KYUNGPOOK Math. J. 62(2022), 509-531
https://doi.org/10.5666/KMJ.2022.62.3.509
pISSN 1225-6951 eISSN 0454-8124

On Ruled Surfaces with a Sannia Frame in Euclidean 3-space

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Abstract. In this paper we define a new family of ruled surfaces using an othonormal Sannia frame defined on a base consisting of the striction curve of the tangent, the principal normal, the binormal and the Darboux ruled surface. We examine characterizations of these surfaces by first and second fundamental forms, and mean and Gaussian curvatures. Based on these characterizations, we provide conditions under which these ruled surfaces are developable and minimal. Finally, we present some examples and pictures of each of the corresponding ruled surfaces.

1. Introduction

A surface is the image of a function with two real variables in three dimensional space. Geometric shapes such as planes, cylinders, cones, and spheres are examples of surfaces. Surfaces are used in such applications as architectural structures, computer graphics, works of art, geometric design, textile and automobile design. Surface theory is an important field of study in differential geometry; the basic theory can be found, for example, [1, 2, 3]. Developable surfaces, in particular, are widely used in industrial applications. Ruled surface have also been widely studied, [4, 10]. Ruled surfaces are called linear surfaces because they are formed by moving a line along a curve, so are represented by an infinite family of straight lines. A generalization of ruled surfaces was introduced by Juza in the 1960s and has since by well studied [5]. The striction point, the striction curve and the dis-

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Received November 25, 2021; revised February 14, 2022; accepted February 22, 2022.
2020 Mathematics Subject Classification: 53A04, 53A05.
Key words and phrases: Ruled surfaces, Sannia frame, striction curve, mean curvature, Gaussian curvature.

tribution parameter (Drall) of a ruled surface with a Frenet frame in 3-dimensional Euclidean space were considered in [6, 7]. Some characteristic properties of a ruled surface with a Frenet frame of a non-cylindrical ruled surface were investigated by Ouarab and Chahdi [8]. On the other hand, Pottmann and Wallner expressed the orthonormal Sannia frame on the striction curve of a ruled surface in 3-dimensional Euclidean space [9].
The aim of this study is to examine a ruled surface with the orthonormal Sannia frame defined on the striction curve of the tangent, normal, binormal and Darboux ruled surfaces.

2. Preliminaries

Let E^3 be a 3-dimensional Euclidean space provided with the standard flat metric given by

$$<,>= dx_1^2 + dx_2^2 + dx_3^2,$$

where $x = (x_1, x_2, x_3)$ is a rectangular coordinate system of E^3. Let α be a space curve with respect to the arclength s in E^3, and let T, N and B be the tangent, principal normal and binormal unit vectors at a point $\alpha(s)$ of the curve α, respectively. There exists an orthogonal frame $\{T, N, B\}$ which satisfies the Frenet-Serret equations,

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, \tag{2.1}$$

where κ is the curvature, τ is the torsion of the curve α [2]. The surface obtained by a line r moving along a differentiable curve α is called a ruled surface and its parametric equation is given by

$$X(s, v) = \alpha(s) + vr(s). \tag{2.2}$$

The curve α is called the base curve and the straight line r is called the ruling of the ruled surface [11]. Specifically, if the Frenet vectors of the curve are taken instead of r, the equations of the surfaces are obtained by

$$\begin{aligned} X_T(s, v) &= \alpha(s) + vT(s), \\ X_N(s, v) &= \alpha(s) + vN(s), \\ X_B(s, v) &= \alpha(s) + vB(s). \end{aligned}$$

The normal vector field, the components of first and second fundamental forms, the Gaussian curvature and the mean curvature of a surface are computed as

$$u_X = \frac{X_s \times X_v}{\|X_s \times X_v\|}, \tag{2.3}$$

$$E = \langle X_s, X_s \rangle, \quad F = \langle X_s, X_v \rangle,$$
$$G = \langle X_v, X_v \rangle, l = \langle X_{ss}, u_X \rangle, \tag{2.4}$$
$$m = \langle X_{sv}, u_X \rangle, \quad n = \langle X_{vv}, u_X \rangle,$$

$$K = \frac{ln - m^2}{EG - F^2}, \quad H = \frac{En - 2Fm + Gl}{2(EG - F^2)}, \tag{2.5}$$

respectively [11]. Frenet vectors of a curve make an instantaneous rotation along the curve and around an axis that is called as the axis of rotation. The Darboux vector W points in the direction of the rotational axis and is calculated by

$$W = \tau T + \kappa B.$$

The unit Darboux vector C, on the other hand, can be computed as following

$$C = \sin\varphi T + \cos\varphi B, \quad \sin\varphi = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}, \quad \cos\varphi = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}$$

where the angle φ is between the Darboux vector W and the binormal vector B of the moving curve [12]. The parametric equation of the ruled surface created by the moving the vector C along the curve α is

$$X_C(s, v) = \alpha(s) + vC(s).$$

If there exist a common perpendicular to two consecutive ruling in the ruled surface, then the foot of the common perpendicular on the main ruling is called a striction point and the set of these points is also defined as the striction curve. The equation of the striction curve of the ruled surface given in (2.2) can be written by

$$\beta(s) = \alpha(s) - \frac{\langle \alpha', r' \rangle}{\|r'\|^2} r \tag{2.6}$$

[6]. Specifically, if the striction curve is taken to be the base curve on the surface, then the parametric equation of the ruled surface is given as

$$X(s, v) = \beta(s) + vr(s).$$

Let the curve β be a striction curve of the ruled surface $X(s, v)$. The Sannia orthonormal frame [9] is the orthanormal frame $\{e_1, e_2, e_3\}$ created by unit vectors along the striction curve β such that

$$e_1 = r, \quad e_2 = \frac{e'_1}{\|e'_1\|}, \quad e_3 = e_1 \wedge e_2, \tag{2.7}$$

where r is the ruling of the ruled surface $X(s, v)$. The Sannia formulae along the striction curve become

$$e_1' = k_1 e_2, \quad e_2' = -k_1 e_1 + k_2 e_3, \quad e_3' = -k_2 e_2,$$

where k_1 and k_2 are the curvature and the torsion of the striction curve of the ruled surface $X(s,v)$ [9].

3. Ruled Surfaces with Sannia Frames

In this section, we examine the ruled surfaces formed by Sannia frames along the striction curves of ruled surfaces generated by Frenet vectors of a curve. The surfaces obtained are called Sannia ruled surfaces. The relation between the Sannia and Frenet frame, the first and second fundamental forms, and the Gaussian and the mean curvatures of each ruled surface are calculated separately.

