KYUNGPOOK Math. J. 62(2022), 455-465
https://doi.org/10.5666/KMJ.2022.62.3.455
pISSN 1225-6951 eISSN 0454-8124

Remark on Some Recent Inequalities of a Polynomial and its Derivatives

Barchand Chanam*, Khangembam Babina Devi
and Thangjam Birkramjit Singh
Department of Mathematics, National Institute of Technology Manipur, Imphal, Langol, 795004, India
e-mail : barchand_2004@yahoo.co.in, khangembambabina@gmail.com
and birkramth@gmail.com

Kshetrimayum Krishnadas
Department of Mathematics, Shaheed Bhagat Singh College, University of Delhi, Delhi 110017, India
e-mail : kshetrimayum.krishnadas@sbs.du.ac.in

Abstract. We point out a flaw in a result proved by Singh and Shah [Kyungpook Math. J., 57(2017), 537-543] which was recently published in Kyungpook Mathematical Journal. Further, we point out an error in another result of the same paper which we correct and obtain integral extension of the corrected form.

1. Introduction and Statement of Results

Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n. We define

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \quad r > 0. \tag{1.1}$$

From a well known fact of analysis ([12],[14]), we know

$$\lim_{r\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|. \tag{1.2}$$

* Corresponding Author.
Received December 6, 2020; revised October 12, 2021; accepted November 8, 2021.
2020 Mathematics Subject Classification: 30C10, 30C15.
Key words and phrases: polynomials, s-fold zero, inequality, integral extension.

Thus, it is appropriate to denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|. \tag{1.3}$$

Further, if we define $\|p\|_0 = \exp\left\{\dfrac{1}{2\pi}\int\limits_0^{2\pi} \log|p(e^{i\theta})|d\theta\right\}$, then it is easy to verify that

$$\lim_{r\to 0^+} \|p\|_r = \|p\|_0.$$

If $p'(z)$ denotes the ordinary derivative of $p(z)$, then

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.4}$$

Inequality (1.4) is known in the literature as Bernstein's Inequality [3] and it is best possible with equality holding for the polynomial $p(z) = \alpha z^n$, where $\alpha \neq 0$ is any complex number.

If $p(z)$ has no zero in $|z| < 1$, then inequality (1.4) can be sharpened and replaced by

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.5}$$

Inequality (1.5) was conjectured by Erdös which was later proved by Lax [7]. Inequality (1.5) is best possible and become equality for polynomials which have all the zeros on $|z| = 1$.

Under the same hypothesis on $p(z)$ as in (1.5), Aziz and Dawood [1] improved it by proving:

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \left\{\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|\right\}. \tag{1.6}$$

Equality holds in (1.6) for $p(z) = \alpha + \beta z^n$, $|\alpha| \ge |\beta|$.

Malik [8] generalized (1.5) for polynomial $p(z)$ of degree n having no zero in $|z| < k$, $k \ge 1$ by proving the inequality,

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.7}$$

Chan and Malik [4] considered more general lacunary type of polynomials $p(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, and generalized (1.7) by proving:

Theorem A. *Let* $p(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{1.8}$$

Equality occurs in (1.8) *for* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

As an improvement of Theorem A, Pukhta [9] proved

Theorem B. *Let* $p(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{1.9}$$

Equality in (1.9) *occurs for* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Singh and Shah [13] proved the following result which apparently is a generalization and an improvement of Theorems A and B for the class of polynomials with s-fold zeros at the origin.

Theorem C. *Let* $p(z) = z^s \left(a_0 + \sum\limits_{j=\mu}^{n-s} a_j z^j \right)$, $1 \le \mu \le n-s$, $0 \le s \le n-1$, *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| \ge k$, $k \ge 1$, *then*

$$\begin{aligned} \max_{|z|=1} |p'(z)| \le & \\ & \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1} + s(n-s)|a_0|(1+k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \\ & \times \max_{|z|=1} |p(z)| - \frac{1}{k^s} \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \min_{|z|=k} |p(z)|. \end{aligned} \tag{1.10}$$

In the same paper [13], the authors further proved the following result as generalization of Theorem C.

