KYUNGPOOK Math. J. 62(2022), 485-495
https://doi.org/10.5666/KMJ.2022.62.3.485
pISSN 1225-6951 eISSN 0454-8124

Generalized Quasi-Einstein Metrics and Contact Geometry

Gour Gopal Biswas
Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India
e-mail : ggbiswas6@gmail.com

Uday Chand De*
Department of Pure Mathematics, University of Calcutta, 35 Ballygaunge Circular Road, Kolkata -700019, West Bengal, India
e-mail : uc_de@yahoo.com

Ahmet Yildiz
Education Faculty, Department of Mathematics, Inonu University, 44280, Malatya, Turkey
e-mail : a.yildiz@inonu.edu.tr

Abstract. The aim of this paper is to characterize K-contact and Sasakian manifolds whose metrics are generalized quasi-Einstein metric. It is proven that if the metric of a K-contact manifold is generalized quasi-Einstein metric, then the manifold is of constant scalar curvature and in the case of a Sasakian manifold the metric becomes Einstein under certain restriction on the potential function. Several corollaries have been provided. Finally, we consider Sasakian 3-manifold whose metric is generalized quasi-Einstein metric.

1. Introduction

If the Ricci tensor S of a Riemannian manifold (M^n, g), $n > 2$, satisfies the condition $Ric = \lambda g$, λ being a constant, then the manifold is named an Einstein manifold. According to Besse [4] this condition is called Einstein metric condition. The study of Einstein manifolds and their generalizations are very interesting in Riemannian and semi-Riemannian geometry. There are several generalizations of Einstein metric such as quasi-Einstein metric [8], m-quasi-Einstein metric [9],

* Corresponding Author.
Received February 15, 2021; revised January 30, 2022; accepted February 10, 2022.
2010 Mathematics Subject Classification: 53C25, 53C15.
Key words and phrases: GQE metrics, Almost contact manifolds, Contact manifolds, K-contact manifolds, Sasakian manifolds.

(m, ρ)-quasi-Einstein metric [18], generalized quasi-Einstein metric [10] and many others.

The idea of generalized quasi-Einstein metric in a Riemannian manifold of dimension n is introduced by Catino [10]. A metric g of an $n(> 2)$ dimensional Riemannian manifold M^n is called a *generalized quasi-Einstein metric* (shortly, GQE metric) if there exist three smooth functions ψ, α, β such that

$$S + H^{\psi} - \alpha d\psi \otimes d\psi = \beta g, \tag{1.1}$$

where H^{ψ} is the Hessian of the function ψ defined by

$$H^{\psi}(E, F) = g(\nabla_E grad\, \psi, F)$$

for all vector fields E, F in M^n. Here ∇ is the Riemannian connection and *grad* denotes the gradient operator. Obviously, when ψ is a constant, the metric becomes an Einstein metric.

For individual values of α and β, we get different type of metrics. They are

i) *Gradient Ricci soliton* [7] for $\alpha = 0$ and $\beta \in \mathbb{R}$,

ii) *Gradient almost Ricci soliton* [1] for $\alpha = 0$ and $\beta \in C^{\infty}(M^n)$,

iii) *Gradient ρ-Einstein soliton* [11] for $\alpha = 0$, $\beta = \rho r + \lambda$ and $\lambda \in \mathbb{R}$, r being the scalar curvature,

iv) *m-quasi-Einstein metric* for $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$,

v) *gradient generalized m-quasi metric* [2] for $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in C^{\infty}(M^n)$,

vi) *(m, ρ)-quasi-Einstein metric* for $\alpha = \frac{1}{m}$, $m > 0$, $\beta = \rho r + \lambda$ and $\lambda \in \mathbb{R}$.

