 0$). Thus, representable rings are strictly between Artinian and semiperfect rings.

In the following, we will determine the structure of representable rings. To this purpose we need the following.

Proposition 2.5. *Let R be a ring and $e \in R$ be an idempotent element. Then we have the following statements.*

(i) *Re is a ring with $1_{Re} = e$.*

(ii) *Every ideal of Re is of the form Ie, where I is an ideal of R.*

(iii) *If m is a maximal (prime) ideal of R and $e \notin m$, then me is a maximal (prime) ideal of Re.*

(iv) *If I is an P-secondary (as R-module) ideal of R, then Ie is either zero or Pe-secondary (as Re-module) ideal of Re.*

Proof. (i), (ii) and (iv) are easily proven by definitions. For (iii), one can consider ring isomorphism $Re/Ie \cong (R/I)(e + I)$. □

Lemma 2.6. *A ring R is representable and local if and only if it is secondary as an R-module.*

Proof. Let (R, m) be a representable and local ring. Then $m = N(R)$, and hence every elements of R is a unit or nilpotent. Thus, R is m-secondary. Conversely, if R is m-secondary, then $R \setminus U(R) = m$, where $U(R)$ is the set of unit elements of R. Thus, R is local with unique maximal ideal m. □

Lemma 2.7. *Let R be a representable ring with $Max(R) = \{m_1, \ldots, m_t\}$ and $e \in R$ be an idempotent element such that $e - 1 \in m_1$ and $e \in \bigcap_{i=2}^{t} m_i$. Then Re is a representable local ring with unique maximal ideal m_1e.*

Proof. According to Lemma 2.6, it is sufficient to show that the ring Re is m_1e-secondary. Let $R = I_1 + I_2 + \cdots + I_s$ be a minimal secondary representation for R, where I_i be m_i-secondary $(i = 1, \ldots, s)$. In fact, $s = t$. Because, if $s < t$ then for $r \in m_t \setminus \bigcup_{i=1}^{s} m_i$, we have $rR = rI_1 + \cdots + rI_s = I_1 + \cdots + I_s = R$. So r is unit, which is a contradiction. Since $e \notin m_1$ and $e \in m_i$ $(2 \leqslant i \leqslant s)$, we have, $Re = I_1e (= I_1)$. Therefore, Re is m_1e-secondary. □

Theorem 2.8. (Structure theorem for representable rings) *A representable ring is uniquely (up to isomorphism) a finite direct product of representable local rings.*

Proof. Let $Max(R) = \{m_1, \ldots, m_t\}$. Since $J(R)$ is a nil ideal, hence by [27, 42.7], every idempotent in $R/J(R)$ can be lifted to an idempotent in R. But by Chines Reminder Theorem, $R/J(R) \cong \prod_{i=1}^{t} R/m_i$. Thus, there exists a set $\{e_1, \ldots, e_t\}$ of orthogonal idempotents (i.e. $e_i^2 = e_i$ for all i and $e_i e_j = 0$ for $i \neq j$) in R such that $e_i - 1 \in m_i$ and $e_i \in \bigcap_{j \neq i}^{t} m_j$, for $i = 1, \ldots, t$. Thus, we have $(e_1 - 1)(e_2 - 1) \ldots (e_t - 1) \in J(R) = N(R)$. So there exists $n \geqslant 1$, such that $(e_1 - 1)^n (e_2 - 1)^n \ldots (e_t - 1)^n = 0$. Hence $e_1 + e_2 + \cdots + e_t = 1$. By [13, Exercise 24, page 135], we have $R \cong Re_1 \times \cdots \times Re_t$. But by Lemma 2.7, each Re_i is representable and local. Therefore, R is a finite direct product of representable local rings.
For uniqueness, suppose $R \cong \prod_{i=1}^{s} R_i$, where the (R_i, m_i) are representable local ($= m_i$-secondary) rings. For each i, we have a natural injective homomorphism $\varphi_i : R_i \longrightarrow R$. Let $I_i = Im(\phi_i)$. Then one can simply see that $R = I_1 + \cdots + I_s$ is a minimal secondary representation of R. On the other hand by proof of Lemma 2.7, $R = Re_1 + \cdots + Re_t$ is a minimal secondary representation of R and all the secondary components Re_i are isolated. Hence by 2nd uniqueness theorem for secondary representation (see [18, 3.2]), we have $s = t$ and $R_i \cong I_i \cong Re_i$ for all i, and the proof is complete. □

Corollary 2.9. *Let R be a representable ring. Then every non-zero ideal of R is representable (as an R-module).*

Proof. By Theorem 2.8, $R = R_1 \times \cdots \times R_n$, where R_i is a representable local ring (or equivalently secondary ring by Lemma 2.6). Let I be a non-zero ideal of R. Then $I = I_1 \times \cdots \times I_n$, where I_i is an ideal of R_i for all i. Since every non-zero ideal of a secondary ring is secondary, I_i is secondary for all i with $I_i \neq 0$. For simplicity, we can assume that all of I_i's are non-zero. Then $I = (I_1 \times 0 \times \cdots \times 0) + \cdots + (0 \times \cdots \times 0 \times I_n)$, is a secondary representation of I and the proof is complete. □

Proposition 2.10. *Let R be a ring. Then R is a representable ring if and only if $Spec(R)$ is Noetherian and for every non-zero ideal I of R and every minimal prime ideal P of $Ann(I)$, there exists $r \in R \setminus P$ such that rI is P-secondary.*