Theorem D. *Let* $p(z) = z^s \left(a_{n-s} z^{n-s} + \sum\limits_{j=\mu}^{n-s} a_{n-s-j} z^{n-s-j} \right)$, $1 \le \mu \le n-s$, $0 \le s \le n-1$, *be a polynomial of degree* n, *having* s*-fold zeros at the origin and* *remaining* $n-s$ *zeros on* $|z|=k$, $k\leq 1$, *then*

(1.11)

$$\max_{|z|=1}|p'(z)|\leq$$

$$\left[\frac{n-s}{k^{n-s-\mu+1}}\left\{\frac{(n-s)|a_{n-s}|k^{2\mu}+\mu|a_{n-s-\mu}|k^{\mu-1}}{\mu|a_{n-s-\mu}|(1+k^{\mu-1})+(n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})}\right\}+s\right]\max_{|z|=1}|p(z)|.$$

2. Lemmas

We need the following lemmas to prove the theorem.

Lemma 2.1. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, *is a polynomial of degree* n *having no zero in* $|z|<k$, $k\geq 1$, *then*

$$|q'(z)|\geq k^{\mu+1}\left(\frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}+1}{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}\right)|p'(z)| \quad on\ |z|=1, \tag{2.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu\leq 1, \tag{2.2}$$

where $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

This lemma is due to Qazi [10].

Lemma 2.2. *If* $p(z)$ *is a polynomial of degree* n *such that* $p(z)\neq 0$ *in* $|z|<k$, $k>0$, *then*

$$|p(z)|\geq m \quad for \quad |z|\leq k,$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [6].

Lemma 2.3. *The function*

$$f(x)=k^{\mu+1}\left\{\frac{\frac{\mu}{n}\left(\frac{|a_\mu|}{x}\right)k^{\mu-1}+1}{\frac{\mu}{n}\left(\frac{|a_\mu|}{x}\right)k^{\mu+1}+1}\right\} \tag{2.3}$$

is a non-decreasing function of x.

This lemma is due to Gardner et al. [6, Lemma 2.6]. However, the authors did not define the quantities x, μ and k for the conclusion to hold. It is of interest for the sake of completeness to define these quantities.
From (2.3), we can show that

$$f'(x) = \frac{\left\{\frac{\mu}{n}\left(\frac{|a_\mu|}{x^2}\right)k^{2\mu}(k^2-1)\right\}}{\left\{\frac{\mu}{n}\left(\frac{|a_\mu|}{x}\right)k^{\mu+1}+1\right\}^2},$$

which implies that, for $k \geq 1$ and μ any real, then $f'(x) \geq 0$ for all non-zero real x. Thus, by the first derivative test, $f(x)$ is non-decreasing for all non-zero real x, any real μ and $k \geq 1$.

Lemma 2.4. *If $p(z)$ is a polynomial of degree n and $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, then for each α, $0 \leq \alpha < 2\pi$ and $r > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi}\left|q'(e^{i\theta})+e^{i\alpha}p'(e^{i\theta})\right|^r d\theta\, d\alpha \leq 2\pi n^r\int_0^{2\pi}\left|p(e^{i\theta})\right|^r d\theta. \tag{2.4}$$

The above lemma was proved by Aziz and Rather [2].

3. Theorem and Comment on Theorem D

In this paper, firstly, we prove the following integral extension whose ordinary version corresponds to the corrected form of Theorem C which we state as the corollary. Secondly, we point out a flaw concerning Theorem D proved by Singh and Shah [13].

Theorem. *Let $p(z) = z^s\left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ be a polynomial of degree n having s-fold zeros at the origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every λ with $|\lambda| < 1$ and $r > 0$,*

$$\|zp'(z) - sp(z)\|_r \leq \frac{n-s}{\|A+e^{i\alpha}\|_r}\left\|\frac{p(z)}{z^s} - \frac{\lambda}{k^s}m\right\|_r, \tag{3.1}$$

where

$$A = k^{\mu+1}\left(\frac{\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0|-|\lambda|\frac{m}{k^s}}k^{\mu-1}+1}{1+\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0|-|\lambda|\frac{m}{k^s}}k^{\mu+1}}\right). \tag{3.2}$$

and $m = \min_{|z|=k} |p(z)|$.

Proof. Let

$$p(z) = z^s H(z), \tag{3.3}$$

where $H(z) = a_0 + \sum_{j=\mu}^{n-s} a_j z^j$, $1 \le \mu \le n-s$ and $0 \le s \le n-1$, is a polynomial of degree $n-s$ having all its zeros in $|z| > k$, $k \ge 1$.
From (3.3) we have

$$\begin{aligned} zp'(z) &= sz^s H(z) + z^{s+1}H'(z) \\ &= sp(z) + z^{s+1}H'(z). \end{aligned}$$

