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On 2-absorbing Primary Ideals of Commutative Semigroups

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Abstract. In this paper we introduce the notion of 2-absorbing primary ideals of a commutative semigroup. We establish the relations between 2-absorbing primary ideals and prime, maximal, semiprimary and 2-absorbing ideals. We obtain various characterization theorems for commutative semigroups in which 2-absorbing primary ideals are prime, maximal, semiprimary and 2-absorbing ideals. We also study some other important properties of 2-absorbing primary ideals of a commutative semigroup.

1. Introduction

The concept of a 2-absorbing ideal for a commutative ring was introduced by Badawi [2] and later extended to commutative semigroups by Cay et al. [4] as follows : a proper ideal I of a commutative semigroup S is said to be a 2-absorbing ideal if $abc \in I$ implies either $ab \in I$ or $ac \in I$ or $bc \in I$ for some $a, b, c \in S$. The notion of a 2-absorbing primary ideal of a commutative ring was introduced as a generelisation of a 2-absorbing ideal by Badawi [3]. Studies of commutative algebraic structures (rings, semirings) via 2-absorbing primary ideals have been made by many authors in ([3],[8],[10]) and 2-absorbing primary ideals in lattices were defined and studied in [15]. The main purpose of this paper is to define the concept of 2-absorbing primary ideal, and to invetigate how this notion can be used in the study of commutative semigroups.
In this paper, we define a 2-absorbing primary ideal in a commutative semigroup ($cf.$ Definition 2.1). Clearly 2-absorbing ideals, prime ideals and maximal ideals are 2-absorbing primay ideals ($cf.$ Theorem 2.2, Corollary 2.6) and we prove that every semiprimary ideal of S is a 2-absorbing primay ideal ($cf.$ Theorem 2.14). The converses is not true ($cf.$ Remark 2.7, 2.8, 2.16). Then we characterize the semigroups for which all 2-absorbing primary ideals are prime, maximal, 2-absorbing

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and semiprimary (cf. Theorem 2.10, 2.12, 2.9, 2.17,), as a result we obtain that a semigroup in which 2-absorbing primary ideals are semiprimary is equivalent to a semiprimary semigroup (cf. Theorem 2.17) and semigroup in which 2-absorbing primary ideals are maximal is either a group or union of two groups (cf. Corollary 2.13). Then we prove that a proper ideal I of a semigroup S is 2-absorbing primary ideal of S if and only if $I[x]$ ($I[x_1, x_2, \ldots, x_n]$) is a 2-absorbing primary ideal of the polynomial semigroup $S[x]$ ($S[x_1, x_2, \ldots, x_n]$) (cf. Theorem 2.31) and for any ideal I of S, $\sqrt{I}$ is a 2-absorbing primary ideal of S if and only if $\sqrt{I}$ is a 2-absorbing ideal of S (cf. Theorem 2.25). As a consequence we prove that the radical of a 2-absorbing primary ideal I is a 2-absorbing ideal, moreover if $\sqrt{I} = P$ is a prime ideal of S, then the residual $(I : x) = \{s \in S : sx \in I\}$ of I by $x \in S - \sqrt{I}$ is a 2-absorbing primary ideal with $\sqrt{(I : x)} = P$ (cf. Theorem 2.24). We prove that the arbitrary union of 2-absorbing primary ideals is 2-absorbing primary but that intersections of 2-absorbing primary ideals need not be a 2-absorbing primary ideal (cf. Example 2.26). Also we find equivalence classes in the semigroup of all 2-absorbing primary ideals of a semigroup S and each class is closed under finite intersections (cf. Theorem 2.27). We observe that under certain conditions 2-absorbing primary ideals remains invariant under homomorphism of semigroups and it's inverse mapping (cf. Theorem 2.35). Lastly we also study 2-absorbing primary ideals in direct product of semigrops (cf. Theorem 2.38, 2.39) .
Before going to the main work we discuss some preliminaries which are necessary:

Definition 1.1. ([13]) A non-empty ideal P of a semigroup S is said to be prime if $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, I, J being ideals of S. An ideal P is said to be completely prime if $ab \in P$ implies either $a \in P$ or $b \in P$ for some $a, b \in S$.

Remark 1.2. These concepts coincide if S is commutative.

Definition 1.3. ([11]) For an ideal I of a semigroup S, radical of I, is defined as $\sqrt{I} = \{x \in S : x^n \in I$ for some natural numbers $n\}$, is the intersections of all prime ideals containing I.

Definition 1.4. ([9]) An ideal I of a semigroup S is called primary if $ab \in I$ implies either $a \in I$ or $b \in \sqrt{I}$. An ideal I of a semigroup S is called semiprimary if $\sqrt{I}$ is a prime ideal of S.