The idea of a gradient ρ-Einstein soliton is introduced by Catino and Mazzieri [11] and studied in the papers ([12], [19]). In the paper [27], Venkatesha and Kumara studied gradient ρ-Einstein solitons on almost Kenmotsu manifolds. In [13], Chen studied m-quasi-Einstein structure in almost cosymplectic manifolds.
In the paper [10], Catino gave a local characterization of GQE metric with harmonic Weyl tensor and $C(grad\, \psi, \cdot, \cdot) = 0$, where C is the Weyl tensor. He proved that if the metric of a manifold (M^n, g), $n \geq 3$ is a GQE metric with harmonic Weyl tensor and $C(grad\, \psi, \cdot, \cdot) = 0$, then M is locally warped product with $(m-1)$-dimensional Einstein fibers around any regular point of ψ. Recently, GQE manifolds have been studied by Mirshafeazadeh and Bidabad ([22], [23]). So far our knowledge goes, contact or paracontact manifolds whose metrics are GQE metric have not been investigated. In the present paper we attempt to characterize K-contact and Sasakian manifolds whose metrics are GQE metric.

At first we obtain the expression of *Riemannian curvature tensor* and *Ricci tensor* in a Riemannian manifold whose metric is GQE metric. Then we provide our main theorems. In the proof of the theorems we assume that the potential function ψ remains invariant under the characteristic vector field ξ, that is, $\pounds_\xi \psi = 0$, which implies that $\xi\psi = 0$, $\pounds_\xi$ being the Lie-derivative in the direction of ξ. Precisely we prove the following theorems:

Theorem 1.1. *The scalar curvature of a K-contact manifold with GQE metric is constant, provided the potential function ψ remains invariant under the characteristic vector field ξ.*

Theorem 1.2. *A Sasakian manifold with GQE metric is an Einstein manifold, provided the potential function ψ remains invariant under the Reeb vector field ξ.*

2. Preliminaries

Let M^{2n+1} be a smooth manifold. Let η be a 1-form, ξ be a vector field and φ be a $(1,1)$-tensor field. The triple (η, ξ, φ) is called an *almost contact structure* (*acs*) if

$$I = -\varphi^2 + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I is the identity map. Obviously $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. The *acs* is called *normal* if the almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J\left(E, \gamma\frac{d}{dt}\right) = \left(\varphi E - \gamma\xi, \eta(E)\frac{d}{dt}\right)$$

for all $E \in \chi(M^{2n+1})$ and $\gamma \in C^\infty(M^{2n+1} \times \mathbb{R})$, is integrable. Here $\chi(M^{2n+1})$ denotes the tangent space of M^{2n+1}. Blair [5] proved that the acs is normal if and only if $[\varphi, \varphi] + 2\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ denotes the *Nijenhuis tensor* of φ defined by

$$[\varphi, \varphi](E, F) = \varphi^2[E, F] + [\varphi E, \varphi F] - \varphi[\varphi E, F] - \varphi[E, \varphi F], \ \forall\, E, F \in \chi(M^{2n+1}).$$

If there exists a Riemannian metric g on M^{2n+1} such that

$$g = g(\varphi\,\cdot, \varphi\,\cdot) + \eta \otimes \eta, \tag{2.2}$$

then the manifold M^{2n+1} together with (η, ξ, φ, g) is said to be an *almost contact metric manifold* (shortly, *acm* manifold). On *acm* manifold we can define the fundamental 2-form Φ defined by $\Phi = g(\cdot, \varphi\,\cdot)$. When $d\eta = \Phi$, the acm manifold is called a contact metric (*cm*) manifold. On a *cm* manifolds $\eta \wedge (d\eta)^n$ is a non-vanishing $(2n+1)$-form. Contact metric manifolds have been studied by several authors such as ([14], [15], [20], [24]-[26], [28]) and many others.