This gives for $|z| = 1$,

$$|p'(z)| \le s|p(z)| + |H'(z)|.$$

The above inequality holds for all points on $|z| = 1$ and hence

$$|p'(z)| \le s|p(z)| + \max_{|z|=1} |H'(z)|. \tag{3.4}$$

Let $m_1 = \min_{|z|=k} |H(z)|$, then $m_1 \le |H(z)|$ for $|z| = k$. As all $n-s$ zeros of $H(z)$ lie in $|z| > k$, $k \ge 1$, therefore, for every complex number λ such that $|\lambda| < 1$, it follows by Rouche's theorem that all zeros of the polynomial $H(z) - \lambda m_1$ lie in $|z| > k$, $k \ge 1$.
Now, the reciprocal polynomial of $H(z) - \lambda m_1$ is

$$\begin{aligned} z^n \left(\overline{H\left(\frac{1}{\overline{z}}\right) - \lambda m_1} \right) &= z^n \overline{H\left(\frac{1}{\overline{z}}\right)} - z^n \overline{\lambda} m_1 \\ &= z^n \left(\overline{\frac{p\left(\frac{1}{\overline{z}}\right)}{\left(\frac{1}{\overline{z}}\right)^s}} \right) - z^n \overline{\lambda} m_1 \\ &= z^n \left(z^s \overline{p\left(\frac{1}{\overline{z}}\right)} \right) - z^n \overline{\lambda} m_1 \\ &= z^s \left(z^n \overline{p\left(\frac{1}{\overline{z}}\right)} \right) - z^n \overline{\lambda} m_1 \\ &= z^s q(z) - z^n \overline{\lambda} m_1 \\ &= G(z) \quad \text{(say)}, \end{aligned}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ is the reciprocal polynomial of p(z).
Applying Lemma 2.1 to the polynomial $H(z) - \lambda m_1$, we have for $|z| = 1$

$$|G'(z)| \geq k^{\mu+1} \left(\frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \lambda m_1|} k^{\mu-1} + 1}{1 + \frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \lambda m_1|} k^{\mu+1}} \right) |H'(z)|$$

$$= A_1 |H'(z)|, \tag{3.5}$$

where $A_1 = k^{\mu+1} \left(\frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \lambda m_1|} k^{\mu-1} + 1}{1 + \frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \lambda m_1|} k^{\mu+1}} \right)$.

Now,

$$m_1 = \min_{|z|=k} |H(z)| = \frac{1}{k^s} \min_{|z|=k} |p(z)| = \frac{m}{k^s}, \quad \text{where} \quad m = \min_{|z|=k} |p(z)|. \tag{3.6}$$

Further, using Lemma 2.2 to the polynomial $H(z)$, we have $|H(z)| > m_1$ for $|z| < k$, i.e., in particular, $|a_0| > m_1$. Then,

$$|a_0 - \lambda m_1| \geq |a_0| - |\lambda| m_1. \tag{3.7}$$

Thus, using the fact of (3.7) to Lemma 2.3, we have $A_1 \geq A$, where A is given by (3.2). Hence, inequality (3.5) gives for $|z| = 1$

$$|G'(z)| \geq A |H'(z)|, \tag{3.8}$$

Now, for real numbers α and $R \geq r_1 \geq 1$, it is easy to verify that

$$|R + e^{i\alpha}| \geq |r_1 + e^{i\alpha}|,$$

which implies, for $r > 0$

$$\int_0^{2\pi} |R + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |r_1 + e^{i\alpha}|^r d\alpha. \tag{3.9}$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $H'(e^{i\theta}) \neq 0$, put $R = \left| \frac{G'(e^{i\theta})}{H'(e^{i\theta})} \right|$ and $r_1 = A$, then from Remark 3.3 and inequality (3.8), we have $R \geq r_1 \geq 1$.

Then, for every $r > 0$

$$\begin{aligned}
\int_0^{2\pi} \left|G'(e^{i\theta}) + e^{i\alpha}H'(e^{i\theta})\right|^r d\alpha &= \left|H'(e^{i\theta})\right|^r \int_0^{2\pi} \left|\frac{G'(e^{i\theta})}{H'(e^{i\theta})} + e^{i\alpha}\right|^r d\alpha \\
&= \left|H'(e^{i\theta})\right|^r \int_0^{2\pi} \left|\left|\frac{G'(e^{i\theta})}{H'(e^{i\theta})}\right| + e^{i\alpha}\right|^r d\alpha \\
&\geq \left|H'(e^{i\theta})\right|^r \int_0^{2\pi} \left|A + e^{i\alpha}\right|^r d\alpha \quad \text{(using (3.9))}, \qquad (3.10)
\end{aligned}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $H'(e^{i\theta}) \neq 0$. Moreover (3.10) holds trivially for points $e^{i\theta}$, $0 \leq \theta < 2\pi$ for which $H'(e^{i\theta}) = 0$. Thus, for $r > 0$