3.1. Sannia ruled surfaces associated with tangent ruled surface

Theorem 3.1. *Let X_T be a tangent ruled surface and $\{e_1, e_2, e_3\}$ be a Sannia frame on the striction curve of X_T. Then, the relationship between the Sannia frame $\{e_1, e_2, e_3\}$ on striction curve and the Frenet frame $\{T, N, B\}$ is as follows:*

$$e_1 = T, \quad e_2 = N, \quad e_3 = B. \tag{3.1}$$

Proof. Let the curve ζ be a striction curve of the tangent ruled surface X_T. Using (2.6), it can be easily shown that the striction curve ζ is equal to the base curve of X_T, i.e. $\zeta(s) = \alpha(s)$. Therefore, the equation (3.1) is satisfied. □

Definition 3.2. A surface Φ_1 is called a e_1 Sannia ruled surface in Euclidean 3-space, if the surface Φ_1 is generated by moving the vector e_1 along the striction curve ζ of X_T and its parametric equation is defined as

$$\Phi_1(s,v) = \zeta(s) + v e_1(s). \tag{3.2}$$

Taking the partial differential of Φ_1 with respect to s and v, we get

$$\Phi_{1s} = T + v\kappa N \text{ and } \Phi_{1v} = T.$$

By (2.3), the normal vector field of Φ_1, which is denoted by u_{e_1}, is found as

$$u_{e_1}(s,v) = -B.$$

Theorem 3.3. *Let Φ_1 be a e_1 Sannia ruled surface in E^3. Then, the first and the second fundamental form, the Gaussian curvature and the mean curvature of Φ_1 are calculated as*

$$\begin{aligned} I_{e_1} &= \left(1 + v^2\kappa^2\right) ds^2 + 2ds dv + dv^2, \\ II_{e_1} &= -v\kappa\tau ds^2, \\ K_{e_1} &= 0, \quad H_{e_1} = \frac{\tau}{2v\kappa}, \end{aligned}$$

$\kappa \neq 0$, *respectively.*

Proof. Taking the second order partial differentials of the surface Φ_1 given by (3.2) with respect to s and v, we get

$$\Phi_{1ss} = \kappa N + v\left(-\kappa^2 T + \kappa' N + \kappa\tau B\right),$$
$$\Phi_{1sv} = \kappa N, \quad \Phi_{1vv} = 0.$$

Using the equation (2.4), the components of the first and the second fundamental form of Φ_1 are obtained as follows:

$$E_{e_1} = 1 + v^2\kappa^2, \quad F_{e_1} = 1, \quad G_{e_1} = 1,$$
$$l_{e_1} = -v\kappa\tau, \quad m_{e_1} = 0, \quad n_{e_1} = 0.$$

From here, if the last equations are substituted in the equation (2.5), the proof is complete. □

Corollary 3.4. *Let X_T and Φ_1 be a tangent ruled surface with base curve α and a e_1 Sannia ruled surface with base curve ζ which is striction curve of X_T, respectively. Then, the surfaces X_T and Φ_1 are the same surfaces.*

Corollary 3.5. *Let X_T and Φ_1 be the tangent ruled surface with base curve α and e_1 Sannia ruled surface with base curve ζ which is striction curve of X_T, respectively. If the striction curve ζ of X_T is planar curve, the e_1 Sannia ruled surface is developable and the minimal surface.*

Definition 3.6. A surface Φ_2 is called a e_2 Sannia ruled surface in Euclidean 3-space, if the surface Φ_2 is generated by moving the vector e_2 along the striction curve ζ of X_T and its parametrical equation is defined as

$$\Phi_2(s,v) = \zeta(s) + v e_2(s). \tag{3.3}$$

Taking the first order partial differentials of Φ_2 with respect to s and v, we have

$$\Phi_{2s} = (1 - v\kappa) T + v\tau B \text{ and } \Phi_{2v} = N.$$

So, by (2.3), the normal vector field u_{e_2} of Φ_2 is obtained as

$$u_{e_2}(s,v) = \frac{-v\tau T + (1 - v\kappa) B}{\sqrt{v^2\tau^2 + (1 - v\kappa)^2}}.$$

Theorem 3.7. *Let Φ_2 be a e_2 Sannia ruled surface in E^3. Then, the first and the second fundamental forms, the Gaussian curvature and the mean curvature of Φ_2*

are given as

$$I_{e_2} = \left((1 - v\kappa)^2 + (v\tau)^2\right) ds^2 + dv^2,$$

$$II_{e_2} = \frac{v^2\left(\tau\kappa' - \tau'\kappa\right) + v\tau'}{\sqrt{v^2\tau^2 + (1 - v\kappa)^2}} ds^2 + \frac{2\tau}{\sqrt{v^2\tau^2 + (1 - v\kappa)^2}} dsdv,$$
$$K_{e_2} = -\frac{\tau^2}{\left(v^2\tau^2 + (1 - v\kappa)^2\right)^2}, \quad H_{e_2} = \frac{v^2\left(\tau\kappa' - \tau'\kappa\right) + v\tau'}{2\left(v^2\tau^2 + (1 - v\kappa)^2\right)^{\frac{3}{2}}},$$

respectively.

Proof. Taking the second order partial differentials of the surface Φ_2 given by (3.3) with respect to s and v, we get

$$\Phi_{2ss} = \kappa N + v\left(-\kappa' T - \left(\kappa^2 + \tau^2\right) N + \tau' B\right),$$
$$\varphi_{2sv} = -\kappa T + \tau B, \quad \varphi_{2vv} = 0.$$

From equations (2.4), the components of the first and the second fundamental form of Φ_2 are obtained as follows:

$$E_{e_2} = (1 - v\kappa)^2 + (v\tau)^2, \quad F_{e_2} = 0, \quad G_{e_2} = 1,$$

$$l_{e_2} = \frac{v^2\left(\tau\kappa' - \tau'\kappa\right) + v\tau'}{\sqrt{v^2\tau^2 + (1 - v\kappa)^2}}, \quad m_{e_2} = \frac{\tau}{\sqrt{v^2\tau^2 + (1 - v\kappa)^2}}, \quad n_{e_2} = 0.$$

From here, if these equations are substituted in the equation (2.5), the proof is complete. □

Corollary 3.8. *Let X_T and Φ_2 a be tangent ruled surface with base curve α and e_2 Sannia ruled surface with base curve ζ which is striction curve of X_T, respectively. If the striction curve ζ of X_T is planar curve, the ruled surface Φ_2 with the Sannia frame is developable and the minimal surface. Also, since $K_{e_2} < 0$, all points of the ruled surface Φ_2 are hyperbolic points.*

Definition 3.9. A surface Φ_3 is called a e_3 Sannia ruled surface in Euclidean 3-space, if the surface Φ_3 is generated by moving the vector e_3 along the striction curve ζ of X_T and its parametrical equation is defined as

$$\Phi_3(s, v) = \zeta(s) + ve_3(s). \tag{3.4}$$

Taking the first order partial differentials of Φ_3 with respect to s and v, we have

$$\Phi_{3_s} = T - v\tau N \text{ and } \Phi_{3v} = B.$$

So, by considering (2.3) the normal vector field u_{e_3} of Φ_3 is obtained as

$$u_{e_3}(s,v) = -\frac{v\tau T + N}{\sqrt{1+(v\tau)^2}}.$$

Theorem 3.10. *Let Φ_3 be a e_3 Sannia ruled surface in E^3. Then, the first and the second fundamental forms, the Gaussian curvature and the mean curvature of Φ_3 are obtained as*

$$I_{e_{31}} = \left(1+v^2\kappa^2\right) ds^2 + dv^2,$$

$$II_{e_3} = -\frac{\kappa\left(1+v^2\tau^2\right) - v\tau'}{\sqrt{1+(v\tau)^2}} ds^2 + \frac{2\tau}{\sqrt{1+(v\tau)^2}} dsdv,$$

$$K_{e_3} = -\frac{\tau^2}{\left(1+v^2\kappa^2\right)^2}, \quad H_{e_3} = -\frac{\kappa\left(1+v^2\tau^2\right) - v\tau'}{2(1+v^2\kappa^2)^{\frac{3}{2}}},$$

respectively.