Definition 1.5. ([11]) A commutative semigroup S is said to be fully prime if every ideal of S is prime and primary if every ideal of S is a primary ideal of S.

Definition 1.6. ([9]) A commutative semigroup S is said to be semiprimary semigroup if every ideal of S is semiprimary. Moreover, S is a semiprimary semigroup if and only if prime ideals of S are linearly ordered.

Definition 1.7. ([13]) An ideal M of a semigroup S is called a maximal ideal of S if $M \subsetneq S$ and there does not exist an ideal M_1 of S such that $M \subsetneq M_1 \subsetneq S$. If S is a semigroup with unity then S has unique maximal ideal, which is the union of all proper ideals of S, is prime also.

Theorem 1.8. ([11]) *If I and J are any two ideals of a commutative semigroup S,*

then the following statements are true
(1) $IJ \subseteq I \cap J$.
(2) $I \subseteq \sqrt{I} = \sqrt{\sqrt{I}}$.
(3) $I \subseteq J \Rightarrow \sqrt{I} \subseteq \sqrt{J}$.
(4) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. (5) $(I \cup J)K = IK \cup JK$, *where* K *is an ideal of* S.
(5) *If* A *is a prime ideal of* S, *then* $A = \sqrt{A}$ *and if* A *is a primary ideal of* S, *then* $\sqrt{A}$ *is a prime ideal of* S.

Definition 1.9.([13]) An ideal I of a semigroup S is said to be a semiprime ideal if $a^2 \in I$ for some $a \in S$ implies $a \in I$.

Definition 1.10. ([5]) Let S be a semigroup and x be an indeterminate. Then $S[x] = \{sx^i : s \in S, i \geq 0\}$ forms a semigroup wiith respect to the multiplication defined as : $(sx^i)(tx^j) = (st)x^{i+j}$, where $s, t \in S$ and $i, j \geq 0$, called the polynomial semigroup over S. Similarly we can define the polynomial semigroup $S[x_1, x_2, \ldots, x_n]$ in n variables.

2. Some Properties of 2-absorbing Primary Ideals

Throughout this paper, unless otherwise mentioned, S stands for a commutative semigroup.

Definition 2.1. A proper ideal I of a commutative semigroup S is said to be 2-absorbing primary if $abc \in I$ implies either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ for some $a, b, c \in S$.

Since $I \subseteq \sqrt{I}$ for any ideal I of a semigroup S so we have the following result

Theorem 2.2. *Let* S *be a commutative semigroup. Then every* 2*-absorbing ideal of* S *is a* 2*-absorbing primary ideal of* S.

The following lemmas are obvious, hence we omit the proof.

Lemma 2.3.([Lemma 2.1, [6]]) *Let* S *be a commutative semigroup. Then every prime ideal of* S *is a* 2*-absorbing ideal of* S.

Lemma 2.4.([Theorem 2.4, [6]]) *Let* S *be a commutative semigroup. Then every maximal ideal of* S *is a* 2*-absorbing ideal of* S.

Lemma 2.5.([Lemma 2.9, [6]]) *Let* P_1 *and* P_2 *be two prime ideals of a semigroup* S. *Then* $P_1 \cap P_2$ *is a* 2*-absorbing ideal of* S.

Corollary 2.6. *Let* S *be a commutative semigroup. Then*
(1) *if* P_1 *and* P_2 *are two prime ideals of* S *then* $P_1 \cap P_2$ *is a* 2*-absorbing primary ideal of* S.
(2) *every maximal ideal of* S *is a* 2*-absorbing primary ideal of* S.
(3) *every prime ideal of* S *is a* 2*-absorbing primary ideal of* S.

Remark 2.7. The following example shows that converse of Lemma 2.3. and 2.4. are not true. Consider the ideal $I_2 = \{n \in \mathbb{N} : n \geq 2\}$ in the semigroup $S = (\mathbb{N} \cup \{0\}, +)$, which is 2-absorbing primary (as well 2-absorbing) but neither prime nor a maximal ideal of S.

Remark 2.8. The following example shows that converse of Theorem 2.2 is not true. Consider the ideal $I = (m \in \mathbb{N} : m \geq 6)$ in the semigroup $S = (\mathbb{N}, +)$. Then $1 + 2 + 3 \in I$ but neither $1 + 2 \in I$ nor $2 + 3 \in I$ nor $1 + 3 \in I$. Clearly, I is a 2-absorbing primary ideal of S but not a 2-absorbing ideal of S.

A semigroup S is called regular if for each element $s \in S$ there exists an element $x \in S$ such that $sxs = s$ ([7], Section 5). Since in a commutative regular semigroup every ideal coincide with its radical ([7], Theorem 5.1), we have the following result.