Proof. Let R be a representable ring. Then by Proposition 2.2 (v), $Spec(R)$ is Noetherian. Let I be a non-zero ideal of R and P be a minimal prime ideal of $Ann(I)$. By Corollary 2.9, I has a minimal secondary representation, say, $I = I_1 + \cdots + I_t$. Thus, $\sqrt{Ann(I)} = \bigcap_{i=1}^{t} P_i \subseteq P$. So, there exists i such that $P_i \subseteq P$, and hence $P_i = P$ because $dim(R) = 0$. By rearranging I_i's (if necessary), we can assume that $i = 1$. If $t = 1$, then $I = I_1$ is P-secondary and the proof is complete for $r = 1$. Otherwise, suppose $s \in (\bigcap_{i=2}^{t} P_i) \setminus P$. Then there exists $k \geqslant 1$ such that $s^k \in Ann(I_i)$ for all i, $2 \leqslant i \leqslant t$. Let $r = s^k$. Then $rI = rI_1 + \cdots + rI_t = rI_1 = I_1$ is P-secondary.
Conversely, let P_0 be a minimal prime ideal of $R = I_0$. Then by assumption, there exists $r_0 \in R \setminus P_0$ such that $r_0 R$ is P_0-secondary. Let $I_1 = (0 :_R r_0) = Ann_R(Rr_0)$. Since $r_0^2 I_0 = r_0 I_0$, so we have $R = r_0 I_0 + I_1$. If $I_1 = 0$, then $R = r_0 I_0$ is a representation of R. Otherwise, let P_1 be a minimal prime ideal of $Ann(I_1)$. Again by assumption, there exists $r_1 \in R \setminus P_1$ such that $r_1 I_1$ is P_1-secondary and hence

$I_1 = r_1 I_1 + I_2$, where $I_2 = (0 :_R r_1)$. Thus, $R = r_0 I_0 + r_1 I_1 + I_2$. If $I_2 = 0$, then $R = r_0 I_0 + r_1 I_1$ is representation of R. Otherwise we continue this process and claim that there exists some $n \geqslant 1$ such that $I_n = 0$. If not, then since $\cdots \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq I_0 = R$ and hence $\sqrt{Ann(R)} \subseteq \sqrt{Ann(I_1)} \subseteq \sqrt{Ann(I_2)} \subseteq \dots$. Since $Spec(R)$ is Noetherian, there exists $n \geqslant 1$ such that $\sqrt{Ann(I_n)} = \sqrt{Ann(I_{n+1})}$. Thus, there exists $t \geqslant 1$ such that $r_n^t I_n = 0$. But $r_n^t I_n = r_n I_n$. So $r_n \in Ann(I_n) \subseteq P_n$, which is a contradiction. Therefore, there exists $n \geqslant 0$ such that $R = r_0 I_0 + r_1 I_1 + \cdots + r_n I_n$, a representation of R and the proof is complete. □

In the next theorem we give some characterization of representable rings. A ring R is called von Neumann regular ring if for each $a \in R$, there exists $b \in R$ such that $a = a^2 b$. A ring R is said to be a Q-ring, if every ideal of R is a finite product of primary ideals. A ring R is called a completely packed ring if whenever $I \subseteq \bigcup_{\alpha \in \Gamma} P_\alpha$, where I is an ideal and P_α's are prime ideals of R then $I \subseteq P_\beta$, for some $\beta \in \Gamma$. It is well known that R is a compactly packed ring if and only if every prime ideal is the radical of a principle ideal [24, Theorem]. We need the following lemma.

Lemma 2.11. *Let $I \subseteq P$ be ideals of a ring R with P a prime ideal. Then the following statements are equivalent.*

(i) *P is a minimal prime ideal of I.*

(ii) *For each $x \in P$, there is $y \in R \setminus P$ and a positive integer n such that $yx^n \in I$.*

Proof. [12, Theorem 2.1, Page 2]. □

Theorem 2.12. *Let R be a ring. The following statements are equivalent.*

(i) *R is a representable ring.*

(ii) *$R/J(R)$ is a semisimple ring and $J(R)$ is a nil ideal.*

(iii) *$R/N(R)$ is a Noetherian and von Neumann regular ring.*

(iv) *R is a zero dimensional compactly packed ring.*

(v) *R has Noetherian spectrum and $dim(R) = 0$.*

(vi) *R is a zero dimensional Q-ring.*

Proof. $(i) \Rightarrow (ii)$ It follows from Propositions 2.2 and 2.4.
$(ii) \Rightarrow (iii)$ A ring is semisimple if and only if it is Noetherian Von Neumann regular ring [17, Theorem 4.25]. Since $J(R)$ is a nil ideal, $J(R) = N(R)$. Hence the result follows.
$(iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ [15, Theorem 2.12].
Now by Proposition 2.10, we show that the equivalent conditions (iv) and (v) imply (i).
Let I be a non-zero ideal of R and P a minimal prime ideal of $Ann(I)$. By [24, Theorem], there exists $a \in R$ such that $P = \sqrt{(a)}$. Now by Lemma 2.11, there exists

$r \in R \backslash P$ and $n \geqslant 1$ such that $ra^n \in Ann(I)$. We claim that rI is P-secondary. Since $r \notin P$, $rI \neq 0$. Let $s \in R$. If $s \in P$, then there exists $m \geqslant 1$ such that $s^m \in (a)$. Thus $s^m = ta$, for some $t \in R$. We have $s^{nm}(rI) = t^n a^n (rI) = t^n (ra^n I) = 0$. If $s \notin P$, since P is maximal, then $R = Rs + P$. Therefore, $1 = ys + x$, for some $y \in R$ and $s \in P$. Since $P = \sqrt{(a)}$, $s^k = za$ for some $k \geqslant 1$ and $z \in R$. Now we have $1 = (ys + x)^{kn} = \alpha s + x^{kn} = \alpha s + z^n a^n$, for some $\alpha \in R$. Thus, $rI = (\alpha sr + z^n a^n r)I = \alpha srI$. So, $s(rI) \subseteq rI = s(\alpha rI) \subseteq s(rI)$. Therefore, $s(rI) = rI$ and the proof is complete. □

The following result is the expected generalization of [3, Theorem 8.5].

