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AN IDENTITY ON STANDARD OPERATOR ALGEBRA

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Abstract. The purpose of this research is to find an extension of the renowned *Chernoff* theorem on standard operator algebra. Infact, we prove the following result: Let H be a real (or complex) Banach space and $\mathcal{L}(H)$ be the algebra of bounded linear operators on H. Let $\mathcal{A}(H) \subset \mathcal{L}(H)$ be a standard operator algebra. Suppose that $D : \mathcal{A}(H) \to \mathcal{L}(H)$ is a linear mapping satisfying the relation

$$D(A^n B^n) = D(A^n)B^n + A^n D(B^n)$$

for all $A, B \in \mathcal{A}(H)$. Then D is a linear derivation on $\mathcal{A}(H)$. In particular, D is continuous. We also present the limitations on such identity by an example.

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1. Introduction

The proposed problem is a generalization of a classical result by Chernoff [5]. Some useful generalization of this idea was obtained in [2, 4, 7, 9, 10, 11]. Let us recall the prerequisite that we need for the development of our main result. We consider R to be an associative ring with identity e throughout. Recall that a ring R is semiprime if for any $x \in R$, $xRx = 0$, implies $x = 0$. A ring R is said to be n-torsion free, if whenever $nx = 0$ for $x \in R$, implies $x = 0$, where $n > 1$ is any integer. A map $D : R \to R$ will be called a derivation if it is additive and fulfils the condition $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. In case D satisfies for all $x \in R$, $D(x^2) = D(x)x + xD(x)$, we call such D a Jordan derivation on R. An additive map $F : R \to R$ is termed as a generalized derivation if there exists a derivation $D : R \to R$ such that $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$. Following Zalar [13], an additive map $h : R \to R$ is termed as left centralizer of R if $h(xy) = h(x)y$ for all $x, y \in R$ and particularly it is called Jordan left centralizer if $h(x^2) = h(x)x$ holds for all $x \in R$. One can think about the structure

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of right centralizer and Jordan right centralizer by same approach. It is obvious fact that every left centralizer and right centralizer is Jordan left centralizer and Jordan right centralizer respectively on R. The converse statement to this fact is not guaranteed. A brief description of the allied matter can be found in [13]. Let $F, D : R \to R$ be two additive mappings which satisfy the following identity $F(x^n y^n) = F(x^n)y^n + x^n D(y^n)$ for all $x, y \in R$. Now, a natural question arises that the additive mappings $F, D : R \to R$ satisfying the above identity having some torsion restrictions on a ring will be a generalized derivation and derivation respectively on R. We have the affirmative answer of this question. The proof of this result can be found in [1]. For more literature review one can look in to [2,3,6,8].

Let H be a real or complex Banach space. $\mathcal{L}(H)$ and $\mathcal{F}(H)$ denote the algebra of all bounded linear operators on H and the ideal of all finite rank operators in $\mathcal{L}(H)$ respectively. An algebra $\mathcal{A}(H) \subseteq \mathcal{L}(H)$ is said to be standard operator algebra in case $\mathcal{F}(H) \subset \mathcal{A}(H)$. Any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. In [5], author obtained that the following:

Theorem 1.1. *Let H be a real or complex Banach space, $\mathcal{A}(H)$ be a standard operator algebra on H and $D : \mathcal{A}(H) \longrightarrow L(H)$ be a linear mapping such that*

$$D(A_1A_2) = D(A_1)A_2 + A_1D(A_2) \textit{for all } A_1, A_2 \in \mathcal{A}(H). \tag{1}$$

Then there exists $B \in L(\mathcal{H})$ such that $D(A) = AB - BA$ for all $A \in \mathcal{A}(H)$.

Another remarkable contribution in this line of investigation found in [11] as below:

Theorem 1.2. *Let H be a real or complex Banach space and let $\mathcal{A}(H)$ be a standard operator algebra on H. Suppose there exists a linear mapping $D : \mathcal{A}(H) \longrightarrow \mathcal{L}(H)$ satisfying the relation*

$$D(A^2) = D(A)A + AD(A) \textit{ for all } A \in \mathcal{A}(H). \tag{2}$$

In this case D is of the form $D(A) = [A, B]$, for all $A \in \mathcal{A}(H)$ and some $B \in \mathcal{L}(H)$.

Our proposed problem deals with the more general case of Theorem 1.1 and Theorem 1.2. In other words, we can say that for $n = 1$ our main theorem is nothing but the Chernoff theorem.