Given a *cm* manifold M^{2n+1} we can define a symmetric $(1,1)$-tensor field $h = \frac{1}{2}\pounds_\xi \varphi$, where $\pounds_\xi$ denotes the Lie derivative along the vector field ξ, which satisfy

$$h\xi = 0, \quad h\varphi + \varphi h = 0 \tag{2.3}$$

$$\nabla_E \xi = -\varphi E - \varphi h E \tag{2.4}$$

$$(\nabla_E \varphi)F + (\nabla_{\varphi E}\varphi)\varphi F = 2g(E,F)\xi - \eta(F)(E + hE + \eta(E)\xi) \tag{2.5}$$

for all $E, F \in \chi(M^{2n+1})$. We denote R for Riemannian curvature tensor and Q for Ricci operator defined by

$$R(E,F) = [\nabla_E, \nabla_F] - \nabla_{[E,F]}, \tag{2.6}$$

$$S(E,F) = g(QE,F).$$

According to Blair [5] $h = 0$ if and only if the Reeb vector field ξ is Killing. If ξ is a Killing vector field, then the *cm* manifold M^{2n+1} is called *K-contact* manifold [5]. On a K-contact manifold M^{2n+1} the following relations hold:

$$\nabla_E \xi = -\varphi E \tag{2.7}$$

$$Q\xi = 2n\xi \tag{2.8}$$

$$R(\xi, E)F = (\nabla_E \varphi)F \tag{2.9}$$

$$(\nabla_E \varphi)F + (\nabla_{\varphi E}\varphi)\varphi F = 2g(E,F)\xi - \eta(F)(E + \eta(E)\xi) \tag{2.10}$$

for all $E, F \in \chi(M^{2n+1})$. Taking covariant derivative of (2.8) along $E \in \chi(M^{2n+1})$, we obtain

$$(\nabla_E Q)\xi = Q\varphi E - 2n\varphi E. \tag{2.11}$$

Since ξ is Killing, $\pounds_\xi Q = 0$. By direct computation

$$(\nabla_\xi Q)E = Q\varphi E - \varphi Q E. \tag{2.12}$$

A normal *cm* manifold is said to be a *Sasakian manifold*. A necessary and sufficient condition for an *acm* manifold M^{2n+1} to be Sasakian is that

$$(\nabla_E \varphi)F = g(E,F)\xi - \eta(F)E \tag{2.13}$$

for all $E, F \in \chi(M^{2n+1})$. A *cm* manifold is Sasakian if and only if

$$R(E,F)\xi = \eta(F)E - \eta(E)F. \tag{2.14}$$

Every Sasakian manifold is K-contact manifold, but the converse is not true, in general. However in 3-dimensional manifold K-contact and Sasakian manifolds are equivalent [21]. The relations (2.7)-(2.10) also hold in Sasakian manifolds. The Ricci tensor in a Sasakian 3-manifold is given by [6]

$$S = \frac{r-2}{2}g + \frac{6-r}{2}\eta \otimes \eta. \tag{2.15}$$

From the above we see that if $r = 6$ then the manifold is an Einstein manifold and conversely. Since in a three dimensional manifold, Einstein and space of constant sectional curvature are equivalent, a Sasakian 3-manifold is of *constant sectional curvature* 1 if and only if $r = 6$.

3. Generalized Guasi-Einstein Metric in a Riemannian Manifold

In this section we deduce the expression of R and S on a Riemannian manifold with GQE metric.

Proposition 3.1. *In a Riemannian manifold* (M^{2n+1}, g) *with GQE metric, the tensors* R *and* S *satisfy*

$$\begin{aligned} R(E,F)grad\,\psi &= (\nabla_F Q)E - (\nabla_E Q)F + (E\beta)F - (F\beta)E \\ &+ \{(E\alpha)(F\psi) - (F\alpha)(E\psi)\}grad\,\psi \\ &- \alpha\{(F\psi)QE - (E\psi)QF\} + \alpha\beta\{(F\psi)E - (E\psi)F\} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} (1-\alpha)S(F, grad\,\psi) &= \frac{1}{2}(Fr) - 2n(F\beta) - g(grad\,\psi, grad\,\psi)(F\alpha) \\ &+ \{g(grad\,\alpha, grad\,\psi) - \alpha r + 2n\alpha\beta\}(F\psi) \end{aligned} \tag{3.2}$$

for all $E, F \in \chi(M^{2n+1})$.