Corollary 2.13. *A ring R is representable if and only if R is Laskerian and $dim(R) = 0$.*

Proof. It follows from Proposition 2.2, [10, Theorem 4] and Theorem 2.12 (v). □

Corollary 2.14. *Let R be a local ring. Then R is representable if and only if $dim(R) = 0$.*

Proof. It follows form Proposition 2.2 (ii) and Theorem 2.12 (iii). □

Corollary 2.15. *A ring R is reduced and representable if and only if R is a Noetherian von Neumann regular ring (or equivalently a semisimple ring).*

Proof. Let R be reduced (i.e. $N(R) = 0$) and representable. Then by Theorem 2.12 (iii), R is a Noetherian von Neumann regular ring. Conversely, if R is a Noetherian von Neumann regular ring, then it is semisimple [17, Theorem 4.25]. So it is a finite direct product of fields and hence is representable and reduced. □

Representability is not a local property. However, we have the following theorem.

Theorem 2.16. *Let R be a ring. Then R_m is a representable ring for all maximal ideal m of R if and only if $dim(R) = 0$.*

Proof. If $dim(R) = 0$ then by Corollary 2.14, R_m is a representable ring, for all maximal ideals m of R. Now let R_m be a representable ring, for all maximal ideal m of R. Let q be a prime ideal and m be a maximal ideal of R such that $q \subseteq m$. Then by Proposition 2.2 (ii), $dim(R_m) = 0$. So $q_m = m_m$, and hence $q = m$. This shows that $dim(R) = 0$. □

Corollary 2.17. *Let R be a ring. Then R is reduced and R_m is a representable ring, for all maximal ideal m of R if and only if R is a von Neumann regular ring.*

Proof. It follows from Theorem 2.16 and [12, Remark, Page 5]. □

Now we characterize Artinian rings in terms of representable rings.

Theorem 2.18. *Let R be a ring. The following statements are equivalent.*

(i) *R is an Artinian ring.*

(ii) *R is a representable ring and locally Noetherian.*

(iii) *R is a Noetherian ring and R_m is a representable ring, for all maximal ideals m of R.*

Proof. $(i) \Rightarrow (ii)$ Follows from [18, 5.2] and [3, Theorem 8.5].
$(ii) \Rightarrow (iii)$ By Theorem 2.12 (vi), R is a Q-ring with $dim(R) = 0$. Now by [14, Theorem 3] and Theorem 2.16, (iii) holds.
$(iii) \Rightarrow (i)$ Follows from Theorem 2.16 and [3, Theorem 8.5]. □

3. Completely Representable Modules

In this section, we consider modules that all non-zero submodules are representable. This is similar to definition of Laskerian modules and, in a sense, a dual of that notion. We call these modules "completely representable". Artinian modules (see [18]), representable modules over regular rings ([8, Theorem 2.3]) and modules over local rings that their maximal ideals are nilpotent, are examples of such modules.

Definition 3.1. An R-module M is said to be completely representable, if $M \neq 0$ and every non-zero submodule of M is representable. A ring R is completely representable, if it is completely representable as a module over itself.

Remark 3.2. Both representable and completely representable modules are generalizations of Artinian modules. But the second seems to be a better generalization, because we know that every submodule of an Artinian module is again Artinian, and we may want this feature to be preserved in generalization and this holds in definition of completely representable modules (but not in representable modules).

For the case of rings, representable and completely representable are the same.

Proposition 3.3. *A ring R is completely representable if and only if it is representable.*

Proof. If R is completely representable, then R is representable by definition. The converse is Corollary 2.9. □

By definition, every completely representable module is representable. But the converse is not true.

Example 3.4. The $\mathbb{Z}$-module $\mathbb{Q}$ is (0)-secondary and hence representable. But it is not completely representable, because the submodule $\mathbb{Z}$ is not representable.

In the next, we give some necessary and some sufficient conditions for a module to be completely representable. An R-module M is said to satisfy (*dccr*) if the descending chain $IN \supseteq I^2N \supseteq \dots$ terminates, for every submodule N of M and every finitely generated ideal I of R. The following result mentioned in [25, Proposition 3] without complete proof. We give its proof.

Proposition 3.5. *Let M be a completely representable R-module. Then M satisfies ($dccr$).*

Proof. By [25, Theorem, Page 2], M satisfies ($dccr$) if and only if for any submodule N of M and $a \in R$, $N = a^k N + (0 :_N a^k)$, for all large k. Let $N = N_1 + \cdots + N_t$ be a minimal secondary representation of N with N_i, P_i-secondary. If $a \notin \bigcup_{i=1}^t P_i$, then $aN_i = N_i$, for all i. Hence $N = aN$ and the proof is complete. Otherwise, let (by rearranging if necessary) $a \in \bigcap_{i=1}^l P_i$ and $a \notin \bigcup_{i=l+1}^t P_i$. Then there exists an integer k such that $a^k N_i = 0$, for all i, $1 \leqslant i \leqslant l$. So, $a^k N = N_{l+1} + \cdots + N_t$ and $N_1 + \cdots + N_l \subseteq (0 :_N a^k)$. Hence $N = a^k N + (0 :_N a^k)$ and the proof is complete. □

The following examples show that the converse of Proposition 3.5, is not true in general.