Corollary 2.9. *Let S be a commutative regular semigroup. Then an ideal I of S is a 2-absorbing primary ideal of S if and only if I is a 2-absorbing ideal of S.*

The following is a characterization of a semigroup in which 2-absorbing primary ideals are prime:

Theorem 2.10. *Let S be a commutative semigroup. Then every 2-absorbing primary ideals of S are prime if and only if prime ideals of S are linearly ordered and $A = \sqrt{A}$ for every 2-absorbing primary ideal A of S.*

Proof. Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is a 2-absorbing primary ideal of S (*cf.* Corollary 2.6(1)) and so prime by hypothesis. Hence prime ideals are linearly ordered. Again let A be a 2-absorbing primary ideal of S and so prime ideal of S. Therefore $A = \sqrt{A}$.
Conversely, Let A be a 2-absorbing primary ideal of S. Since prime ideals are linearly ordered so $A = \sqrt{A} = \bigcap_{\alpha \in \Lambda} P_\alpha = P_\beta$ for some $\beta \in \Lambda$, where $\{P_\alpha\}_{\alpha \in \Lambda}$ are prime ideals of S containing A. Hence the result follows. □

Since every primary ideals of a commutative semigroup S is 2-absorbing primary, we have the following result by using Theorem 3.1 of [12].

Corollary 2.11. *Let S be a commutative semigroup with zero and identity in which nonzero 2-absorbing primary ideals are prime. Then S satisfies one of the following conditions.*
(i) $S = H \cup M$, where H is the group of units in S and $M = \{0, ah : a \in M, a^2 = 0, h \in H\}$.
(ii) $M^n = M$ for every positive integer n.

The following is a characterization of a semigroup in which 2-absorbing primary ideals are maximal:

Theorem 2.12. *Let S be a commutative semigroup with unity. Then 2-absorbing primary ideals of S are maximal if and only if S is either a group or has a unique 2-absorbing primary ideals A such that $S = A \cup H$, where H is the group of units of S.*

Proof. Let S be commutative semigroup with unity in which 2-absorbing primary ideals are maximal. If S is not a group, it has unique maximal ideal say A and since maximal ideals are 2-absorbing primary (*cf.* Corollary 2.6(2)) so has unique

2-absorbing primary ideal A. Therefore $S = A \cup H$, where H is the group of units of S.
Conversely, if S is a group then it has no 2-absorbing primary ideal so the condition satisfied vacously. Again if S has unique 2-absorbing primary ideal then clearly it is maximal.

Moreover, we prove that A is also a group. Clearly A is the unique prime ideal of S. Then for any $a \in A$, $\sqrt{aS} = A$. Hence aS is a 2-absorbing primary ideal of S. Hence $aS = A$ for every $a \in A$, by hypothesis. Then $aS = a^2S = A$ implies $a = a^2x \Rightarrow ax = a^2x^2$. Thus ax is an idempotent element of A. If possible let e, f be two idempotent element of S. Then $eS = fS \Rightarrow e = fe = ef = f$. Consequently $eS = aS = A$. Therefore A is a group. So we can conclude the corollary □

Corollary 2.13. *Let S be a commutative semigroup with unity. Then 2-absorbing primary ideals are maximal if and only if either S is a group or S is a union of two groups.*

Theorem 2.14. *Let S be a commutative semigroup. Then every semiprimary ideal of S is a 2-absorbing primary ideal of S.*

Proof. Let I be a semiprimary ideal of a semigroup S and $abc \in I \subseteq \sqrt{I}$ with $ab \notin I$ for some $a, b, c \in S$. Hence $\sqrt{I}$ is a prime ideal of S.
Case(1). Suppose $ab \notin \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of S so $c \in \sqrt{I}$. Hence $ac \in \sqrt{I}$ and $bc \in \sqrt{I}$.
Case(2). Suppose $ab \in \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal, we have either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Hence either $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.
Therefore I is a 2-absorbing primary ideals of S.

The following are obvious consequence of above theorem:

Corollary 2.15. *Let I be an ideal of a commutative semigroup S. Then*
(1) *if $\sqrt{I}$ is a prime ideal of S, then I is a 2-absorbing primary ideal of S.*
(2) *if I is a prime ideal of S, then I^n is a 2-absorbing primary ideal of S for each natural number n.*
(3) *every primary ideal of S is a 2-absorbing primary ideal of S.*

Remark 2.16. The converse of Theorem 2.14 is not true. Consider the principal ideal $I = (6)$ generated by 6 in the semigroup $S = \{\mathbb{Z}, .\}$, which is clearly 2-absorbing primary but $\sqrt{I} = (6)$ is not a prime ideal of S and hence not a semiprimary ideal of S.