Proof. Taking the second order partial differentials of the surface Φ_3 given by (3.4) with respect to s and v, we reach

$$\Phi_{3ss} = v\tau\kappa T + (\kappa - v\tau') N - v\tau^2 B,$$
$$\Phi_{3sv} = -\tau N, \quad \Phi_{3vv} = 0.$$

So, by recalling the equation (2.4), the components of the first and the second fundamental form of Φ_3 are given as follows:

$$E_{e_3} = 1+v^2\tau^2, \quad F_{e_3} = 0, \quad G_{e_3} = 1,$$

$$l_{e_3} = -\frac{\kappa\left(1+v^2\tau^2\right) - v\tau'}{\sqrt{1+(v\tau)^2}}, \quad m_{e_3} = \frac{\tau}{\sqrt{1+(v\tau)^2}}, \quad n_{e_3} = 0.$$

From here, if these equations are substituted in the equation (2.5), the proof is complete. □

Example 3.11. Consider the curve

$$\alpha(s) = \frac{3}{4}\left(cos\,(s) + \frac{cos\,(3s)}{9}, sin\,(s) + \frac{sin\,(3s)}{9}, \frac{-2cos\,(s)}{\sqrt{3}}\right)$$

with Frenet vectors and the curvatures as follows:

$$\begin{aligned}
&T=\tfrac{3}{4}\left(-sin\left(s\right)-\tfrac{sin(3s)}{3},cos\left(s\right)+\tfrac{cos(3s)}{3},\tfrac{2sin(s)}{\sqrt{3}}\right),\\
&N=\left(-\tfrac{\sqrt{3}cos(2s)}{2},-\tfrac{\sqrt{3}sin(2s)}{2},\tfrac{1}{2}\right),\\
&B=\left(\tfrac{3cos(s)-cos(3s)}{4},sin(s)^{3},\tfrac{\sqrt{3}cos(s)}{2}\right),\\
&\kappa=\sqrt{3}cos\left(s\right),\tau=\sqrt{3}sin\left(s\right)
\end{aligned}$$

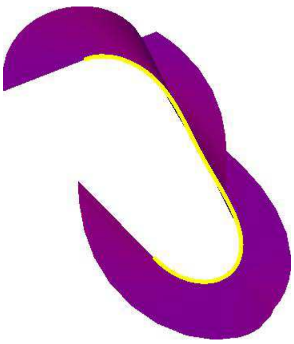
[10]. Since the striction curve and the base curve of tangent ruled surface are the same curve, the equations of the ruled surfaces with the Sannia frame $\{e_1, e_2, e_3\}$ are

$$\begin{aligned}
&\Phi_1\left(s,v\right)=\frac{3}{4}\left(cos\left(s\right)+\frac{cos\left(3s\right)}{9},sin\left(s\right)+\frac{sin\left(3s\right)}{9},\frac{-2cos\left(s\right)}{\sqrt{3}}\right)\\
&+\frac{3}{4}v\left(-sin\left(s\right)-\frac{sin\left(3s\right)}{3},cos\left(s\right)+\frac{cos\left(3s\right)}{3},\frac{2sin\left(s\right)}{\sqrt{3}}\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi_2\left(s,v\right)=\frac{3}{4}\left(cos\left(s\right)+\frac{cos\left(3s\right)}{9},sin\left(s\right)+\frac{sin\left(3s\right)}{9},\frac{-2cos\left(s\right)}{\sqrt{3}}\right)\\
&+v\left(-\frac{\sqrt{3}cos\left(2s\right)}{2},-\frac{\sqrt{3}sin\left(2s\right)}{2},\frac{1}{2}\right),
\end{aligned}$$

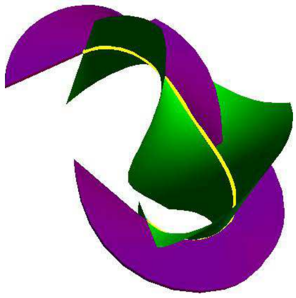
$$\begin{aligned}
&\Phi_3\left(s,v\right)=\frac{3}{4}\left(cos\left(s\right)+\frac{cos\left(3s\right)}{9},sin\left(s\right)+\frac{sin\left(3s\right)}{9},\frac{-2cos\left(s\right)}{\sqrt{3}}\right)\\
&+v\left(\frac{3cos\left(s\right)-cos\left(3s\right)}{4},sin(s)^{3},\frac{\sqrt{3}cos\left(s\right)}{2}\right),
\end{aligned}$$

respectively, (Figure.1).



(a) The tangent ruled surface(Φ_1 Sannia ruled surface) $X_T = \Phi_1$ Sannia ruled surface (in purple), striction curve ζ (in yellow)

(b) The tangent ruled surface X_T (in purple), Φ_2 Sannia ruled surface (in red), striction curve ζ (in yellow)



(c) The tangent ruled surface X_T (in purple), Φ_3 Sannia ruled surface (in green), striction curve ζ (in yellow)

Figure 1: Sannia ruled surfaces associated with tangent ruled surface with $s \in (-1, 3)$ and $v \in (-1, 1)$

3.2. Sannia ruled surfaces associated with normal ruled surface

Theorem 3.12. *Let X_N be a normal ruled surface and $\{f_1, f_2, f_3\}$ be the Sannia frame on the striction curve of X_N, denoted by β. Then, the relationship between the Sannia frame $\{f_1, f_2, f_3\}$ on striction curve and the Frenet frame $\{T, N, B\}$ is as follows:*

$$f_1 = N, \quad f_2 = \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B,$$
$$f_3 = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B$$

where $\kappa^2 + \tau^2 \neq 0$.

Proof. Considering the equation (2.6), we can easily calculate the striction curve of the normal ruled surface by fallowing:

$$\beta(s) = \alpha(s) + \frac{\kappa}{\kappa^2+\tau^2}N.$$

By the definition of X_N, we say $f_1 = N$ and also, by using the equations (2.1) and (2.7), the vectors f_2 and f_3 are computed as

$$f_2 = \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B \quad \text{and} \quad f_3 = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B.$$

□

Definition 3.13. A surface Γ_1 is called a f_1 Sannia ruled surface in E^3, if the surface Γ_1 is generated by moving the vector f_1 along the striction curve β of X_N. The parametrical equation of f_1 ruled surface is defined as

$$\Gamma_1(s,v) = \beta(s) + v f_1(s) \tag{3.1}$$

where $\beta(s) = \alpha(s) + \dfrac{\kappa}{\kappa^2+\tau^2}N$ and $f_1 = N$.