Example 3.6. Let R be a non Noetherian von Neumann regular ring (e.g. $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$). Then $dim(R) = 0$. So, by [26, Proposition 1.2], R satisfies ($dccr$). By Theorem 2.12, R is not representable. (Note that any von Neumann regular ring is reduced).

Example 3.7. Let $M = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ is the set of all prime numbers. Then by [26, Remark 1.10], M satisfies ($dccr$) condition as a $\mathbb{Z}$-module. But this module is not representable (and therefore, is not completely representable). Because if N is a $p\mathbb{Z}$-secondary submodule of M, for some $p \in \mathbb{P}$, then every component of every element of N is zero except probably the component that belongs to $\mathbb{Z}_p$. Obviously, the finite sum of this submodules can not be equal to M.

Therefore, the class of completely representable modules is strictly between modules that satisfies (dcc) (i.e. Artinian modules) and modules that satisfies ($dccr$) condition. Also, by Proposition 3.3 and [26, Proposition 1.2], representable rings are strictly between Artinian rings and rings with zero dimension.

Although, the converse of Proposition 3.5, is true under some additional conditions. A module M over a Noetherian ring R is said to have finite Goldie dimension, if M does not contain an infinite direct sum of non-zero submodules.

Proposition 3.8. *Let M be a module of finite Goldie dimension over a commutative Noetherian ring R. Then M is completely representable if and only if M satisfies ($dccr$).*

Proof. [25, Proposition 3]. □

Bourbaki in [4, Chap. IV, Sect. 2, Exercise 23, Page 295], give a necessary and sufficient conditions for a finitely generated module to be Laskerian. In the following we dualize these conditions for a module to be completely representable.

Let N be an R-module and S be a multiplicatively closed subset of R. We consider $S(N) = \bigcap_{r \in S} rN$. If P be a prime ideal of R and $S = R \setminus P$, we denote $S(N)$ by $S_P(N)$.

Proposition 3.9. *Let M be a completely representable R-module. Then we have*

(i) *For every non-zero submodule N of M and every multiplicatively closed subset S of R, there exists $r \in S$ such that $S(N) = rN$.*

(ii) *For every non-zero submodule N of M, every increasing sequence $(S_n(N))_{n\geqslant 1}$ is stationary, where $(S_n)_{n\geqslant 1}$ is any decreasing sequence of multiplicatively closed subset of R.*

Proof. (i) Let $N = \sum_{i=1}^{t} N_i$ be a minimal secondary representation with $\sqrt{Ann(N_i)} = P_i$, $(1 \leqslant i \leqslant t)$. If $S \cap P_i = \emptyset$ for all i, then for every $s \in S$, $sN_i = N_i$ for all i, and hence $sN = N$. Thus, $S(N) = N$ and (i) holds, for $r = 1 \in S$. Otherwise, there exists l, $1 \leqslant l < t$, such that $P_i \cap S = \emptyset$, $(1 \leqslant i \leqslant l)$ and $P_i \cap S \neq \emptyset$ $(l + 1 \leqslant i \leqslant t)$. Thus, for every $s \in S$, $sN = N_1 + \cdots + N_l + sN_{l+1} + \cdots + sN_t$. On the other hand, there exists $r \in S$ such that $rN_i = 0$, $(l + 1 \leqslant i \leqslant t)$. So $rN = N_1 + \cdots + N_l \subseteq S(N) \subseteq rN = N_1 + \cdots + N_l$. Thus, $S(N) = rN$ and (i) holds.
(ii) Suppose the contrary is true; i.e., there exists a decreasing sequence $(S_n)_{n\geqslant 1}$ such that $S_1(N) \subsetneqq S_2(N) \subsetneqq \dots$. Let $P_i \cap S_1 = \emptyset$ $(i = 1, \dots, l)$ and $P_i \cap S_1 \neq \emptyset$ $(i = l + 1, \dots, t)$. Since $S_2 \subseteq S_1$ and $S_1(N) \subsetneqq S_2(N)$, we have $S_2 \cap P_k = \emptyset$, for some k, $(l + 1 \leqslant k \leqslant t)$. If similarly continue this, then there exists some i, such that $S_i \cap P_j = \emptyset$ $(j = 1, 2, \dots, t)$, and so $N = S_i(N) = S_{i+1}(N) = \dots$, which is a contradiction. □

Corollary 3.10. *Let M be a completely representable R-module. Then for every non-zero submodule N of M and every minimal prime ideal P over $Ann(N)$, there exists $r \in R \setminus P$ such that rN is P-secondary.*

Proof. Let $S = R \setminus P$. Then by Proposition 3.9 (i), there exists $r \in R \setminus P$ such that $S_P(N) = rN$. Let $s \in R$. If $s \notin P$, $rN = S_P(N) \subseteq srN \subseteq rN$. Thus, $s(rN) = rN$. If $s \in P$, by Lemma 2.11, there exists $t \in R \setminus P$ and $n \geqslant 1$ such that $ts^n \in Ann(N)$. So, $ts^nN = 0$, and hence $ts^nrN = 0$. But $trN = rN$. Thus, $s^n(rN) = 0$. Therefore, rN is P-secondary. □

Conditions of Proposition 3.9 are sufficient for a finitely cogenerated $AB5^*$ module to be completely representable.