$$\begin{aligned}
\int_0^{2\pi} \left|A + e^{i\alpha}\right|^r d\alpha \int_0^{2\pi} \left|H'(e^{i\theta})\right|^r d\theta &\leq \int_0^{2\pi}\int_0^{2\pi} \left|G'(e^{i\theta}) + e^{i\alpha}H'(e^{i\theta})\right|^r d\alpha\, d\theta \\
&\leq 2\pi(n-s)^r \int_0^{2\pi} \left|H(e^{i\theta}) - \lambda m_1\right|^r d\theta,
\end{aligned}$$

(using Lemma 2.4).

Substituting the value of m_1 from equation (3.6) in the above inequality, we have

$$\int_0^{2\pi} \left|H'(e^{i\theta})\right|^r d\theta \leq \frac{(n-s)^r}{\frac{1}{2\pi}\int_0^{2\pi} \left|A + e^{i\alpha}\right|^r d\alpha} \int_0^{2\pi} \left|H(e^{i\theta}) - \lambda \frac{m}{k^s}\right|^r d\theta.$$

Multiplying by $\frac{1}{2\pi}$ on both sides of the above inequality and taking r^{th} root, we get

$$\|H'\|_r \leq \frac{n-s}{\left\|A + e^{i\alpha}\right\|_r} \left\|H(z) - \lambda \frac{m}{k^s}\right\|_r,$$

which is equivalent to

$$\|zp'(z) - sp(z)\|_r \leq \frac{(n-s)}{\|A + e^{i\alpha}\|_r} \left\|\frac{p(z)}{z^s} - \lambda \frac{m}{k^s}\right\|_r,$$

which completes the proof of the theorem. □

Remark 3.1. If we let $r \to \infty$ in (3.1), we have

$$\max_{|z|=1} |zp'(z) - sp(z)| \leq \left(\frac{n-s}{1+A}\right) \max_{|z|=1} \left|\frac{p(z)}{z^s} - \frac{\lambda}{k^s} m\right|,$$

where A is defined by (3.2) in the theorem. Then

$$\max_{|z|=1}|p'(z)| - \max_{|z|=1}|sp(z)| \le \left(\frac{n-s}{1+A}\right)\max_{|z|=1}\left|\frac{p(z)}{z^s} - \frac{\lambda}{k^s}m\right|. \tag{3.11}$$

Let z_0 on $|z|=1$ be such that

$$\max_{|z|=1}\left|\frac{p(z)}{z^s} - \frac{\lambda}{k^s}m\right| = \left|\frac{p(z_0)}{z_0^s} - \frac{\lambda}{k^s}m\right|. \tag{3.12}$$

Choose the argument of λ suitably such that

$$\begin{aligned}\left|\frac{p(z_0)}{z_0^s} - \frac{\lambda}{k^s}m\right| &= \left|\frac{p(z_0)}{z_0^s}\right| - |\lambda|\frac{m}{k^s}\\ &= |p(z_0)| - |\lambda|\frac{m}{k^s}\\ &\le \max_{|z|=1}|p(z)| - |\lambda|\frac{m}{k^s}. \end{aligned}\tag{3.13}$$

Using (3.13) in (3.12), we have

$$\max_{|z|=1}\left|\frac{p(z)}{z^s} - \frac{\lambda}{k^s}m\right| \le \max_{|z|=1}|p(z)| - |\lambda|\frac{m}{k^s}. \tag{3.14}$$

Combining (3.11) and (3.14), we get

$$\max_{|z|=1}|p'(z)| - s\max_{|z|=1}|p(z)| \le \left(\frac{n-s}{1+A}\right)\left\{\max_{|z|=1}|p(z)| - |\lambda|\frac{m}{k^s}\right\},$$

which on simplification gives

$$\max_{|z|=1}|p'(z)| \le \left(\frac{n-s}{1+A}+s\right)\max_{|z|=1}|p(z)| - \frac{n-s}{(1+A)k^s}|\lambda|m. \tag{3.15}$$