2. Main results

We begin with the below mentioned theorem:

Theorem 2.1. *Let H be a real (or complex) Banach space and $\mathcal{L}(H)$ be the algebra of bounded linear operators on H. Let $\mathcal{A}(H) \subset \mathcal{L}(H)$ be a standard* *operator algebra. Suppose that $D : \mathcal{A}(H) \to \mathcal{L}(H)$ is a linear mapping satisfying the relation*

$$D(A^n B^n) = D(A^n)B^n + A^n D(B^n)$$

for all $A, B \in \mathcal{A}(H)$. Then D is a linear derivation on $\mathcal{A}(H)$. In particular, D is continuous.

Proof. We have

$$D(A^n B^n) = D(A^n)B^n + A^n D(B^n). \tag{3}$$

Let $A, B \in \mathcal{F}(H)$, the ideal of all finite rank operators in $\mathcal{L}(H)$ and $P \in \mathcal{F}(H)$ be projections with $AP = PA = A$ and $BP = PB = B$. Substituting P in place of A, B in (3) to get

$$D(P) = D(P)P + PD(P).$$

Multiplying P from left in the above equation to find

$$PD(P)P = 0. \tag{4}$$

Replacing A by $A + P$ in (3), we have

$$D((A+P)^n B^n) = D((A+P)^n)B^n + (A+P)^n D(B^n). \tag{5}$$

On simplification we have

$$\begin{aligned} D\Big(\big(\sum_{i=0}^{n}\tbinom{n}{i}A^{n-i}P^i\big)B^n\Big) \quad &= \quad \Big[\sum_{i=0}^{n}\tbinom{n}{i}D(A^{n-i}P^i)\Big]B^n \\ &\quad + \Big[\sum_{i=0}^{n}\tbinom{n}{i}A^{n-i}P^i\Big]D(B^n) \end{aligned} \tag{6}$$

$$\sum_{i=0}^{n}\tbinom{n}{i}\Big[D(A^{n-i}P^iB^n) - D(A^{n-i}P^i)B^n - A^{n-i}P^iD(B^n)\Big] = 0. \tag{7}$$

Which can be expressed as $\sum_{i=0}^{n} f_i(A, P, B) = 0$, where

$$f_i(A, P, B) = D(A^{n-i}P^iB^n) - D(A^{n-i}P^i)B^n - A^{n-i}P^iD(B^n).$$

By reinstating $A + kP$ in place of A in (6), we find a system of n homogeneous equations of variables as $f_i(A, P, B)$, where $i = 1, 2, ..., n$ and $k = 1, 2, ..., n$. The coefficient matrix of this resulting system will be Van-der Monde matrix as below

$$\mathbb{V} = \begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^n \\ . & . & ... & \\ . & . & ... & \\ . & . & ... & \\ n & n^2 & ... & n^n \end{bmatrix}.$$

Since $\mathbb{V}$ is equal to the product of positive integers and every element of that product is less than n, then $f_i(A, P, B) = 0$, for all $i = 1, 2, ..., n$. Particularly,

take $i = n$ and $i = n-1$ to obtain following:
$f_n(A,P,B) = 0$, that is,

$$\binom{n}{n}[D(A^0P^nB^n) - D(A^0P^n)B^n - A^0P^nD(B^n)] = 0.$$

This implies that

$$D(B^n) = D(P)B^n + PD(B^n). \tag{8}$$

Multiplying (8) by P from left to get $PD(B^n) = PD(P)B^n + PD(B^n)$. Which lead us

$$PD(P)B^n = 0. \tag{9}$$

Next, we have $f_{n-1}(A,P,B) = 0$, which implies that

$$\binom{n}{n-1}[D(AP^{n-1}B^n) - D(AP^{n-1})B^n - AP^{n-1}D(B^n)] = 0.$$

This implicit that

$$D(AB^n) = D(A)B^n + AD(B^n). \tag{10}$$

Substitute B by $B+P$ in (10) to obtain

$$\begin{aligned} & D\Big(A\sum_{i=0}^{n}\tbinom{n}{i}B^{n-i}P^i\Big) \\ = \; & D(A)\Big[\sum_{i=0}^{n}\tbinom{n}{i}B^{n-i}P^i\Big] + AD\Big[\sum_{i=0}^{n}\tbinom{n}{i}B^{n-i}P^i\Big] \\ = \; & \sum_{i=0}^{n}\tbinom{n}{i}\Big[D(AB^{n-i}P^i) - D(A)B^{n-i}P^i - AD(B^{n-i}P^i)\Big] \\ = \; & 0. \end{aligned} \tag{11}$$

Using same arguments as the above, we conclude that $f'_{n-1}(A,P,B) = 0$,

$$\binom{n}{n-1}[D(ABP) - D(A)BP - AD(B)] = 0.$$

Also we can reword above expression as $D(AB) = D(A)B + AD(B)$. In particular we can have $D(A^2) = D(A)A + AD(A)$. Let us define a mapping $D_1 : \mathcal{A}(H) \longrightarrow \mathcal{L}(H)$ such that $D_1(A) = AB - BA$ and consider $D_1 = D - D_0$. Using same techniques as above, we have
$S = A + PAP - (AP + PA)$, $D_1(S) = D_1(A)$, $SP = PS = 0$.