Taking the first order partial differentials of Γ_1 with respect to s and v, we get

$$\Gamma_{1s} = \lambda_1 T + \lambda_2 N + \lambda_3 B, \quad \Gamma_{1v} = N$$

such that

$$\lambda_1 = \frac{\tau^2}{\kappa^2+\tau^2} - v\kappa,\ \lambda_2 = \left(\frac{\kappa}{\kappa^2+\tau^2}\right)' \text{ and } \quad \lambda_3 = \tau\left(\frac{\kappa}{\kappa^2+\tau^2} + v\right).$$

So, by considering (2.3) the normal vector field of Γ_1 which is denoted by u_{f_1} is found as

$$u_{f_1}(s,v) = \frac{-\lambda_3}{\sqrt{\lambda_3^2+\lambda_1^2}}T + \frac{\lambda_1}{\sqrt{\lambda_3^2+\lambda_1^2}}B.$$

Theorem 3.14. *Let Γ_1 be a f_1 Sannia ruled surface. Then the Gaussian curvature and the mean curvature of Γ_1 are*

$$K_{f_1} = \frac{-\tau^2}{\lambda_3^2+\lambda_1^2} \text{ and } H_{f_1} = \frac{\lambda_1\lambda'_3 - \lambda_2\tau - \lambda'_1\lambda_3}{2(\lambda_1^2+\lambda_3^2)^{\frac{3}{2}}},$$

respectively.

Proof. Taking the second order partial differential of Γ_1 given by (3.1), we get

$$\begin{aligned}
\Gamma_{1ss} &= (\lambda'_1 - \lambda_2\kappa)T + (\lambda'_2 + \lambda_1\kappa - \lambda_3\tau)N + (\lambda'_3 + \lambda_2\tau)B,\\
\Gamma_{1sv} &= -\kappa T + \tau B, \quad \Gamma_{1vv} = 0
\end{aligned}$$

By using these equations, the components of the first and the second fundamental form of Γ_1 are found as

$$E_{f_1} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad F_{f_1} = \lambda_2, \quad G_{f_1} = 1,$$

$$l_{f_1} = \frac{\lambda_3(-\lambda'_1 + \lambda_2\kappa) + \lambda_1(\lambda'_3 + \lambda_2\tau)}{\sqrt{\lambda_1^2+\lambda_3^2}}, \quad m_{f_1} = \tau, \quad n_{f_1} = 0.$$

From the equation (2.5), we reach the desired. □

Corollary 3.15. *Let X_N be a normal ruled surface in E^3. if the base curve α of*

X_N is planar curve, then the f_1 Sannia ruled surface is developable and minimal surface.

Definition 3.16. A surface Γ_2 is called a f_2 Sannia ruled surface in E^3, if the surface Γ_2 is generated by moving the vector f_2 along the striction curve β of X_N. The parametric equation of f_2 Sannia ruled surface is defined as

$$\Gamma_2(s,v)=\beta(s)+vf_2(s) \tag{3.2}$$

where $\beta(s)=\alpha(s)+\dfrac{\kappa}{\kappa^2+\tau^2}N$ and $f_2=\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}B$.

Taking the first order partial differentials of Γ_2 with respect to s and v, we get

$$\begin{aligned}\Gamma_{2s}&=\eta_1T+\eta_2N+\eta_3B,\\ \Gamma_{2v}&=-\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}B\end{aligned}$$

such that

$$\begin{aligned}\eta_1&=\frac{\tau^2}{\kappa^2+\tau^2}-v\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)',\\ \eta_2&=\left(\frac{\kappa}{\kappa^2+\tau^2}\right)'-v\sqrt{\kappa^2+\tau^2},\\ \eta_3&=\frac{\kappa\tau}{\kappa^2+\tau^2}+v\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'.\end{aligned}$$

So, by considering (2.3) the normal vector field of Γ_2 which is denoted by u_{f_2} is found as

$$u_{f_2}=\frac{\eta_2\tau T-(\eta_1\tau+\eta_3\kappa)N+\eta_2\kappa B}{\sqrt{\eta_2{}^2(\kappa^2+\tau^2)+(\eta_1\tau+\eta_3\kappa)^2}}.$$

Theorem 3.17. *Let Γ_2 be a f_2 Sannia ruled surface in E^3, then the Gaussian curvature and the mean curvature of Γ_2 are*

$$K_{f_2}=-\frac{(\kappa^2+\tau^2)\left((\eta_3\kappa+\eta_1\tau)\eta'_2-\eta_2(\eta'_1\tau+\eta'_3\kappa)\right)^2}{\left((\eta_3\kappa+\eta_1\tau)^2+\eta_2{}^2(\kappa^2+\tau^2)\right)^2},$$

$$H_{f_2}=\frac{(\kappa^2+\tau^2)\begin{pmatrix}\kappa\tau(\eta_3{}^2-\eta_1{}^2)+\eta_2(\tau\eta'_1+\kappa\eta'_3)\\ -\eta_1\eta_3(\kappa^2-\tau^2)-\eta'_2(\eta_3\kappa+\eta_1\tau)\end{pmatrix}-2(\eta_1\kappa-\eta_3\tau)\sqrt{\kappa^2+\tau^2}\begin{pmatrix}(\eta_3\kappa+\eta_1\tau)\eta'_2\\ -\eta_2(\tau\eta'_1+\kappa\eta'_3)\end{pmatrix}}{2\left((\eta_1\tau+\eta_3\kappa)^2+\eta_2^2(\kappa^2+\tau^2)\right)^{\frac{3}{2}}},$$

respectively.

Proof. Taking the second order partial differential of Γ_2, we have

$$\Gamma_{2ss} = (\eta'_1 - \eta_2\kappa)\, T + (\eta'_2 + \eta_1\kappa - \eta_3\tau)\, N + (\eta'_3 + \eta_2\tau)\, B,$$

$$\Gamma_{2sv} = \eta'_1 T + \eta'_2 N + \eta'_3 B, \quad \Gamma_{2vv} = 0.$$

From here, the component of the first and the second fundamental forms of Γ_2 are computed as

$$E_{f_2} = \eta_1^2 + \eta_2^2 + \eta_3^2, \quad F_{f_2} = \frac{\eta_3\tau - \eta_1\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad G_{f_2} = 1,$$

$$l_{f_2} = \frac{\begin{pmatrix} \eta_2\tau\,(\eta'_1 - \eta_2\kappa) + \eta_2\kappa\,(\eta'_3 + \eta_2\tau) \\ -\,(\eta_1\tau + \eta_3\kappa)\,(\eta'_2 + \eta_1\kappa - \eta_3\tau) \end{pmatrix}}{\sqrt{\eta_2{}^2\,(\kappa^2+\tau^2) + (\eta_1\tau + \eta_3\kappa)^2}},$$

$$m_{f_2} = \frac{\eta_2\,(\eta'_3\kappa + \eta'_1\tau) - \eta'_2\,(\eta_3\kappa + \eta_1\tau)}{\sqrt{\eta_2{}^2\,(\kappa^2+\tau^2) + (\eta_1\tau + \eta_3\kappa)^2}},$$

$$n_{f_2} = 0.$$

So, substituting these equations into (2.5), the proof is complete. □

Definition 3.18. A surface Γ_3 is called a f_3 Sannia ruled surface in E^3, if the surface Γ_3 is generated by moving the vector f_3 along the striction curve β of X_N. The parametric equation of f_3 Sannia ruled surface is defined as

(3.3) $$\Gamma_3\,(s,v) = \beta(s) + v f_3\,(s)$$

where $\beta(s) = \alpha\,(s) + \frac{\kappa}{\kappa^2+\tau^2} N$ and $f_3 = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B$.