An R-module M is said to be finitely cogenerated if for every family $\{M_i\}_{i\in I}$ of submodules of M with $\bigcap_{i\in I} M_i = 0$, there is a finite subset $F \subseteq I$ with $\bigcap_{i\in F} M_i = 0$. It is clear that every submodule of a finitely cogenerated module is finitely cogenerated. Also if N and M/N are finitely cogenerated then so is M. Hence every direct sum of finitely cogenerated modules is finitely cogenerated.

A family $\{M_i\}_{i\in I}$ of submodules of a module M is called inverse (direct) if, for all $i, j \in I$ there exists $k \in I$ such that $M_k \subseteq M_i \cap M_j$ $(M_i + M_j \subseteq M_k)$. For example, every chain of submodules is an inverse and direct family. The module M is said to be satisfy the $AB5^*$ (AB5) condition (and is called an $AB5^*$ (AB5) module) if, for every submodule K of M and every inverse (direct) family $\{M_i\}_{i\in I}$ of submodules of M, $K + \bigcap_{i\in I} M_i = \bigcap_{i\in I}(K + M_i)$ $(K \cap (\sum_{i\in I} M_i) = \sum_{i\in I} K \cap M_i)$.

Every Artinian module is finitely cogenerated and $AB5^*$. For more information about this class of modules one can see [6, 9, 19, 27].

Theorem 3.11. *Let M be a finitely cogenerated $AB5^*$ module. Then M is completely representable if and only if*

(i) *For every non-zero submodule N of M and every minimal prime P over $Ann(N)$, there exists $r \in R \setminus P$ such that $S_P(N) = rN$ (or equivalently rN is P-secondary).*

(ii) *For every non-zero submodule N of M, every increasing sequence $(S_n(N))_{n\geqslant 1}$ is stationary, where $(S_n)_{n\geqslant 1}$ is any decreasing sequence of multiplicatively closed subset of R.*

Proof. ($\Rightarrow$) It follows by Proposition 3.9. ($\Leftarrow$) Let conditions (i) and (ii) holds and let N be a non-zero submodule of M. Let P_1 be a minimal prime of $Ann(N)$. Then by (i), and by proof of Corollary 3.10, there exists $r_1 \in R \setminus P_1$ such that $Q_1 = r_1 N$ is P_1-secondary. Let $N'_1 = (0 :_N r_1)$. Since $r_1^2 N = r_1 N$, so we have $N = Q_1 + N'_1$. If $N'_1 = 0$, then $N = Q_1$ is representable. Otherwise, let $\Sigma = \{0 \neq G \leq M \mid N = Q_1 + G, G \subseteq N'_1\}$. $N'_1 \in \Sigma$, and hence $\Sigma \neq \emptyset$. Since M is finitely cogenerated and $AB5^*$ module, so every chain in Σ has a lower bound. Hence by Zorn's lemma, Σ has a minimal element with respect to inclusion, N_1 say. Let P_2 be a minimal prime of $Ann(N_1)$ and $r_2 \in R \setminus P_2$ such that $Q_2 = r_2 N_1$ is P_2-secondary. Then $N_1 = Q_2 + N'_2$ where $N'_2 = (0 :_{N_1} r_2)$. Thus, $N = Q_1 + Q_2 + N'_2$. If $N'_2 = 0$, $N = Q_1 + Q_2$ is representable. Otherwise, let $\Sigma = \{0 \neq G \leq M \mid N = Q_1 + Q_2 + G, G \subseteq N'_2\}$. Again by Zorn's lemma, Σ has a minimal element N_2 with respect to inclusion. We continue this process and claim that there exists $n \geqslant 1$ such that $N'_n = 0$. Suppose on the contrary that $N'_n \neq 0$ for all n. Let $S_n = R \setminus \bigcup_{i=1}^{n} P_i$ $(n = 1, 2, 3, \dots)$. We show that $S_n \cap Ann(N_n) \neq \emptyset$. If $S_n \cap Ann(N_n) = \emptyset$, then $Ann(N_n) \subseteq \bigcup_{i=1}^{n} P_i$ and by Prime Avoidance Theorem, $Ann(N_n) \subseteq P_i$ for some i, $1 \leqslant i \leqslant n$. But, $\cdots \subseteq N_2 \subseteq N'_2 \subseteq N_1 \subseteq N'_1 \subseteq N$. So, $N_n \subseteq N_i \subseteq N'_i$, and hence, $Ann(N'_i) \subseteq Ann(N_i) \subseteq Ann(N_n) \subseteq P_i$. Thus, $r_i \in P_i$, a contradiction. Therefore, $S_n \cap Ann(N_n) \neq \emptyset$. Let $s \in S_n \cap Ann(N_n)$. Then $Q_1 + \cdots + Q_n \subseteq S_n(N) \subseteq sN = Q_1 + \cdots + Q_n$, so; $S_n(N) = Q_1 + \cdots + Q_n$ $(n = 1, 2, \dots)$. Now we have a decreasing sequence $(S_n)_{n\geqslant 1}$ of multiplicatively closed subsets of R, such that the sequence $(S_n(N))_{n\geqslant 1}$ is strictly increasing. Because if $S_n(N) = S_{n+1}(N)$ for some n; then $Q_{n+1} \subseteq Q_1 + \cdots + Q_n$. Hence, $N = Q_1 + \cdots + Q_n + N_{n+1}$. But, $N_{n+1} \subseteq N_n$, and hence by minimality of N_n, $N_{n+1} = N_n$. Since, $r_{n+1} N'_{n+1} = 0$, thus $r_{n+1} N_{n+1} = 0$. So, $r_{n+1} N_n = 0$, and hence $r_{n+1} \in Ann(N_n) \subseteq P_{n+1}$. Thus, $r_{n+1} \in P_{n+1}$, which is a contradiction. Therefore, the sequence $(S_n(N))_{n\geqslant 1}$ is not stationary, which contradicts condition (ii). Thus, there exists $n \geqslant 1$ such that $N'_n = 0$, and hence $N = Q_1 + \cdots + Q_n$ is representable. □

Remark 3.12. We note that the notion of "completely representable" is the dual of the notion "primary decomposition". In some basic theorems for primary decomposition, the authors assume that the condition "finitely generated" (see [4, Chap.