The following theorem is a characterization of a semigroup in which 2-absorbing primary ideals are semiprimary:

Theorem 2.17. *Let S be a commutative semigroup. Then the following statements are equivalent:*
(1) *2-absorbing primary ideals of S are semiprimary.*
(2) *Prime ideals of S are linearly ordered.*
(3) *S is a semiprimary semigroup.*
(4) *Semiprime ideals are linearly ordered.*

(5) *Semiprime ideals of S are prime.*

Proof. (1) $\Rightarrow$ (2) Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is a 2-absorbing primary ideal of S (*cf*. Corollary 2.6(1)) and so semiprimary ideal of S, by hypothesis. Therefore $\sqrt{P_1 \cap P_2} = \sqrt{P_1} \cap \sqrt{P_2} = P_1 \cap P_2$, is a prime ideal of S. Therefore either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.
(2) $\Rightarrow$ (1) Since prime ideals of S are linearly ordered, then for any ideal I of S, $\sqrt{I}$ is a prime ideal of S. Consequently, 2-absorbing primary ideals of S are semiprimary.
(2) $\Leftrightarrow$ (3) follows from ([9], Theorem 1).
(2) $\Rightarrow$ (4) Let S_1 and S_2 are two distinct semiprime ideals of S. Then $S_1 \cap S_2$ is a semiprime ideal of S. Hence $\sqrt{S_1 \cap S_2} = S_1 \cap S_2$, is a prime ideal of S, since prime ideals are linearly ordered. Hence semiprime ideals of S are linearly ordered.
(4) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (5) Let I be a semiprime ideal of S. Then $I = \sqrt{I}$, is a prime ideal of S.
(5) $\Rightarrow$ (2) Let P_1 and P_2 be two distinct prime ideals of S. Then $P_1 \cap P_2$ is a semiprime ideals of S, hence prime ideals of S. Consequently, prime ideals of S are linearly ordered. □

Definition 2.18. A commutative semigroup S is said to be 2-absorbing primary if every proper ideal of S is a 2-absorbing primary ideal of S.

Example 2.19. Consider the commuttive semigroup $S = \{\mathbb{N}, +\}$, which has no proper prime ideal. Clearly S is a primary semigroup and hence 2-absorbing primary semigroup.

Theorem 2.20. *Let S be a commutative semigroup with unity. If*
(i) *proper prime ideals are maximal then S is a 2-absorbing primary semigroup.*
(ii) *2-absobing ideals are semiprimary then S is a 2-absorbing primary semigroup.*

Proof. (i) If S is commutative semigroup with unity in which proper prime ideals are maximal. Then S has unique proper prime as well as maximal ideal say M. Clearly for any ideal I of S, $\sqrt{I} = M$, a prime ideal of S. Hence S is a 2-absorbing primary semigroup (*cf*. Corollary 2.15(1)).
(ii) Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is 2-absorbing ideal of S and hence semiprimary ideal of S. Then $\sqrt{P_1 \cap P_2} = \sqrt{P_1} \cap \sqrt{P_2} = P_1 \cap P_2$, is a prime ideal of S and hence prime ideals of S are linearly ordered. Then for any ideal I of S, $\sqrt{I}$ is a prime ideal of S and hence S is a 2-absorbing primary ideal of S (*cf*. Corollary 2.15(1)). □

Theorem 2.21. *Let S be a commutative semigroup. Then S can be written as disjoint union of 2-absorbing primary semigroup.*

Proof. Let S be a commutative semigroup. Then S can be written as disjount union of archimedian subsemigroups of S ([12], Corollary 1.6). Clearly every archimedian semigroup is primary and every primary semigroup is 2-absorbing primary. Therefore $S = \bigcup S_\alpha$, where S_α is a 2-absorbing primary subsemigroup of S. □

Theorem 2.22. *Let S be a commutative regular semigroup. Then the following*

statements about S are equivalent:
(1) 2-*absorbing primary ideals of S are semiprimary.*
(2) *prime ideals of S are linearly ordered.*
(3) *idempotents of S form a chain under natural ordering.*
(4) *All ideals of S are linearly ordered.*
(5) *S is a fully prime semigroup.*
(6) *S is a primary semigroup.*
(7) *S is a semiprimary semigroup.*
(8) 2-*absorbing ideals of S are prime.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2.17.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) follows from ([9], Theorem 1).
(7) $\Rightarrow$ (8) Let I be a 2-absorbing ideal of S. Since S is semiprimary and regular so $I = \sqrt{I} = P$, a prime ideal of S, as desired.
(8) $\Rightarrow$ (1) Let I be a 2-absorbing primary ideal of a regular semigroup S. Then $I = \sqrt{I}$ is a 2-absorbing ideal of S (cf. Theorem 2.24(1)) and hence prime by hypothesis. Therefore I is a semiprimary ideal of S, as desired. $\square$

Lemma 2.23. *Let I be a 2-absorbing ideal of a semigroup S with unity. Then $(I : x)$ is a 2-absorbing ideal of S for all $x \in S - I$.*

Proof. Let $abc \in (I : x)$ for some $a, b, c \in S$. Since $x \notin I$, we have $(I : x) \neq S$. Then $abcx \in I$. Since I is a 2-absorbing ideal of S so either $ab \in I$ or $bcx \in I$ or $acx \in I$ that is $ab \in (I : x)$ or $ac \in (I : x)$ or $bc \in (I : x)$. Consequently $(I : x)$ is a 2-absorbing ideal of S.