$$D_1(A^{2n}) = D_1(A^n)A^n + A^nD_1(A_1^n), \text{ for all } A \in \mathcal{A}(H). \tag{12}$$

This implies that

$$\begin{aligned}
D_1(S^n)S^n + S^n D_1(S^n) &= D_1(S^{2n}) \\
&= D_1(S^{2n} + P) \\
&= D_1((S+P)^{2n}) \\
&= D_1(S+P)^n(S+P)^n + (S+P)^n D_1(S+P)^n \\
&= D_1(S^n+P)(S^n+P) + (S^n+P)D_1(S^n+P) \\
&= (D_1(S^n) + D_1(P))(S^n+P) + (S^n+P)(D_1(S^n) \\
&\quad +D_1(P)) \\
&= D_1(S^n)S^n + D_1(S^n)P + D_1(P)S^n + D_1(P)P \\
&\quad +S^n D_1(S^n) + S^n D_1(P) + PD_1(S^n) + PD_1(P) \\
&= 0
\end{aligned}$$

Making use of (12), we can find

$$D_1(A^n)P + PD_1(A^n) = 0. \tag{13}$$

Since $D_1(S) = D_1(A)$, we use the same steps as earlier to get

$$D_1(A)P + PD_1(A) = 0. \tag{14}$$

Multiplying (14) by P from right to obtain

$$PD_1(A)P = 0. \tag{15}$$

Multiplying (14) by P from left to get

$$D_1(A)P = 0. \tag{16}$$

Since P is an arbitrary 1-dimensional projection, hence in view of (15) and (16) it gives us $D_1(A) = 0$ for any $A \in \mathcal{A}(H)$. Thus, we have $D(A) = D_0(A) = AB - BA$ for all $A \in \mathcal{A}(H)$, as desired.

Example 2.2. *Notice that as a limitation of our main theorem, there exists a densely defined unbounded and closed operator $\mathcal{A}$ on a Hilbert space such that the domain $D(\mathcal{A}^2) = 0$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two unbounded self-adjoint operators such that*

$$D(\mathcal{T}_1) \cap D(\mathcal{T}_2) = D(\mathcal{T}_1^{-1}) \cap D(\mathcal{T}_2^{-1}) = \{0\}.$$

Clearly $\mathcal{T}_1^{-1}$ and $\mathcal{T}_2^{-1}$ are not bounded. Let us think about an operator defined as

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & \mathcal{T}_1^{-1} \\ \mathcal{T}_2 & 0 \end{pmatrix} \right\}.$$

Since $D(\mathcal{A}) := D(\mathcal{T}_2) \oplus D(\mathcal{T}_1{}^{-1}) \subseteq \mathrm{L}^2(\mathbb{R}) \oplus \mathrm{L}^2(\mathbb{R})$, we have $\mathcal{T}_1{}^{-1}$ and $\mathcal{T}_2$ are closed. Now, consider

$$\mathcal{A}^2 = \left\{ \begin{pmatrix} \mathcal{T}_1^{-1}\mathcal{T}_2 & 0 \\ 0 & \mathcal{T}_2\mathcal{T}_1^{-1} \end{pmatrix} \right\}.$$

This implies that $D(\mathcal{A}^2) = D(\mathcal{T}_1^{-1}\mathcal{T}_2) \oplus D(\mathcal{T}_2\mathcal{T}_1^{-1}) = \{0\} \oplus \{0\} = \{(0,0)\}$. Hence, we come up with an operator whose square has a trivial domain.

3. Conclusion

We conclude that the theorem of Chernoff can be obtained by setting $n = 1$ in Theorem 2.1. This interpretation might be of some interest from the automatic continuity point of view. Specifically, we obtain some consequences in the context of automatic continuity theory. Interested researcher for continuity of derivations and operators are referred to [8, 10]. It can also be helpful for those who want to pursue research work in reference of range inclusion results for derivations on rings and algebras as in [12].

4. Conjecture

Our interpretation and calculation enable us to propose following conjecture for the case of a ring without identity.

Theorem 4.1. *Let $\mathfrak{R}$ be a semiprime ring (not necessarily having identity) with suitable torsion condition and $\mathfrak{F}, \mathfrak{D} : R \longrightarrow R$ be two additive mappings satisfying the algebraic identity*

$$\mathfrak{F}(x^n y^n) = \mathfrak{F}(x^n) y^n + x^n \mathfrak{D}(y^n)$$

for all $x, y \in \mathfrak{R}$. Then $\mathfrak{F}$ is a generalized derivation on $\mathfrak{R}$ associated with a derivation $\mathfrak{D}$.

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