Now,

$$\begin{aligned}\left(\frac{n-s}{1+A}+s\right) = &\left[\frac{1}{(n-s)\left(|a_0| - |\lambda|\frac{m}{k^s}\right)(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]\\ &\times\Big[(n-s)^2\left(|a_0| - |\lambda|\frac{m}{k^s}\right) + (n-s)\mu|a_\mu|k^{\mu+1}\\ &+s(n-s)\left(|a_0| - |\lambda|\frac{m}{k^s}\right)(1+k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})\Big]\end{aligned}\tag{3.16}$$

and

$$\frac{n-s}{(1+A)k^s} = \frac{1}{k^s}\,\frac{(n-s)^2\left(|a_0| - |\lambda|\frac{m}{k^s}\right) + (n-s)\mu|a_\mu|k^{\mu+1}}{(n-s)\left(|a_0| - |\lambda|\frac{m}{k^s}\right)(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}. \tag{3.17}$$

Using (3.16) and (3.17) in (3.15) and considering the limit as $|\lambda| \to 1$, we have the following corrected form of (1.10) of Theorem C.

Corollary. *Let* $p(z) = z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| \geq k$, $k \geq 1$, *then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \leq & \left[\frac{1}{(n-s)\left(|a_0| - \frac{m}{k^s}\right)(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right] \\ & \times \Big[(n-s)^2\left(|a_0| - \frac{m}{k^s}\right) + (n-s)\mu|a_\mu|k^{\mu+1} \\ & +s(n-s)\left(|a_0| - \frac{m}{k^s}\right)(1+k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})\Big] \max_{|z|=1}|p(z)| \\ & - \frac{1}{k^s} \frac{(n-s)^2\left(|a_0| - \frac{m}{k^s}\right) + (n-s)\mu|a_\mu|k^{\mu+1}}{(n-s)\left(|a_0| - \frac{m}{k^s}\right)(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \min_{|z|=k}|p(z)|.\end{aligned}$$

Remark 3.2. In fact, inequality (1.10) of Theorem C is not the correct form it should have been. It must be noted that in the correct form of(1.10), as is given by the corollary, every factor $|a_0|$ wherever it appears, is replaced by $|a_0| - \frac{m}{k^s}$, where $m = \min_{|z|=k}|p(z)|$.

Remark 3.3. In the theorem, A given by (3.2) is such that $A \geq 1$. To see this, we have from (3.2)

$$A = \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{\left(|a_0| - |\lambda|\frac{m}{k^s}\right)} k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n-s} \frac{|a_\mu|}{\left(|a_0| - |\lambda|\frac{m}{k^s}\right)} k^{\mu+1} + 1}.$$

As $k \geq 1$ and $\mu \geq 1$, we have $k^{2\mu} \geq k^{\mu+1} \geq 1$. Thus, $A \geq 1$.

Comment on Theorem D. Theorem D is incorrect as it is based on an incorrect lemma (for reference see [5, Lemma 2.2]) as pointed out by Qazi [11].

References

[1] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivatives*, J. Approx. Theory, **54**(1988), 306–313.

[2] A. Aziz and N. A. Rather, *Some Zygmund type L^q inequalities for polynomials*, J. Math. Anal. Appl., **289**(2004), 14–29.

[3] S. Bernstein, *Lecons sur les propriétés extrémales et la meilleure approximation des-fonctions analytiques d'une variable réelle*, Gauthier Villars, Paris(1926).

[4] T. N. Chan and M. A. Malik, *On Erdös-Lax Theorem*, Proc. Indian Acad. Sci. Math. Sci., **92(3)**(1983), 191–193.

[5] K. K. Dewan and S. Hans, *On maximum modulus for the derivative of a polynomial*, Ann. Univ. Mariae Curie-Skodowska Sect. A, **63**(2009), 55–62.

[6] R. B. Gardner, N. K. Govil and S. R. Musukula, *Rate of growth of polynomials not vanishing inside a circle*, J. Inequal. Pure Appl. Math., **6(2)**(2005), 1–9.

[7] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50**(1944), 509–513.

[8] M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc., **2(1)**(1969), 57–60.

[9] M. S. Pukhta, *On extremal properties and location of zeros of polynomials*, Ph.D Thesis, Jamia Millia Islamia, New Delhi(2002).

[10] M. A. Qazi, *On the maximum modulus of polynomials*, Proc. Amer. Math. Soc., **115**(1992), 337–343.

[11] M. A. Qazi, *Remarks on some recent results about polynomials with restricted zeros*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **67**(2013), 59–64.

[12] W. Rudin, *Real and Complex Analysis*, Tata Mcgraw-Hill Publishing Company (Reprinted in India)(1977).

[13] G. Singh and W. M. Shah, *Some inequalities for derivatives of polynomials*, Kyungpook Math. J., **57**(2017), 537–543.

[14] A. E. Taylor, *Introduction to Functional Analysis*, John Wiley and Sons, Inc. New York(1958).