Proof. From (1.1) it follows that

$$\nabla_F grad\,\psi = -QF + \beta F + \alpha g(grad\,\psi, F)grad\,\psi. \tag{3.3}$$

where H^ψ= Hessian of the function ψ is defined by

$$H^\psi(E,F) = g(\nabla_E grad\,\psi, F)$$

for all vector fields E, F in M^{2n+1}. Taking covariant derivative of (3.3) in the direction $E \in \chi(M^{2n+1})$, we obtain

$$\begin{aligned} \nabla_E \nabla_F grad\,\psi &= -\nabla_E(QF) + (E\beta)F + \beta\nabla_E F + (E\alpha)g(grad\,\psi, F)grad\,\psi \\ &+ \alpha(Eg(grad\,\psi, F))grad\,\psi + \alpha g(grad\,\psi, F)\nabla_E grad\,\psi. \end{aligned} \tag{3.4}$$

Interchanging E and F in the foregoing equation, we have

$$\begin{aligned} \nabla_F \nabla_E grad\,\psi &= -\nabla_F(QE) + (F\beta)E + \beta\nabla_F E + (F\alpha)g(grad\,\psi, E)grad\,\psi \\ &+ \alpha(Fg(grad\,\psi, E))grad\,\psi + \alpha g(grad\,\psi, E)\nabla_F grad\,\psi. \end{aligned} \tag{3.5}$$

Using (3.3)-(3.5) in (2.6), we get (3.1). Contracting the equation (3.1) and applying the well known formulas $Er = tr\{F \to (\nabla_E Q)F\}$ and $\frac{1}{2}Er = div\, Q$, we get the second result. □

4. Proof of the Main Results

Proof of the Theorem 1.1. Replacing E by ξ in (3.1) and using (2.11) and (2.12), we have

$$\begin{aligned} R(\xi, F)grad\,\psi &= \varphi QF - 2n\varphi F + (\xi\beta)F - (F\beta)\xi \\ &+ \{(\xi\alpha)(F\psi) - (F\alpha)(\xi\psi)\}grad\,\psi \\ &- \alpha\{2n(F\psi)\xi - (\xi\psi)QF\} + \alpha\beta\{(F\psi)\xi - (\xi\psi)F\}. \end{aligned} \tag{4.1}$$

Taking inner product of the foregoing equation with E and using (2.9), we infer

$$\begin{aligned} -g((\nabla_F\varphi)E, grad\,\psi) &= g(\varphi QF, E) - 2ng(E, \varphi F) \\ &+ (\xi\beta)g(E, F) - (F\beta)\eta(E) \\ &+ (\xi\alpha)(E\psi)(F\psi) - (\xi\psi)(E\psi)(F\alpha) \\ &- \alpha\{2n(F\psi)\eta(E) - (\xi\psi)g(QF, E)\} \\ &+ \alpha\beta\{(F\psi)\eta(E) - (\xi\psi)g(E, F)\}. \end{aligned} \tag{4.2}$$

Replacing E by φE and F by φF in (4.2), entail that

$$\begin{aligned} -g((\nabla_{\varphi F}\varphi)\varphi E, grad\,\psi) &= g(Q\varphi F, E) - 2ng(E, \varphi F) \\ &+ (\xi\beta)g(\varphi E, \varphi F) + (\xi\alpha)g(\varphi E, grad\,\psi)g(\varphi F, grad\,\psi) \\ &- (\xi\psi)g(\varphi E, grad\,\psi)g(\varphi F, grad\,\alpha) \\ &+ \alpha(\xi\psi)g(Q\varphi F, \varphi E) - \alpha\beta(\xi\psi)g(\varphi E, \varphi F). \end{aligned} \tag{4.3}$$