Theorem 2.24. *Let I be a 2-absorbing primary ideal of a semigroup S with unity. Then the following statements are true*
(1) *$\sqrt{I}$ is a 2-absorbing ideal of S.*
(2) *If $\sqrt{I}$ is a prime ideal P of S then $(I : x)$ is a 2-absorbing primary ideal of S with $\sqrt{(I : x)} = P$ for all $x \in S - \sqrt{I}$.*
(3) *If $\sqrt{I}$ is a maximal ideal M of S then $(I : x)$ is a 2-absorbing primary ideal of S with $\sqrt{(I : x)} = M$ for all $x \in S - \sqrt{I}$.*
(4) *$(\sqrt{I} : x)$ is a 2-absorbing ideal of S for all $x \in S - \sqrt{I}$.*
(5) *$(\sqrt{I} : x) = (\sqrt{I} : x^2)$ for all $x \in S - \sqrt{I}$.*

Proof. (1) Proof is similar to ([3], Theorem 2.2).
(2) Let $x \in S - \sqrt{I}$ and $p \in (I : x)$. Then $px \in I \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of S and $x \notin \sqrt{I}$ so $p \in \sqrt{I}$. Hence $I \subseteq (I : x) \subseteq \sqrt{I} = P$, which implies $P = \sqrt{I} \subseteq \sqrt{(I : x)} \subseteq \sqrt{I} = P$. Consequently, $\sqrt{(I : x)} = P$, a prime ideal of S and hence $(I : x)$ is a 2-absorbing primary ideal of S (cf. Corollary 2.15.(1)).
(3) If $\sqrt{I} = M$ and maximal ideal of a semigroup with unity is prime, hence the proof follows from (2).
(4) Since I is a 2-absorbing primary ideal of S so $\sqrt{I}$ is a 2-absorbing ideal of S. Hence $(\sqrt{I} : x)$ is a 2-absorbing ideal of S (cf. Lemma 2.23).
(5) Let $p \in (\sqrt{I} : x^2)$. Then $x^2p \in \sqrt{I}$ and hence either $x^2 \in \sqrt{I}$ or $xp \in \sqrt{I}$, since $\sqrt{I}$ is a 2-absorbing ideal of S. If $x^2 \in \sqrt{I}$ then $x \in \sqrt{I}$, a contradiction.

Hence $xp \in \sqrt{I}$ implies $p \in (\sqrt{I} : x)$. Therefore $(\sqrt{I} : x^2) \subseteq (\sqrt{I} : x)$. Clearly, $(\sqrt{I} : x) \subseteq (\sqrt{I} : x^2)$. Hence, $(\sqrt{I} : x) = (\sqrt{I} : x^2)$ for all $x \in S - \sqrt{I}$. □

Theorem 2.25. *Let I be an ideal of a semigroup S. Then $\sqrt{I}$ is a 2-absorbing primary ideal of S if and only if $\sqrt{I}$ is a 2-absorbing ideal of S.*

Proof. Suppose $\sqrt{I}$ is a 2-absorbing primary ideal of S and $abc \in \sqrt{I}$ for some $a, b, c \in S$. So either $ab \in \sqrt{I}$ or $bc \in \sqrt{\sqrt{I}} = \sqrt{I}$ or $ca \in \sqrt{\sqrt{I}} = \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of S.
Conversely, let $\sqrt{I}$ be a 2-absorbing ideal of S. Since 2-absorbing ideals are 2-absorbing primary (cf. Theorem 2.2) so clearly $\sqrt{I}$ is a 2-absorbing primary ideal of S. □

Clearly arbitary union of 2-absorbing primary ideals of a semigroup S is 2-absorbing primary ideal but intersection of two 2-absorbing primary ideals need not be 2-absorbing primary ideal of S. We have the following example

Example 2.26. Let $I_1 = 5\mathbb{Z}$ and $I_2 = 6\mathbb{Z}$ be two ideals of the semigroup $(\mathbb{Z}, .)$. Clearly I_1 and I_2 are 2-absorbing primary ideal of S. Then $\sqrt{I_1 \cap I_2} = 30\mathbb{Z}$, which is not a 2-absorbing ideal of S. Hence $I_1 \cap I_2$ is a not a 2-absorbing primary ideal of S (cf. Theorem 2.24(1)).