IV, Sect. 2, Exercise 23]). Also the AB5 condition is true for all modules (see [19, Lemma 6.22]). Since the "cofinitely generated" and "AB5*" are the dual notions of "finitely generated" and "AB5" conditions respectively, so it is natural to use these conditions in the proof or results about "completely representable".

Proposition 3.13. *Let M be a finitely cogenerated $AB5^*$ module and N be a non-zero submodule of M. Then M is completely representable if and only if both N and M/N are completely representable.*

Proof. If M is completely representable, then it is straightforward to show that N and M/N are completely representable. Conversely, Let N and M/N be completely representable. Let K be a non-zero submodule of M and P be a minimal prime of $Ann(K)$. By condition (i) of Proposition 3.9, there exist r_1, $r_2 \in R \setminus P$ such that $S_P(K \cap N) = r_1(K \cap N)$ and $S_P(K/K \cap N) = r_2(K/K \cap N)$. Let $r = r_1r_2$. We show that $S_P(K) = rK$. If $\alpha \in rK$, then there exists $x \in K$ such that $\alpha = rx$. We have

$$r_2(x + K \cap N) \in r_2(K/K \cap N) = S_P(K/K \cap N).$$

So,

$$r_2x + K \cap N \in \bigcap_{s \in R\setminus P} s(K/K \cap N) = \bigcap_{s\in R\setminus P}((sK + K \cap N)/K \cap N) = \bigcap_{s\in R\setminus P}(sK + K \cap N)/K \cap N.$$

Thus, $r_2x \in \bigcap_{s\in R\setminus P}(sK + K \cap N)$. Hence,

$$\alpha = r_1r_2x \in r_1(\bigcap_{s\in R\setminus P}(sK + K \cap N)) \subseteq \bigcap_{s\in R\setminus P}(r_1sK + r_1(K \cap N)).$$

On the other hand, it is obvious that for every $s \in R \setminus P$, $r_1(K \cap N) = r_1s(K \cap N)$. So

$$\alpha \in \bigcap_{s\in R\setminus P}(sK + r_1sK) = \bigcap_{s\in R\setminus P} sK = S_P(K).$$

Thus, $\alpha \in S_P(K)$. Now we check condition (ii) of Theorem 3.11. Let $(S_n)_{n\geqslant 1}$ be a decreasing sequence of multiplicatively closed subset of R. Then there exists $n \geqslant 1$ and $r_1, r_2 \in S_{n+1}$ such that $S_{n+1}(K \cap N) = S_n(K \cap N) = r_1(K \cap N)$ and $S_{n+1}(K/K \cap N) = S_n(K/K \cap N) = r_2(K/K \cap N)$. Let $r = r_1r_2$. Then $S_n(K) = rK = S_{n+1}(K)$. Thus, conditions (i) and (ii) of Theorem 3.11 are hold. So M is completely representable. □

Proposition 3.14. *Let M_1, M_2,. . . ,M_n be R-modules such that are finitely cogenerated, $AB5^*$ and completely representable. If $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is $AB5^*$, then M is completely representable.*

Proof. We prove this by induction on n. For $n = 1$, $M = M_1$ is completely representable. Now assume the assertion is true for $n = k$. For $n = k + 1$, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k \oplus M_{k+1}$. Since M is $AB5^*$, $M' = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ is $AB5^*$. Hence by induction hypothesis, it is completely representable. Since

$M' \cong M/M_{k+1}$ and M_{k+1} are completely representable, by Proposition 3.13, M is completely representable and the proof is complete by induction. □

Note that finite direct sum of $AB5^*$ modules need not to be $AB5^*$ ([6, Lemma 2.5]).

A family of sets has the finite intersection property if every finite subfamily has a nonempty intersection. A module M is linearly compact (with respect to the discrete topology) if every collection of cosets of submodules of M which has the finite intersection property has non-empty intersection. Linearly compact modules are $AB5^*$ (see [9, Lemma 7.1]). Also every finite direct sum of linearly compact modules is linearly compact (see [27, 29.8(2)]). So, we have the following Corollary.

Corollary 3.15. *Let M_1, M_2,..., M_n be linearly compact, finitely cogenerated and completely representable R-modules. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is completely representable.*

For relation between linearly compact and $AB5^*$ modules, see ([1],[2],[6]).

Heinzer, in [11, Proposition 2.1], gives a new restatement of Bourbaki's conditions. Inspired by this, we have the following results.