Adding (4.2) and (4.3) and using (2.10), we get

$$\begin{aligned} &-2g(E, F)(\xi\psi) + \eta(E)((F\psi) + \eta(F)(\xi\psi)) \\ &= g(\varphi QF + Q\varphi F, E) - 4ng(E, \varphi F) \\ &+ (\xi\beta)(g(E, F) + g(\varphi E, \varphi F)) - (F\beta)\eta(E) \\ &+ (\xi\alpha)((E\psi)(F\psi) + g(\varphi E, grad\,\psi)g(\varphi F, grad\,\psi)) \\ &- (\xi\psi)((E\psi)(F\alpha) + g(\varphi E, grad\,\psi)g(\varphi F, grad\,\alpha)) \\ &- \alpha\{2n(F\psi)\eta(E) - (\xi\psi)g(QF, E)\} \\ &+ \alpha\beta\{(F\psi)\eta(E) - (\xi\psi)g(E, F)\} \\ &+ \alpha(\xi\psi)g(Q\varphi F, \varphi E) - \alpha\beta(\xi\psi)g(\varphi E, \varphi F). \end{aligned} \tag{4.4}$$

Anti-symmetrizing the above equation, it follows that

$$\begin{aligned} &(1 + 2n\alpha - \alpha\beta)((F\psi)\eta(E) - (E\psi)\eta(F)) \\ &= 2g(\varphi QF + Q\varphi F, E) - 8ng(E, \varphi F) \\ &+ (E\beta)\eta(F) - (F\beta)\eta(E) + (\xi\psi)((E\alpha)(F\psi) - (F\alpha)(E\psi)) \\ &+ (\xi\psi)(g(\varphi E, grad\,\alpha)g(\varphi F, grad\,\psi) - g(\varphi F, grad\,\alpha)g(\varphi E, grad\,\psi)). \end{aligned} \tag{4.5}$$

Now we assume that the potential function ψ remains invariant under the characteristic vector field ξ, that is, $\xi\psi = 0$. Then the equation (4.5) reduces to

$$\begin{aligned} & (1+2n\alpha-\alpha\beta)((F\psi)\eta(E)-(E\psi)\eta(F)) \\ = \ & 2g(\varphi QF+Q\varphi F,E)-8ng(E,\varphi F)+(E\beta)\eta(F)-(F\beta)\eta(E). \end{aligned} \tag{4.6}$$

Replacing E by φE and F by φF in the equation (4.6), we infer

$$\varphi QF + Q\varphi F = 4n\varphi F \tag{4.7}$$

for all vector field F on M^{2n+1}. Suppose $\{e_1, e_2, \cdots, e_n, \varphi e_1, \varphi e_2, \cdots, \varphi e_n, \xi\}$ is a φ-basis of (M^{2n+1}, g). Then $g(\varphi Q e_i, \varphi e_i) = g(Qe_i, e_i)$ for $i = 1, 2, \cdots, n$. We compute

$$\begin{aligned} r &= \sum_{i=1}^{n} g(Qe_i, e_i) + \sum_{i=1}^{n} g(Q\varphi e_i, \varphi e_i) + g(Q\xi, \xi) \\ &= \sum_{i=1}^{n} g(\varphi Q e_i + Q\varphi e_i, \varphi e_i) + 2n \\ &= 2n(2n+1). \end{aligned}$$

This finishes the proof.