Let M be the set of all 2-absorbing primary ideals of a semigroup S and we define a relation ρ on M by $I_1 \rho I_2$ if and only if $\sqrt{I_1} = A = \sqrt{I_2}$ for some 2-absorbing ideal A of S. Clearly ρ is a congrurence on M. So every element of a ρ-equivalence class A is a A-2-absorbing primary ideal for some 2-absorbing ideal A of S. Clearly M forms a semigroup with respect to usual set union and each ρ-equivalence class of M is an element of the factor semigroup M/ρ.

Theorem 2.27. *Each ρ-class of M is closed under finte intersections.*

Proof. Let I_1,I_2,....I_n be elements of a ρ-class A for some 2-absorbing ideal A of S. Then $\sqrt{(I_1 \cap I_2 \cap ..... \cap I_n)} = A$. Let $abc \in I = \bigcap I_i$ and $ab \notin I$ for some $a, b, c \in S$. Then $ab \notin I_i$ for some $i \in \{1, 2, ..., n\}$. Hence $bc \in \sqrt{I_i} = A$ or $ac \in \sqrt{I_i} = A$. Hence the result follows.

Therefore the semigroup M can be written as disjoint union of semigroups that is $M = \bigcup\{A : A \in M/\rho\}$. □

Theoprerm 2.28. *Let I_1 and I_2 be two P_1-primary and P_2- primary ideals for some prime ideals P_1 and P_2 of a commutative a semigroup S. Then $I_1 \cap I_2$ and I_1I_2 are 2-absorbing primary ideals of S.*

Proof. The proof is similar to ([3], Theorem 2.4).

Proposition 2.29. *Let I be a proper ideal of a semigroup S. Then the following statements are equivalent*
(1). *For every ideals J, K, L of S such that $I \subseteq J$, $JKL \subseteq I$ implies $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$.*
(2). *For every ideals J,K,L of S, $JKL \subseteq I$ implies either $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$.*

Proof. (1) $\Rightarrow$ (2) Let $JKL \subseteq I$ for some ideals J, K, L of S. Then $(J \cup I)KL = JKL \cup IKL \subseteq I$. Setting $P = I \cup J$ we have $I \subseteq P$ and $PKL \subseteq I$ implies either $PK \subseteq I$ or $PL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$. Therefore either $(J \cup I)K \subseteq I$ or $(J \cup I)L \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$ implies $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$, as desired.
(2) $\Rightarrow$ (1) Straightforward. □

Let $S[x]$ be the polynomial semigroup of a semigroup S. Then if I is an ideal of S, then $I[x] = \{ax^i : a \in I, i \geq 0\}$ is an ideal of $S[x]$ and also we have the following result:

Lemma 2.30. *Let I be an ideal of a semigroup S. Then $\sqrt{I[x]} = \sqrt{I}[x]$.*
Proof. Let $f(x) = ax^i \in \sqrt{I[x]}$ for some $a \in S$ and $i \geq 0$. Then for some $n \in \mathbb{N}$, $a^n x^{in} \in I[x] \Rightarrow a^n \in I \Rightarrow a \in \sqrt{I} \Rightarrow f(x) = ax^i \in \sqrt{I}[x]$. Therefore $\sqrt{I[x]} \subseteq \sqrt{I}[x]$.
Again let $g(x) = bx^i \in \sqrt{I}[x]$. Then $b \in \sqrt{I} \Rightarrow b^n \in I$ for some $n \in \mathbb{N}$. Hence $(g(x))^n = b^n x^{in} \in I[x] \Rightarrow g(x) \in \sqrt{I[x]}$. Therefore $\sqrt{I}[x] \subseteq \sqrt{I[x]}$. Consequently, $\sqrt{I[x]} = \sqrt{I}[x]$. □

Proposition 2.31. *Let S be a commutative semigroup. Then a proper ideal I is a 2-absorbing primary ideal of S if and only if $I[x]$ is a 2-absorbing primary ideal of $S[x]$.*