Taking the first order partial differentials of Γ_3 with respect to s and v, we get

$$\Gamma_{3s} = \mu_1 T + \mu_2 N + \mu_3 B,$$
$$\Gamma_{3v} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B$$

such that

$$\mu_1 = \frac{\tau^2}{\kappa^2+\tau^2} + v\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)',$$

$$\mu_2 = \left(\frac{\kappa}{\kappa^2+\tau^2}\right)', \ \mu_3 = \frac{\kappa\tau}{\kappa^2+\tau^2} + v\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'.$$

So, considering (2.3) the normal vector field of Γ_3 which is denoted by u_{f_3} is found as

$$u_{f_3} = \frac{\mu_2\kappa T + (\mu_3\tau - \mu_1\kappa)\, N - \mu_2\tau B}{\sqrt{\mu_2{}^2\,(\kappa^2+\tau^2) + (\mu_3\tau - \mu_1\kappa)^2}}.$$

Theorem 3.19. *Let Γ_3 be a f_3 Sannia ruled surface in E^3, then the Gaussian curvature and the mean curvature of Γ_3 are*

$$K_{f_3} = -\frac{\left(\kappa^2+\tau^2\right)\left(-\mu_1{\mu'}_2\kappa+\mu_3{\mu'}_2\tau+\kappa\mu_2{\mu'}_1-\tau\mu_2{\mu'}_3\right)^2}{\left(\mu_3\tau-\mu_1\kappa\right)^2+\mu_2^2\left(\kappa^2+\tau^2\right)},$$

$$H_{f_3} = \frac{\begin{array}{c}\left(\kappa^2+\tau^2\right)\left(\begin{array}{c}\left(\mu_1\kappa-\mu_3\tau\right)^2+\tau\mu_2{\mu'}_3-{\mu'}_1\mu_2\kappa\\ +{\mu_2}^2\kappa^2+{\mu_2}^2\tau^2+\mu_1\mu'\kappa-\mu_3\mu'\tau\end{array}\right)\\ +2\left(\mu_3\kappa+\mu_1\tau\right)\sqrt{\kappa^2+\tau^2}(\mu_1{\mu'}_2\kappa-\mu_3{\mu'}_2\tau+\tau\mu_2{\mu'}_3-\mu'\mu_{21}\kappa)\end{array}}{2\left(\left(\mu_1\kappa-\mu_3\tau\right)^2+{\mu_2}^2\left(\kappa^2+\tau^2\right)\right)^{\frac{3}{2}}},$$

respectively.

Proof. Taking the second order partial differential of Γ_3, we have

$$\begin{aligned}\Gamma_{3ss} &= \left({\mu'}_1-\mu_2\kappa\right)T+\left({\mu'}_2+\mu_1\kappa-\mu_3\tau\right)N+\left({\mu'}_3+\mu_2\tau\right)B,\\ \Gamma_{3sv} &= {\mu'}_1T+{\mu'}_2N+{\mu'}_3B, \quad \Gamma_{3vv}=0.\end{aligned}$$

From here, the components of the first and the second fundamental forms of Γ_3 are computed as

$$\begin{aligned}E_{f_3} &= \mu_1^2+\mu_2^2+\mu_3^2, \quad F_{f_3}=\frac{\mu_1\tau+\mu_3\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad G_{f_3}=1,\\ l_{f_3} &= \frac{\left(\begin{array}{c}\mu_2\kappa\left({\mu'}_1-\mu_2\kappa\right)-\mu_2\tau\left({\mu'}_3+\mu_2\tau\right)\\ +\left(\mu_3\tau-\mu_1\kappa\right)\left({\mu'}_2+\mu_1\kappa-\mu_3\tau\right)\end{array}\right)}{\sqrt{{\mu_2}^2\left(\kappa^2+\tau^2\right)+\left(\mu_3\tau-\mu_1\kappa\right)^2}},\\ m_{f_3} &= \frac{\left(-\mu_1\kappa+\mu_3\tau\right){\mu'}_2+\mu_2\left(\kappa{\mu'}_1-\tau{\mu'}_3\right)}{\sqrt{{\mu_2}^2\left(\kappa^2+\tau^2\right)+\left(\mu_3\tau-\mu_1\kappa\right)^2}}, \quad n_{f_3}=0.\end{aligned}$$

Substituting these into (2.5) completes the proof. □

Example 3.20. Considering the curve α given by example 3.1, the striction curve and Sannia frame vectors of X_N are found as

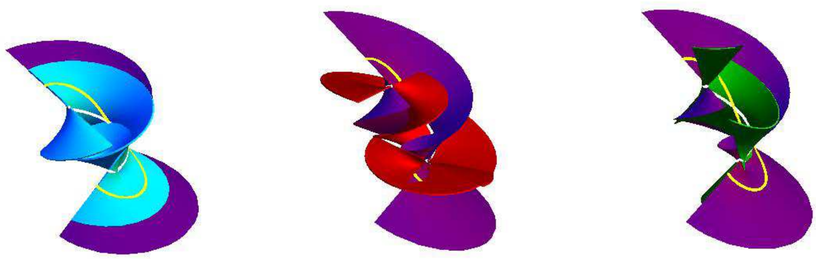
$$\begin{aligned}\beta(s) &= \left(-\frac{1}{3}cos\left(s\right)\left(-2+cos\left(2s\right)\right), \frac{2sin(s)^3}{3}, -\frac{cos\left(s\right)}{\sqrt{3}}\right),\\ f_1 &= \left(-\frac{1}{2}\sqrt{3}cos\left(2s\right), -\frac{1}{2}\sqrt{3}sin\left(2s\right), \frac{1}{2}\right),\\ f_2 &= \left(sin\left(2s\right), -cos\left(2s\right), 0\right),\\ f_3 &= \left(\frac{1}{2}cos\left(2s\right), cos\left(s\right)sin\left(s\right), \frac{\sqrt{3}}{2}\right).\end{aligned}$$