Suppose $d\alpha \wedge d\psi = 0$. Then

$$(E\alpha)(F\psi) - (F\alpha)(E\psi) = 0 \tag{4.8}$$

for all $E, F \in \chi(M^{2n+1})$, which implies $(E\alpha)grad\,\psi - (E\psi)grad\,\alpha = 0$, that is, $grad\,\alpha$ and $grad\,\psi$ are collinear. Conversely, if $grad\,\alpha$ and $grad\,\psi$ are collinear then $d\alpha \wedge d\psi = 0$. Using (4.8) in (4.5), we get

$$\begin{aligned} & (1+2n\alpha-\alpha\beta)((F\psi)\eta(E)-(E\psi)\eta(F)) \\ = \ & 2g(\varphi QF+Q\varphi F,E)-8ng(E,\varphi F) \\ + \ & (E\beta)\eta(F)-(F\beta)\eta(E). \end{aligned}$$

Proceeding in the similar way as in the proof of Theorem 1.1, it follows that the manifold is of constant scalar curvature. Hence, we can state the following:

Corollary 4.1. *The scalar curvature of a K-contact manifold with GQE metric is constant, provided $grad\,\alpha$ and $grad\,\psi$ are collinear.*

Proof of the Theorem 1.2. Let (M^{2n+1}, g) be a Sasakian manifold with GQE metric. In a Sasakian manifold the relation $\varphi Q = Q\varphi$ holds. Therefore $\nabla_\xi Q = 0$.

Using (2.13) in (4.2), we get

$$\begin{aligned}-g(E,F)(\xi\psi)+(F\psi)\eta(E) &= g(\varphi QF,E)-2ng(E,\varphi F)\\ &+ (\xi\beta)g(E,F)-(F\beta)\eta(E)\\ &+ (\xi\alpha)(E\psi)(F\psi)-(\xi\psi)(E\psi)(F\alpha)\\ &- \alpha\{2n(F\psi)\eta(E)-(\xi\psi)g(QF,E)\}\\ &+ \alpha\beta\{(F\psi)\eta(E)-(\xi\psi)g(E,F)\}. \end{aligned} \tag{4.9}$$

Anti-symmetrizing the equation (4.9), we infer

$$\begin{aligned} &(1+2n\alpha-\alpha\beta)((F\psi)\eta(E)-(E\psi)\eta(F))\\ =\ & 2g(\varphi QF,E)-4ng(E,\varphi F)\\ +\ & (E\beta)\eta(F)-(F\beta)\eta(E)\\ +\ & (\xi\psi)\{(E\alpha)(F\psi)-(F\alpha)(E\psi)\}. \end{aligned} \tag{4.10}$$

Replacing E by φE and F by φF in (4.10), we have

$$\begin{aligned} 0 &= 2g(\varphi QF,E)-4ng(E,\varphi F)\\ &+ (\xi\psi)\{g(\varphi E, grad\,\alpha)g(\varphi F, grad\,\psi)-g(\varphi F, grad\,\alpha)g(\varphi E, grad\,\psi)\}. \end{aligned} \tag{4.11}$$

Again replacing E by φE in (4.11) and applying (2.8), we obtain

$$\begin{aligned} S(E,F) &= 2ng(E,F)-\frac{1}{2}(\xi\psi)\{g(\varphi^2E, grad\,\alpha)g(\varphi F, grad\,\psi)\\ &- g(\varphi F, grad\,\alpha)g(\varphi^2 E, grad\,\psi)\} \end{aligned} \tag{4.12}$$

for all vector fields E,F on M^{2n+1}. Contracting the equation, we get

$$r=2n(2n+1)+(\xi\psi)g(\varphi(grad\,\alpha), grad\,\psi).$$

Suppose that, the potential function ψ remains invariant under the characteristic vector field ξ, that is, $\xi\psi=0$. Then from (4.12), we see that $S=2ng$.
This finishes the proof.