Proof. Let I be a 2-absorbing primary ideal of S and $(ax^i)(bx^j)(cx^k) \in I[x]$, where $a, b, c \in S$ and $i, j, k \geq 0$. Then $abcx^{i+j+k} \in I[x] \Rightarrow abc \in I \Rightarrow ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, since I is a 2-absorbing primary ideal of S. Hence $(ax^i)(bx^j) = abx^{i+j} \in I[x]$ or $(ax^i)(cx^j) = acx^{i+j} \in \sqrt{I}[x] = \sqrt{I[x]}$ or $(bx^j)(cx^k) = bcx^{j+k} \in \sqrt{I}[x] = \sqrt{I[x]}$, since $\sqrt{I[x]} = \sqrt{I}[x]$ (*cf.* Lemma 2.30). Therefore $I[x]$ is a 2-absorbing primary ideal of $S[x]$.
Conversely, let $I[x]$ be a 2-absorbing primary of $S[x]$ and $abc \in I$ for some $a, b, c \in S$. Then $(ax^i)(bx^j)(cx^k) = abcx^{i+j+k} \in I[x]$, for some $i, j, k \geq 0$. Since $\sqrt{I[x]} = \sqrt{I}[x]$ (*cf.* Lemma 2,30) so $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Therefore I is a 2-absorbing primary ideal of S. □

The following result is a simple consequence of Proposition 2.31:

Corollary 2.32. *Let I be a proper ideal of S. Then I is a 2-absorbing primary ideal of S if and only if $I[x_1, x_2, \ldots, x_n]$ is a 2-absorbing primary ideal of $S[x_1, x_2, \ldots, x_n]$.*

Let C be a non-empty subsemigroup of a commutative semigroup S. Let ρ be the relation defined on $S \times C$ by $(x, a)\rho(y, b)$ if and only if $cay = cxy$ for some $c \in C$. Clearly ρ is a congrurence on $S \times C$. Then $C^{-1}S = \{\frac{s}{c} : s \in S, c \in C\}$ is the quoitent semigroup of $(S \times C)$ modulo ρ, called the semigroup of fractions.
The composition on $C^{-1}S$ is defined as $(\frac{x}{a})(\frac{y}{b}) = \frac{xy}{ab}$. Also if I is an ideal of S, then $C^{-1}I = \{\frac{i}{c} : i \in I, c \in C\}$ is an ideal of $C^{-1}S$, moreover $\sqrt{C^{-1}I} = C^{-1}\sqrt{I}$.

Theorem 2.33. *Let C be a subsemigroup of a semigroup S and I be an ideal of S such that $C \cap I = \phi$. If I is a 2-absorbing primary ideal of S then $C^{-1}I$ is a 2-absorbing primary ideal of $C^{-1}S$.*

proof. Let I be a 2-absorbing primary ideal of S and $(\frac{a}{s})(\frac{b}{r})(\frac{c}{t}) \in C^{-1}I$ for some $a, b, c \in S$ and $s, r, t \in C$. Hence there exists some $p \in C$ such that $abcp \in I$. Therefore either $ab \in I$ or $bcp \in \sqrt{I}$ or $acp \in \sqrt{I}$. Now $ab \in I$ implies $(\frac{a}{s})(\frac{b}{r}) = \frac{ab}{sr} \in C^{-1}I$, $bcp \in \sqrt{I}$ implies $(\frac{b}{r})(\frac{c}{t}) = \frac{bcp}{rtp} \in C^{-1}\sqrt{I} = \sqrt{C^{-1}I}$ and $acp \in \sqrt{I}$ implies $(\frac{a}{s})(\frac{c}{t}) = \frac{acp}{stp} \in C^{-1}\sqrt{I} = \sqrt{C^{-1}I}$, consequently, $C^{-1}I$ is a 2-absorbing primary ideal of $C^{-1}S$. □

Lemma 2.34. *Let $f: S \to S'$ be a homomorphism of semigroups. Then the following statements holds*
(1) $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$, *where I' is an ideal of S'.*
(2) *If f is an isomorphism, then $f(\sqrt{I}) = \sqrt{f(I)}$, where I is an ideal of S.*

Theorem 2.35. *Let $f: S \to S'$ be a homomorphism of semigroups. Then the following statements holds:*
(1) *If I' is a 2-absorbing primary ideal of S', then $f^{-1}(I')$ is a 2-absorbing primary ideal of S.*
(2) *Let I be a proper ideal of S such that $\{(x, y) \in ker f : x \neq y\} \subseteq I \times I$. Then*
(i) *If $f(I)$ is a 2-absorbing primary ideal of S', then I is a 2-absorbing primary ideal of S.*
(ii) *If f is onto and I is a 2-absorbing primary ideal of S, then $f(I)$ is a 2-absorbing primary ideal of S'.*
(3) *If f is an isomorphism and I is a 2-absorbing primary ideal of S, then $f(I)$ is a 2-absorbing primary ideal of S'.*

Proof. (1) The proof is similar as that of ([3],Theorem 2.20(1)).
(2) The proof of (i) and (ii) are similar as that of Theorem 2.18 of [4] by replacing 2-absorbing ideal I as 2-absorbing primary ideal of S and then use the result of (1).
(3) It is trivial. □