So, the ruled surfaces with Sannia frame are given by the following forms:

$$\Gamma_1(s,v) = \left(-\frac{1}{3}cos(s)(-2+cos(2s)), \frac{2sin(s)^3}{3}, -\frac{cos(s)}{\sqrt{3}}\right) + v\left(-\frac{1}{2}\sqrt{3}cos(2s), -\frac{1}{2}\sqrt{3}sin(2s), \frac{1}{2}\right),$$

$$\Gamma_2(s,v) = \left(-\frac{1}{3}cos(s)(-2+cos(2s)), \frac{2sin(s)^3}{3}, -\frac{cos(s)}{\sqrt{3}}\right) + v(sin(2s), -cos(2s), 0),$$

$$\Gamma_3(s,v) = \left(-\frac{1}{3}cos(s)(-2+cos(2s)), \frac{2sin(s)^3}{3}, -\frac{cos(s)}{\sqrt{3}}\right) + v\left(\frac{1}{2}cos(2s), cos(s)\,sin(s), \frac{\sqrt{3}}{2}\right).$$



(a) The normal ruled surface X_N (in purple), Γ_1 Sannia ruled surface (in cyan), base curve α (in white), striction curve β (in yellow)

(b) The normal ruled surface X_N (in purple), Γ_2 Sannia ruled surface (in red), base curve α (in white), striction curve β (in yellow)

(c) The normal ruled surface X_N (in purple), Γ_3 Sannia ruled surface (in green), base curve α (in white), striction curve β (in yellow)

Figure 2: Sannia ruled surfaces associated with normal ruled surface with $s \in (-1,3)$ and $v \in (-1,1)$

3.3. Sannia ruled surfaces associated with binormal ruled surface

Theorem 3.21. *Let X_B be a binormal ruled surface and $\{g_1, g_2, g_3\}$ be Sannia frame on the striction curve of X_B . Then, the relationship between the Sannia frame and the Frenet frame $\{T, N, B\}$ is as follows:*

$$g_1 = B, \quad g_2 = -N, \quad g_3 = T. \tag{3.1}$$

Proof. Let the curve δ be a striction curve of the binormal ruled surface X_B. By using (2.6), it can be easily shown that the striction curve δ is equal to the base curve of X_B, i.e., $\delta(s) = \alpha(s)$. From the definition of X_B, we say $g_1 = B$ and by using the equations (2.1) and (2.7), the vectors we compute g_2 and g_2 as

$$g_2 = -N \text{ and } g_3 = T.$$

□

Definition 3.22. The surfaces Ψ_1, Ψ_2 and Ψ_3 are called g_1, g_2 and g_3 Sannia ruled surfaces in E^3, if the surfaces Ψ_1, Ψ_2 and Ψ_3 are generated by moving the vectors g_1, g_2 and g_3 along the striction curve δ of X_B, respectively. The parametrical equations of Ψ_1, Ψ_2 and Ψ_3 Sannia ruled surfaces are defined as

$$\begin{aligned}\Psi_1(s,v) &= \delta(s) + v g_1(s),\\ \Psi_2(s,v) &= \delta(s) + v g_2(s),\\ \Psi_3(s,v) &= \delta(s) + v g_3(s)\end{aligned}$$

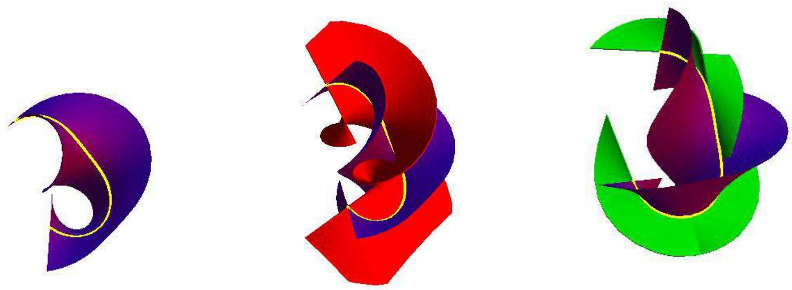
where $g_1 = B$, $g_2 = -N$ and $g_3 = T$.

Corollary 3.23. *Let e_1 and e_3 be Sannia surfaces of the tangent ruled surface and g_1 and g_3 be Sannia ruled surfaces of the binormal ruled surface, then there are the following expressions:*

1. *The g_1 and e_3 Sannia ruled surfaces are the same surfaces.*
2. *The g_3 and e_1 Sannia ruled surfaces are the same surfaces.*

Example 3.24. Let us consider the curve α given by example 3.1. As proved above, the striction curve δ and the base curve α of X_B are the same curve and $g_1 = B$, $g_2 = -N$ and $g_3 = T$. In that case, The equations of ruled surfaces with Sannia frame $\{g_1, g_2, g_3\}$ of X_B are expressed as

$$\begin{aligned}\Psi_1(s,v) =& \frac{3}{4}\left(cos(s) + \frac{cos(3s)}{9}, sin(s) + \frac{sin(3s)}{9}, \frac{-2cos(s)}{\sqrt{3}}\right)\\ &+ v\left(\frac{3cos(s) - cos(3s)}{4}, sin(s)^3, \frac{\sqrt{3}cos(s)}{2}\right),\\ \Psi_2(s,v) =& \frac{3}{4}\left(cos(s) + \frac{cos(3s)}{9}, sin(s) + \frac{sin(3s)}{9}, \frac{-2cos(s)}{\sqrt{3}}\right)\\ &- v\left(-\frac{\sqrt{3}cos(2s)}{2}, -\frac{\sqrt{3}sin(2s)}{2}, \frac{1}{2}\right),\\ \Psi_3(s,v) =& \frac{3}{4}\left(cos(s) + \frac{cos(3s)}{9}, sin(s) + \frac{sin(3s)}{9}, \frac{-2cos(s)}{\sqrt{3}}\right)\\ &+ \frac{3}{4}v\left(-sin(s) - \frac{sin(3s)}{3}, cos(s) + \frac{cos(3s)}{3}, \frac{2sin(s)}{\sqrt{3}}\right).\end{aligned}$$



(a) The binormal ruled surface (Sannia ruled surface) $X_B = \Psi_1$ (in purple), striction curve δ (in yellow).

(b) The binormal ruled surface X_B (in purple), Ψ_2 Sannia ruled surface (in red), striction curve δ (in yellow).

(c) The binormal ruled surface X_B (in purple), Ψ_3 Sannia ruled surface (in green), striction curve δ (in yellow).

Figure 3: Sannia ruled surfaces associated with binormal ruled surface with $s \in (-1, 3)$ and $v \in (-1, 1)$.

3.4. Sannia ruled surfaces associated with Darboux ruled surface

Theorem 3.25. *Let X_C be the Darboux ruled surface and $\{q_1, q_2, q_3\}$ be Sannia frame on the striction curve ϖ of X_C in E^3. Then the relation between the Sannia frame and the Frenet frame $\{T, N, B\}$ is given as*

$$q_1 = \sin\varphi T + \cos\varphi B,$$
$$q_2 = -\cos\varphi T + \sin\varphi B, \quad q_3 = N$$

where the angle φ is between the Darboux vector W and the binormal vector B.