If $\alpha=0$ and $\beta\in\mathbb{R}$, from (4.12) we see that $S=2ng$ and the manifold is an Einstein manifold. Also the equation (4.10) reduces to $(F\psi)\eta(E)-(E\psi)\eta(F)=0$, that is $grad\,\psi=(\xi\psi)\xi$. Now applying $g(\nabla_{\varphi E} grad\,\psi, \varphi F)=g(\nabla_{\varphi F} grad\,\psi, \varphi E)$, we obtain $(\xi\psi)g(E,\varphi F)=0$. This implies $\xi\psi=0$. Therefore $grad\,\psi=0$, that is, ψ is constant. Thus, we can state that:

Corollary 4.2. *A Sasakian manifold whose metric satisfies gradient Ricci soliton equation is an Einstein manifold and the potential function is constant.*

The corollary 4.2 has been proved by He and Zhu [17].
If $\alpha=0$ and $\beta\in C^\infty(M)$, from the equation (4.12) we have $S=2ng$. Then the equation becomes $(F\psi)\eta(E)-(E\psi)\eta(F)=(E\beta)\eta(F)-(F\beta)\eta(E)$. Thus for any $E\perp\xi$, we have $g(grad\,\psi,E)=-(E\beta)$. Since β is a non-zero function, ψ is non-constant. Also $grad\,\psi$ is not perpendicular to $E\perp\xi$. Thus we get the following:

Corollary 4.3. *A Sasakian manifold whose metric satisfies gradient almost Ricci soliton equation is an Einstein manifold. Moreover, neither ψ is a constant function nor grad ψ is perpendicular to the vector field $E \perp \xi$.*

The second part of the Corollary 4.3 is also proved in the paper [3].
If $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\beta \in C^{\infty}(M)$, from (4.12), we see that $S = 2ng$. Thus, we can state that:

Corollary 4.4. *A Sasakian manifold with m-quasi-Einstein metric is an Einstein manifold.*

The above result has also been obtained in [16].
Now we consider GQE metric on Sasakian 3-manifold. Using (2.15) in (4.11), it follows that

$$\begin{aligned} 0 &= (r-6)g(E,\varphi F) + (\xi\psi)\{g(\varphi E, grad\,\alpha)g(\varphi F, grad\,\psi) \\ &- g(\varphi F, grad\,\alpha)g(\varphi E, grad\,\psi)\}. \end{aligned} \tag{4.13}$$

If the potential function ψ remains invariant under the characteristic vector field ξ, from the above equation we have $r = 6$. Thus, we can state that

Corollary 4.5. *A Sasakian 3-manifold with GQE metric is a manifold of constant sectional curvature 1, provided the potential function ψ remains invariant under the Reeb vector field ξ.*

Remark 1. It can be easily shown that in a 3-dimensional Sasakian manifold the φ-sectional curvature is equal to $\frac{r-4}{2}$. Under the hypothesis of Corollary 4.5, we can prove that the scalar curvature of a 3-dimensional Sasakian manifold is constant. Therefore the φ-sectional curvature is constant and the manifold becomes a 3-dimensional Sasakian space form [5], provided the potential function remains invariant under the Reeb vector field ξ.

Acknowledgements. The authors would like to thank the referees and editor for reviewing the paper carefully and their valuable comments to improve the quality of the paper.

References

[1] A. Barros, R. Batista and E. Ribeiro, Jr, *Rigidity of gradient almost Ricci solitons,* Illinois J. Math., **56**(2012), 1267–1279.

[2] A. Barros and J. N. Gomes, *A compact gradient generalized quasi-Einstein metric with constant scalar curvature,* J. Math. Anal. Appl., **401**(2013), 702–705.

[3] N. Basu and B. Bhattacharyya, *Gradient Ricci almost soliton in Sasakian manifolds,* Acta Mathematica Academiae Paedagogicae Nyíregyháziensis, **32**(2016), 161–164.

[4] A. L. Besse, *Einstein manifolds,* Springer-Verlag, Berlin(1987).

[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in math., **203**(2010), Birkhauser.

[6] D. E. Blair, T. Koufogiorgos and R. Sharma, *A classification of 3-dimensional contact metric manifold with* $Q\varphi = \varphi Q$*,* Kodai Math. J., **33**(2010), 361–368.