As a simple consequence of above theorem, we have the following result

Corollary 2.36. (1) *Let $S \subseteq S'$ be an extension of semigroup S and I be a 2-absorbing primary ideal of S'. Then $I \cap S$ is a 2-absorbing primary ideal of S.*
(2) *Let $I \subseteq J$ be two ideals of S. Then J is a 2-absorbing primary ideal of S if and only if J/I is a 2-absorbing primary ideal of S/I.*

Lemma 2.37. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Then the following statement holds*
(1) *If I_1 is an ideal of S_1, then $\sqrt{I_1 \times S_2} = \sqrt{I_1} \times S_2$.*
(2) *If I_2 is an ideal of S_2, then $\sqrt{S_1 \times I_2} = S_1 \times \sqrt{I_2}$.* (3) $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$.

Theorem 2.38. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Then the following statements holds*
(1) *I_1 is a 2-absorbing primary ideal of S_1 if and only if $I_1 \times S_2$ is a 2-absorbing primary ideal of S.*
(2) *I_2 is a 2-absorbing primary ideal of S_2 if and only if $S_1 \times I_2$ is a 2-absorbing primary ideal of S.*

Proof. (1) Suppose $I_1 \times S_2$ is a 2-absorbing primary ideals of S. Let $abc \in I_1$ for some $a, b, c \in S_1$. Then $(abc, x^3) \in I_1 \times S_2$ for some $x \in S_2$. So $(a, x)(b, x)(c, x) \in I_1 \times S_2$ and hence either $(a, x)(b, x) \in I_1 \times S_2$ or $(b, x)(c, x) \in \sqrt{I_1} \times S_2$ or $(a, x)(c, x) \in \sqrt{I_1} \times S_2$ and so either $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$, which implies I_1 is a 2-absorbing primary ideal of S_1.
Conversely, Suppose I_1 is a 2-absorbing primary ideal of S_1. Let $(a, x)(b, x)(c, x) \in I_1 \times S_2$ for some $a, b, c \in I_1$ and $x \in S$. Then $abc \in I_1$ implies $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$ and hence either $(a, x)(b, x) \in I_1 \times S_2$ or $(b, x)(c, x) \in \sqrt{I_1} \times S_2$ or $(a, x)(c, x) \in \sqrt{I_1} \times S_2$, which implies $I_1 \times S_2$ is a 2-absorbing primary ideal of $S_1 \times S_2$.
(2) The proof is similar to (1). □

Theorem 2.39. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Let I_1 and I_2 are ideals of S_1 and S_2 respectively. If $I = I_1 \times I_2$ is a 2-absorbing primary ideal of S then I_1 and I_2 are 2-absorbing primary ideals of S_1 and S_2 respectively.*

Proof. Let $abc \in I_1$ for some $a, b, c \in S$. Then $(a, x)(b, x)(c, x) = (abc, x^3) \in I_1 \times I_2$ for some $x \in I_2$. Since $I_1 \times I_2$ is a 2-absorbing primary ideal of S, either $(a, x)(b, x) = (ab, x^2) \in I_1 \times I_2$ or $(b, x)(c, x) = (bc, x^2) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ or $(a, x)(c, x) = (ac, x^2) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ (*cf.* Lemma 2.37(3)). Hence $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$. Consequently I_1 is a 2-absorbing primary ideal of S_1. Similarly, we can prove that I_2 is a 2-absorbing primary ideal of S_2. □

Remark 2.40. The following example shows that converse of Theorem 2.39 is not true. Consider the 2-absorbing primary ideals $I_1 = 6\mathbb{Z}$ and $I_2 = 5\mathbb{Z}$ of the semigroup $(\mathbb{Z}, .)$. Then $(2, 3)(3, 2)(5, 5) \in I_1 \times I_2$ but nether $(2, 3)(3, 2) \in I_1 \times I_2$ nor $(2, 3)(5, 5) \in \sqrt{I_1 \times I_2}$ nor $(3, 2)(5, 5) \in \sqrt{I_1 \times I_2}$. Hence $I_1 \times I_2$ is not a 2-absorbing primary ideal of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 2.41. *Let $S = S_1 \times S_2$, where S_1, S_2 are commutative semigroup with zero and identity. Let I be a proper ideal of S. Then the following statements are equivalent*
(1) *I is a 2-absorbing primary ideal of S.*
(2) *Either $I = I_1 \times S_2$ for some 2-absorbing primary ideal I_1 of S_1 or $I = S_1 \times I_2$ for some 2-absorbing primary ideal I_2 of S_2 or $I = I_1 \times I_2$ for some primary ideals I_1 of S_1 and I_2 of S_2 respectively.*

Proof. The proof is similar to ([14], Theorem 2.17).

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