Proof. By considering the equation (2.6), the striction curve of X_C can be written as

$$\varpi(s) = \alpha(s) - \frac{\langle \alpha', C' \rangle}{\langle C', C' \rangle} C = \alpha(s) - \frac{\cos\varphi}{\varphi'} C.$$

By the definition of the surface X_C, the Sannia frame vectors on the striction curve of X_C are computed as

$$q_1 = C = \sin\varphi T + \cos\varphi B,$$
$$q_2 = \frac{C'}{\|C'\|} = -\cos\varphi T + \sin\varphi B,$$
$$q_3 = q_1 \times q_2 = -N.$$

□

Definition 3.26. A surface Δ_1 is called q_1 Sannia ruled surfaces in E^3, if the surface Δ_1 is generated by moving the vector q_1 along the striction curve ϖ of X_C. The parametric equation of q_1 Sannia ruled surface is defined as

$$\Delta_1(s,v) = \varpi(s) + vq_1(s) \tag{3.1}$$

where $\varpi(s) = \alpha(s) - \dfrac{\cos\varphi}{\varphi'}C$ and $q_1 = \sin\varphi T + \cos\varphi B$.

Theorem 3.27. *Let Δ_1 be a q_1 Sannia ruled surface, then the normal vector field of Δ_1 and the principal normal vector of the curve α are linearly dependent.*

Proof. When substituted the equations $\varpi(s) = \alpha(s) - \dfrac{\cos\varphi}{\varphi'}C$ and $q_1 = \sin\varphi T + \cos\varphi B$ into the parametric form of Δ_1 given in (3.1), we get

$$\Delta_1(s,v) = \alpha(s) + \frac{v\varphi' - \cos\varphi}{\varphi'}C.$$

Taking the first order partial differential of this equation with respect to s and v, and by performing the necessary operation, we can write

$$\Delta_{1s} \times \Delta_{1v} = -\varphi' vN.$$

From here, the normal vector field denoted by u_{q_1} of Δ_1 is found as

$$u_{q_1} = \pm N.$$

□

Theorem 3.28. *Let Δ_1 be a q_1 Sannia ruled surface, then the Gaussian curvature and the mean curvature of Δ_1 are*

$$Kq_1 = 0 \text{ and } Hq_1 = \frac{-v\varphi'.\|W\|}{2}\left(2\cos^2\varphi + 2\sin\varphi\left(\frac{\cos\varphi}{\varphi'}\right)' + (v\varphi')^2\right)^{-1},$$

respectively.

Proof. Taking the second order partial differentials of Δ_1 results

$$\Delta_{1ss} = \varpi''(s) + vq_1'', \quad \Delta_{1sv} = q_1' \text{ and } \Delta_{1vv} = 0.$$

By using the equation (2.4), the components of the first and second fundamental forms of Δ_1 are computed as

$$E_{q_1} = 1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + (v\varphi')^2 + \cos^2\varphi, \quad F_{q_1} = \sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)', \quad G_{q_1} = 1,$$

$$l_{q_1} = -v\varphi'(\kappa - \cos\varphi\|W\|) - (v\varphi')^2\|W\|, \quad m_{q_1} = 0, \quad n_{q_1} = 0.$$

By substituting these equations into (2.5), the proof is complete. □

Corollary 3.29. *The q_1 Sannia ruled surface is always a developable surface.*

Definition 3.30. A surface Δ_2 is called q_2 Sannia ruled surfaces in E^3, if the surface Δ_2 is generated by moving the vector q_2 along the striction curve ϖ of X_C. The parametric equation of q_2 Sannia ruled surface is defined as

$$\Delta_2(s,v) = \varpi(s) + vq_2(s) \tag{3.2}$$

where $\varpi(s) = \alpha(s) - \frac{\cos\varphi}{\varphi'}C$ and $q_2 = -\cos\varphi T + \sin\varphi B$.

By substituting the latter equations ϖ and q_2 into (3.2), we get

$$\Delta_2(s,v) = \alpha(s) - \frac{\cos\varphi}{\varphi'}C + v(-\cos\varphi T + \sin\varphi B).$$

Taking the first order partial differentials of this equation with respect to s and v, we simply calculate

$$\Delta_{2s} \times \Delta_{2v} = \left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)N - v\left(\varphi' N + \|W\| C\right).$$

So, the normal vector field u_{q_2} of Δ_2 is computed as

$$u_{q_2} = \frac{\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)N - v\left(\varphi' N + \|W\| C\right)}{\sqrt{\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + v^2\left((\varphi')^2 + \|W\|^2\right)}}.$$

Theorem 3.31. *Let Δ_2 be a q_2 Sannia ruled surface, then the Gaussian curvature and the mean curvature of Δ_2 are*

$$Kq_2 = \frac{-\|W\|^2\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)' - v\varphi' + v\varphi''\right)^2}{\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + v^2\left((\varphi')^2 + \|W\|^2\right)},$$

$$Hq_2 = -\|W\|\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)' - v\varphi' + v\varphi''\right).$$

respectively.

Proof. The second order partial differentials of Δ_2 are given as

$$\Delta_{2ss} = \varpi''(s) + vq_2'', \quad \Delta_{2sv} = q_2', \quad \Delta_{2vv} = 0.$$

By using (2.4), the components of the first and second fundamental forms of Δ_2 are computed as

$$E_{q_2} = -\left(\frac{\cos\varphi}{\varphi'}\right)'(\sin\varphi + \cos\varphi) + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + (v\varphi')^2$$
$$+ \sin^2\varphi + 2\varphi'\left(-\sin\varphi + \left(\frac{\cos\varphi}{\varphi'}\right)'\right) + v^2\left((\varphi')^2 + \|W\|^2\right),$$
$$F_{q2} = 1, \quad G_{q2} = 0,$$
$$l_{q2} = \frac{v\|W\|\left(\left(\frac{\cos\varphi}{\varphi'}\right)'' - \left(\frac{\cos\varphi}{\varphi'}\right)'\right) - v\kappa\varphi' + v\tau}{\sqrt{\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + v^2\left((\varphi')^2 + \|W\|^2\right)}},$$
$$m_{q_2} = \frac{\|W\|\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)' - v\varphi' + v\varphi''\right)}{\sqrt{\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + v^2\left((\varphi')^2 + \|W\|^2\right)}},$$
$$n_{q2} = 0.$$

By substituting these equations into (2.5), the proof is complete. □

Definition 3.32. A surface Δ_3 is called q_3 Sannia ruled surface in E^3, if the surface Δ_3 is generated by moving the vector q_3 along the striction curve ϖ of X_C. The parametric equation of q_3 Sannia ruled surface is defined as

$$\Delta_3(s, v) = \varpi(s) + vq_3$$

where $\varpi(s) = \alpha(s) - \frac{\cos\varphi}{\varphi'}C$ and $q_3 = -N$.