[7] H. D. Cao, *Recent progress on Ricci soliton,* Adv. Lect. Math., **11**(2009), 1–38.

[8] J. Case, *The non-existence of quasi-Einstein metrics,* Pacific J. Math., **248**(2010), 277–284.

[9] J. Case, Y. -J. Shu and G. Wei, *Rigidity of quasi-Einstein metrics,* Diff. Geom. Appl., **29**(2011), 93–100.

[10] G. Catino, *Generalized quasi-Einstein manifolds with harmonic Weyl tensor,* Math. Z., **271**(2012), 751–756.

[11] G. Catino and L. Mazzieri, *Gradient Einstein solitons,* Non-linear anal., **132**(2016), 66–94.

[12] G. Catino, L. Mazzieri and S. Mongodi, *Rigidity of gradient Einstein shrinkers,* Commun. Contemp. Math., **17**(2015), 1550046, 18pp.

[13] X. M. Chen, *Quasi-Einstein structures and almost cosymplectic manifolds,* RACSAM, **72**(2020), 14pp.

[14] U. C. De and P. Majhi, *On a type of contact metric manifolds,* Lobachevskii J. Math., **34**(2013), 89–98.

[15] U. C. De and A. Sarkar, *On the quasi-conformal curvature tensor of a* (κ, μ)*-contact metric manifold,* Math. Reports, **14**(2012), 115–129.

[16] A. Ghosh, *Generalized* m*-quasi-Einstein metric within the framework of Sasakian and* K*-contact manifolds,* Ann. Polon. Math., **115**(2015), 33–41.

[17] C. He and M. Zhu, *Ricci solitons on Sasakian manifolds,* arXiv : 1109.4407V2 [Math.DG] 26 sep 2011.

[18] G. Huang and Y. Wei, *The classification of* (m, ρ)*-quasi-Einstein manifolds,* Ann. Global Anal. Geom., **44**(2013), 269–282.

[19] G. Huang, *Integral pinched gradient shrinking* ρ*-Einstein solitons,* J. Math. Anal. Appl., **451**(2017), 1025–1055.

[20] J. B. Jun and A. De, *On* $N(\kappa)$*-contact metric manifolds satisfying certain curvature conditions,* Kyungpook Math. J., **51**(2011), 457–468.

[21] J. B. Jun, I. -B. Kim and U. K. Kim, *On 3-dimensional almost contact metric manifolds,* Kyungpook Math. J., **34**(1999), 293–301.

[22] M. A. Mirshafeazadeh and B. Bidabad, *On generalized quasi-Einstein manifolds,* Adv. Pure Appl. Math., **10**(2019), 193–202.

[23] M. A. Mirshafeazadeh and B. Bidabad, *On the Rigidity of generalized quasi-Einstein manifolds,* Bull. Malays. Math. Sci. Soc., **43**(2020), 2029–2042.

[24] C. Özgür, *Contact metric manifolds with cyclic-parallel Ricci tensor,* Differ. Geom. Dyn. Syst., **4**(2002), 21–25.

[25] C. Özgür and S. Sular, *On* $N(\kappa)$*-contact metric manifolds satisfying certain conditions,* SUT J. Math., **44**(2008), 89–99.

[26] R. Sharma, *Certain results on K-contact and (κ, μ)-contact manifolds,* J. Geom., **89**(2008), 138–147.

[27] V. Venkatesha and H. A. Kumara, *Gradient ρ-Einstein soliton on almost Kenmotsu manifolds,* Ann. Univ. Ferrara Sez. VII Sci. Mat., **65**(2019), 375–388.

[28] A. Yıldız, U. C. De, C. Murathan and K. Arslan, *On the Weyl projective curvature tensor of an $N(\kappa)$-contact metric manifold,* Mathematica Pannonica, **21**(2010), 129–142.