Theorem 3.16. *Let M be an R-module such that $Spec(R/Ann(M))$ is Noetherian and for all non-zero submodule N of M, there exists a minimal prime P of $Ann(N)$ and $r \in R \setminus P$ such that rN is P-secondary. Then M is completely representable.*

Proof. Let N be a non-zero submodule of M. Then by assumption, there exists a minimal prime P_1 of $Ann(N)$ and $r_1 \in R \setminus P_1$ such that r_1N is P_1-secondary. Let $N_1 = (0 :_N r_1)$. Since $r_1^2N = r_1N$, so we have $N = r_1N + N_1$. If $N_1 = 0$, then $N = r_1N$ is representation of N. Otherwise, there exists a minimal prime P_2 of $Ann(N_1)$ and $r_2 \in R \setminus P_2$ such that r_2N_1 is P_2-secondary and $N_1 = r_2N_1 + N_2$, where $N_2 = (0 :_{N_1} r_2)$. Thus, $N = r_1N + r_2N_1 + N_2$. If $N_2 = 0$, then $N = r_1N + r_2N_1$ is representation of N. Otherwise, we continue this process and claim that there exists some $n \geqslant 1$ such that $N_n = 0$. If not, then since $\cdots \subseteq N_3 \subseteq N_2 \subseteq N_1 \subseteq N \subseteq M$ and hence $Ann(M) \subseteq \sqrt{Ann(M)} \subseteq \sqrt{Ann(N)} \subseteq \sqrt{Ann(N_1)} \subseteq \sqrt{Ann(N_2)} \subseteq \ldots$. Since $Spec(R/Ann(M))$ is Noetherian so there exists $n \geqslant 1$ such that $\sqrt{Ann(N_n)} = \sqrt{Ann(N_{n+1})}$. Thus, there exists $t \geqslant 1$ such that $r_{n+1}^t N_n = 0$. But $r_{n+1}^t N_n = r_{n+1}N_n$. So $r_{n+1} \in Ann(N_n) \subseteq P_{n+1}$, $r_{n+1} \in P_{n+1}$, which is a contradiction. Therefore, there exists $n \geqslant 1$ such that $N = r_1N + r_2N_1 + \cdots + r_nN_{n+1}$, a representation of N and the proof is complete. □

For the case of rings, these conditions are also necessary.

Theorem 3.17. *A ring R is completely representable if and only if $Spec(R)$ is Noetherian and for every non-zero ideal I of R, there exists minimal prime P of $Ann(I)$ and $r \in R \setminus P$ such that rI is P-secondary.*

Proof. ($\Leftarrow$) It follows from Theorem 3.16.
($\Rightarrow$) Let R be completely representable. Then R is representable and by Proposition 2.2 (*iv*), it is Laskerian. So by [10, Theorem 4], R has Noetherian spectrum. Now by Corollary 3.10, the proof is complete. □

Remark 3.18. Note that by Corollary 3.10, Theorem 3.17, is the same as Proposition 2.10.

In [7], authors have considered representable linearly compact modules. In the next theorem we give a necessary and sufficient conditions for a finitely cogenerated linearly compact module to be completely representable.

Theorem 3.19. *Let M be a finitely generated, linearly compact and finitely cogenerated R-module. Then M is completely representable if and only if $Spec(R/Ann(M))$ is Noetherian and for all non-zero submodule N of M, there exists a minimal prime ideal P of $Ann(N)$ and $r \in R \setminus P$ such that rN is P-secondary.*

Proof. ($\Leftarrow$) Theorem 3.16.
($\Rightarrow$) By Corollary 3.10, second condition is true. We show that $R/Ann(M)$ has Noetherian spectrum. Let $\{x_1, x_2, ..., x_n\}$ be a set of generators of M. Let $\varphi : R \longrightarrow M \oplus M \oplus \cdots \oplus M$ by $r \longmapsto (rx_1, rx_2, \ldots, rx_n)$. Then φ is an R-homomorphism, and hence $R/Ann(M)$ is a submodule of $M \oplus M \oplus \cdots \oplus M$. But by Corollary 3.15, $M \oplus M \oplus \cdots \oplus M$ is completely representable. So, $R/Ann(M)$ is a representable ring. Therefore, by [10, Theorem 4], $R/Ann(M)$ has Noetherian spectrum and the proof is complete. □

In the following, we show that, if $dim(R) = 0$, then the converse of Theorem 3.16 is true. For this purpose, we need the following lemmas.

Lemma 3.20. *Let I be a primary ideal of a ring R such that every regular element of R/I is unit. Then R/I is a secondary ring.*

Proof. By [23, Lemma 4.3], R/I is non-zero and every zero divisor in R/I is nilpotent. Thus, every element of R/I is unit or nilpotent. Hence R/I is a secondary ring. □

Lemma 3.21. *Let R be a ring with $dim(R) = 0$. Then every regular element of R is unit.*

Proof. Let $r \in R$ be a regular element. By [26, Proposition 1.2], R satisfies $(dccr)$. Thus, there exists integer $n \geqslant 1$ such that $Rr^n = Rr^{n+1}$. So $r^n = r^{n+1}s$, for some $s \in R$. Since r is regular, we have $rs = 1$. □

Proposition 3.22. *Let M be a representable R-module and $M = N_1 + \cdots + N_t$ be a minimal secondary representation of M, with N_i, P_i-secondary $(1 \leqslant i \leqslant t)$. Let $I_i = Ann(N_i)$, $(1 \leqslant i \leqslant t)$. Suppose, P_i's are pairwise comaximal and every regular element of R/I_i, $(1 \leqslant i \leqslant t)$, is unit. Then $Spec(R/Ann(M))$ is Noetherian.*

Proof. Since $(\sqrt{I_i} =)P_i$'s are pairwise comaximal, by [3, Proposition 1.16], I_i's are also pairwise comaximal. Therefore, by Chines Reminder Theorem, $R/Ann(M) \cong \bigoplus_{i=1}^{t} R/I_i$. So, by Lemma 3.20, $R/Ann(M)$ is representable. Thus, by Proposition 2.2 (v), $Spec(R/Ann(M))$ is Noetherian. □

Corollary 3.23. *Let R be a ring with $dim(R) = 0$. Then R-module M is completely representable if and only if $Spec(R/Ann(M))$ is Noetherian and for every non-zero* *submodule N of M and every minimal prime ideal P over $Ann(N)$, there exists $r \in R \setminus P$ such that rN is P-secondary.*

Proof. ($\Rightarrow$) Follows form Proposition 3.22 and Corollary 3.10.
($\Leftarrow$) Follows form Theorem 3.16. □

4. Strongly Representable Modules

By definition of representable modules, we can also consider the following definition.