We take derivate of this equation with respect to s and v, it is found that

$$\Delta_{3_s} = \varpi'(s) + vN', \quad \Delta_{3_v} = -N.$$

Therefore, the normal vector field of Δ_3 can be written as

$$u_{q3} = \frac{\cos\varphi\left(\frac{\cos\varphi}{\varphi'}\right)'T + \left(1 - \sin\varphi\left(\frac{\cos\varphi}{\varphi'}\right)'\right)B - \left(\cos\varphi - \frac{v}{\|W\|}\right)C}{\sqrt{1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + \cos^2\varphi + \left(\frac{v}{\|W\|}\right)^2}}.$$

Theorem 3.33. *Let Δ_3 be a q_3 Sannia ruled surface, then the Gaussian curvature* *and the mean curvature of* Δ_3 *are*

$$K_{q_3} = 0,$$

$$H_{q_3} = \frac{\begin{pmatrix} 2\varphi'\left(\frac{\cos\varphi}{\varphi'}\right)'\left(\sin\varphi - \left(\frac{\cos\varphi}{\varphi'}\right)'\right) \\ +\frac{v}{\|W\|}\left(-\left(\frac{\cos\varphi}{\varphi'}\right)'' + \varphi'\cos\varphi\right) \\ +v\left(\kappa'\sin\varphi + \tau'\cos\varphi\right)\left(\frac{v}{\|W\|} - \cos\varphi\right) \\ -v\left(\frac{\cos\varphi}{\varphi'}\right)'\left(\kappa'\cos\varphi + \tau'\sin\varphi\right) + v\tau' \end{pmatrix}}{\begin{pmatrix} 2\sqrt{1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + \cos^2\varphi + \left(\frac{v}{\|W\|}\right)^2} \\ \sqrt{1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + \cos^2\varphi + v^2\|W\|^2} \end{pmatrix}}$$

where $\varphi' \neq 0$.

Proof. Taking the second order partial differential of Δ_3, it follows that

$$\Delta_{3ss} = \varpi''(s) - vN'', \quad \Delta_{3sv} = \kappa T - \tau B, \quad \Delta_{3vv} = 0.$$

By using the equation (2.4), the components of the first and second fundamental forms of Δ_3 are computed as

$$E_{q3} = 1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + \cos^2\varphi + (v\,\|W\|)^2, F_{q3} = 0, \quad G_{q3} = 1,$$

$$l_{q3} = \frac{\begin{pmatrix} 2\varphi'\sin\varphi\left(\frac{\cos\varphi}{\varphi'}\right)' - 2\varphi'\left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 \\ +\frac{v}{\|W\|}\left(-\left(\frac{\cos\varphi}{\varphi'}\right)'' + \varphi'\cos\varphi\right) + v\tau' \\ +v\left(\kappa'\sin\varphi + \tau'\cos\varphi\right)\left(\frac{v}{\|W\|} - \cos\varphi\right) \\ -v\left(\frac{\cos\varphi}{\varphi'}\right)'\left(\kappa'\cos\varphi + \tau'\sin\varphi\right) \end{pmatrix}}{\sqrt{1 + \left(\left(\frac{\cos\varphi}{\varphi'}\right)'\right)^2 + \cos^2\varphi + \left(\frac{v}{\|W\|}\right)^2}}, \quad m_{q_3} = 0, \quad n_{q3} = 0.$$

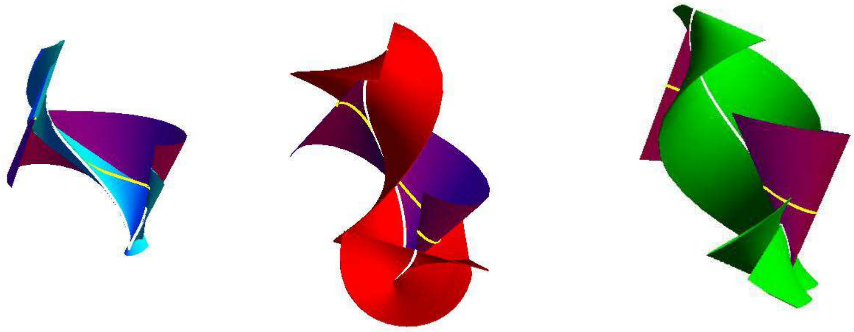
When substituted these into (2.5), the proof is complete. □

Example 3.34. Considering the curve α given in example 3.1, the striction curve and the Sannia frame vectors of X_C are found as

$$\begin{aligned}
\varpi(s) &= \left(\frac{2cos\,(s) - cos\,(s)\,cos\,(2s)}{3}, \frac{2Sin(s)^3}{3}, -\sqrt{3}Cos\,(s) \right), \\
q_1 &= \left(\frac{1}{2}cos\,(2s)\,, \frac{1}{2}sin\,(2s)\,, \frac{\sqrt{3}}{2} \right), \\
q_2 &= (sin\,(2s)\,, -cos\,(2s)\,, 0)\,, \\
q_3 &= \left(\frac{\sqrt{3}cos\,(2s)}{2}, \frac{\sqrt{3}sin\,(2s)}{2}, -\frac{1}{2} \right).
\end{aligned}$$

So, the q_1, q_2 and q_3 Sannia ruled surfaces are given by the following forms:

$$\begin{aligned}
\Delta_1\,(s,v) &= \left(\begin{array}{l} \frac{3cos\,(s) + 3vcos\,(2s) - cos\,(3s)}{6}, \\ \frac{v3sin\,(2s) + 4sin(s)^3}{6}, \frac{\sqrt{3}\,(v - 2cos\,(s))}{2} \end{array} \right), \\
\Delta_2\,(s,v) &= \left(\begin{array}{l} \frac{2cos\,(s) - cos\,(s)\,cos\,(2s) + 3vsin\,(2s)}{3}, \\ \frac{2sin(s)^3 - 3vcos(2s)}{3}, -\sqrt{3}cos\,(s) \end{array} \right), \\
\Delta_3\,(s,v) &= \left(\begin{array}{l} \frac{3cos\,(s) + 3\sqrt{3}vcos\,(2s) - cos\,(3s)}{6}, \\ \frac{4sin(s)^3 + 3\sqrt{3}vsin\,(2s)}{6}, -\frac{v + 2\sqrt{3}cos\,(s)}{2} \end{array} \right).
\end{aligned}$$



(a) The Darboux ruled surface X_C (in purple), Δ_1 Sannia ruled surface (in cyan), base curve α (in white), striction curve ϖ (in yellow).

(b) The graphs of Darboux ruled surface X_C (in purple), Δ_2 Sannia ruled surface (in red), base curve α (in white), striction curve ϖ (in yellow).

(c) The graphs of Darboux ruled surface X_C (in purple), Δ_3 Sannia ruled surface (in green), base curve α (in white), striction curve ϖ (in yellow).

Figure 4: Sannia ruled surfaces associated with Darboux ruled surface with $s \in (-1, 3)$ and $v \in (-1, 1)$.

Acknowledgements. We would like to thank the anonymous reviewers for their suggestions which helped us to improve the quality of manuscript. We would also like to acknowledge the invaluable contributions provided by the editorial team of KMJ.

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