Definition 4.1. Let M be an R-module. We say M is a strongly representable module if there exists secondary submodules $N_1, \ldots, N_t$ such that $M = N_1 \oplus \cdots \oplus N_t$.

Obviously, every strongly representable module is representable. But the converse is not true in general, see [21, Example 2.4]. We give some condition such that representable modules be strongly representable.

Theorem 4.2. *Let M be a representable R-module such that the elements of $Att(M)$ are pairwise comaximal. Then M is strongly representable.*

Proof. Let $M = N_1 + \cdots + N_t$ be a minimal secondary representation of M with N_i, P_i-secondary ($1 \leqslant i \leqslant t$). Thus, $Att(M) = \{P_1, \ldots, P_t\}$. Let $I_i = Ann(N_i)$ for $i = 1, \ldots, t$. Since ($\sqrt{I_i} =$)P_i's are pairwise comaximal, I_i's are also pairwise comaximal by [3, Proposition 1.16]. We show that $N_i \bigcap(\sum_{j \neq i} N_j) = 0$, for all $i = 1, \ldots, t$. For simplicity, let $i = 1$. Suppose $x \in N_1 \bigcap(\sum_{j=2}^{t} N_j)$. By [23, Proposition 3.59], $I_1 + \bigcap_{j=2}^{t} I_j = R$. So, there exists $\alpha \in I_1$ and $\beta \in \bigcap_{j=2}^{t} I_j$ such that $\alpha + \beta = 1$. Thus, $x = 1x = (\alpha + \beta)x = \alpha x + \beta x = 0 + 0 = 0$. Therefore, $M = N_1 \oplus \cdots \oplus N_t$. Hence M is strongly representable. □

Corollary 4.3. *Let M be a representable R-module such that $Att(M) \subseteq Max(R)$. Then M is strongly representable.*

Proof. Every two distinct maximal ideals are comaximal. Hence Corollary follows form Theorem 4.2. □

Corollary 4.4. *Let R be a ring with $dim(R) = 0$. Then every representable R-module is strongly representable. In particular, every representable ring is strongly representable.*

Proof. If $dim(R) = 0$, then $Max(R) = Spec(R)$ and result follows from Corollary 4.3. The "in particular" statement follows from Proposition 2.2 (*ii*). □

Remark 4.5. According to Proposition 3.3 and Corollary 4.4, representable, completely representable and strongly representable rings are the same.

Proposition 4.6. *Let R be a domain with $dim(R) = 1$ (e.g. R be a Dedekind domain) and M be a representable R-module such that contains no non-zero divisible submodule. Then M is strongly representable.*

Proof. Over a domain, divisible modules and 0-secondary modules are the same. Since M contains no divisible submodule, $Att(M) \subseteq Spec(R) \setminus \{0\} = Max(R)$. Hence, result follows from Corollary 4.3. □

Corollary 4.7. *Every representable module over a Dedekind domain is strongly representable.*

Proof. By [16, Theorem 8], every module M over a Dedekind domain R can be decomposed as $M = D \oplus E$, where D is divisible and E has no non-zero divisible submodules. If M is representable, then $E(\cong M/D)$ is also representable (if non-zero). Hence result follows form Proposition 4.6. □

Finally, we show that every finitely generated Artinian module is strongly representable. For this, we need the following lemma.

Lemma 4.8. *Let P be a maximal ideal of a ring R and M be an R-module such that $P^nM = 0$, for some integer $n \geqslant 1$. Then M is a P-secondary R-module.*

Proof. Let $r \in R$. If $r \in P$, then $r^nM = 0$. If $r \notin P$, then since P is maximal, $P + Rr = R$. Hence $1 = \alpha + sr$, for some $\alpha \in P$ and $s \in R$. Thus, $1 = \alpha^n + \gamma r$, for some $\gamma \in R$. So, for every $x \in M$, $x = \alpha^n x + \gamma rx = \gamma rx$. Hence $rM = M$. □

Proposition 4.9. *Every finitely generated Artinian module is strongly representable.*

Proof. By [18, 6.3], if M is an Artinian R-module, then there exist (distinct) maximal ideals $m_1, \ldots, m_t$ of R such that $M = M(m_1) \oplus \cdots \oplus M(m_t)$, where $M(I) = \bigcup_{n=1}^{\infty}(0 :_M I^n) = \{x \in M | I^n x = 0, \exists n \geqslant 1\}$, for every ideals I of R. If M is finitely generated, then every $M(m_i)$ is finitely generated and hence will be annihilate by some power of m_i. So by Lemma 4.8, $M(m_i)$ is m_i-secondary $(1 \leqslant i \leqslant t)$. Therefore, $M = M(m_1) \oplus \cdots \oplus M(m_t)$ is a secondary representation of M and M is strongly representable. □

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