SOME IDENTITIES FOR MULTIPLE \((h, p, q)\)-HURWITZ-EULER ETA FUNCTION†

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ABSTRACT. In this paper, we construct the multiple \((h, p, q)\)-Hurwitz-Euler eta function by generalizing the multiple Hurwitz-Euler eta function. We get some explicit formulas and properties of the higher-order \((h, p, q)\)-Euler numbers and polynomials.

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1. Introduction

The field of the special functions such as the gamma and beta functions, special polynomials, the hypergeometric functions, the zeta and related functions, \(q\)-series, \((p, q)\)-series, and series representations is an ever expanding area in advanced mathematics, applied mathematics, probability, mathematical statistics, and physics. In particular, special polynomials play a fundamental role in applied mathematics, physics, science, and industry (see [1-15]). Choi and Srivastava presented a generalized Hurwitz formula and Hurwitz-Euler eta function (see [5, 6]). It is the purpose of this paper to introduce and investigate a new some generalizations of the \((p, q)\)-Euler numbers and polynomials, \((p, q)\)-Euler zeta function, \((p, q)\)-Hurwitz-Euler zeta function. We call them multiple \((h, p, q)\)-Euler numbers and polynomials, multiple \((h, p, q)\)-Euler zeta function, and multiple \((h, p, q)\)-Hurwitz-Euler eta function. The structure of the paper is as follows: In Sect. 2, we define higher-order \((h, p, q)\)-Euler numbers and polynomials and derive some of their properties involving elementary properties, distribution relation, and so on. In Sect. 3, by using the higher-order \((h, p, q)\)-Euler numbers and polynomials, multiple \((h, p, q)\)-Euler zeta function
and multiple \((h, p, q)\)-Hurwitz-Euler eta function are defined. We also contains some connection formulae between the higher-order \((h, p, q)\)-Euler polynomials and the multiple \((h, p, q)\)-Hurwitz-Euler eta function.

Throughout this paper, we always make use of the following notations: \(\mathbb{N}\) denotes the set of natural numbers, \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\) denotes the set of nonnegative integers, \(\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}\) denotes the set of nonpositive integers, \(\mathbb{Z}\) denotes the set of integers, \(\mathbb{R}\) denotes the set of real numbers, and \(\mathbb{C}\) denotes the set of complex numbers. We use the notation

\[
\sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} = \sum_{k_1, \ldots, k_r=0}^{\infty}.
\]

We would like to review definitions related to \(q\)-number and \((p, q)\)-number used in this paper. For any \(m \in \mathbb{N}\), \(q\)-number can be defined as follows

\[
[m]_q = \frac{1 - q^m}{1 - q} = \sum_{i=0}^{m-1} q^i = 1 + q + q^2 + \cdots + q^{m-1}.
\]

For \(z \in \mathbb{C}\), the \((p, q)\)-number is defined by

\[
[z]_{p,q} = \frac{p^z - q^z}{p - q}, \quad (p \neq q).
\]

With the \((p, q)\)-number, the necessary elements of the \((p, q)\)-calculus, namely, \((p, q)\)-integration, \((p, q)\)-differentiation, \((p, q)\)-exponential, were worked by many mathematicians. Many \((p, q)\)-extensions of some special functions and polynomials have been studied (see [1, 2, 7, 12, 13, 14, 15]).

The binomial formulae are known as

\[
(1 - b)^n = \sum_{k=0}^{n} \binom{n}{k} (-b)^k, \quad \text{where} \quad \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!},
\]

and

\[
\frac{1}{(1 - b)^n} = (1 - b)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-b)^k = \sum_{k=0}^{\infty} \binom{n + k - 1}{i} b^k.
\]

Choi and Srivastava [5] constructed and studied the multiple Hurwitz-Euler eta function \(\eta_r(s, a)\) defined by following \(r\)-ple series:

\[
\eta_r(s, a) = \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{(k_1 + \cdots + k_r + a)^s}, \quad (Re(s) > 0; a > 0; r \in \mathbb{N}).
\]

It is known that \(\eta_r(s, a)\) can be continued analytically to be whole complex \(s\)-plane. Inspired by their work, the \((h, p, q)\)-extension of the multiple Hurwitz-Euler eta function can be defined as follows: For \(s, x \in \mathbb{C}\) with \(Re(x) > 0\) and \(r \in \mathbb{N}\), the multiple \((h, p, q)\)-Hurwitz-Euler eta function \(\eta_{p,q}^{(r,h)}(s,x)\) is define by

\[
\eta_{p,q}^{(r,h)}(s,x) = [2]_q^r \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r}}{[k_1 + \cdots + k_r + x]^s_{p,q}}.
\]
Some identities for multiple \((h, p, q)\)-Hurwitz-Euler eta function

Note that if \(p = 1, q \to 1\), then \(\eta^{(r,h)}_{p,q}(s, a) = 2^r \eta_{r}(s, a)\). Another type of multiple \((h, p, q)\)-Euler zeta function \(\zeta^{(r,h)}_{p,q}(s)\) can be defined as follows. For \(s \in \mathbb{C}\), we define

\[
\zeta^{(r,h)}_{p,q}(s) = [2]^q \sum_{m=1}^{\infty} \left( \frac{m + r - 1}{m} \right) (-1)^{m} p^h m^q m^{r-1} \frac{1}{[m]_{p,q}}.
\]

Observe that if \(r = 1\), then \(\zeta^{(1,h)}_{p,q}(s) = \zeta_{p,q}(s)\) (see [13]). By using the symmetric properties about the multiple \((h, p, q)\)-Hurwitz-Euler eta function, we obtain symmetric identities about the higher-order \((h, p, q)\)-Euler numbers and polynomials. Firstly, we introduce the basic definitions related to higher-order \((h, p, q)\)-Euler numbers and polynomials.

**Definition 1.1.** The classical Euler polynomials \(E_n(x)\) are defined by the following generating function

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi).
\]

When \(x = 0\), \(E_n = E_n(0)\) are called the Euler numbers \(E_n\).

**Definition 1.2.** For \(r \in \mathbb{N}\), the classical higher-order Euler polynomials \(E^{(r)}_n(x)\) are defined by the following generating function:

\[
\left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E^{(r)}_n(x) \frac{t^n}{n!}, \quad (|t| < \pi).
\]

As usual, the numbers \(E^{(r)}_n = E^{(r)}_n(0)\) are called higher-order Euler numbers.

Much research has been done in the area of special functions by using \((p, q)\)-number (see [1, 2, 7, 12, 13, 15]). Some interesting properties of the \((h, p, q)\)-Euler numbers \(E^{(h)}_{n,p,q}\) polynomials \(E^{(h)}_{n,p,q}(x)\) were first investigated by Ryoo [13].

**Definition 1.3.** For \(0 < q < p \leq 1\) and \(h \in \mathbb{Z}\), \((h, p, q)\)-Euler numbers \(E^{(h)}_{n,p,q}\) and \((h, p, q)\)-Euler polynomials \(E^{(h)}_{n,p,q}(x)\) are defined by means of the generating functions

\[
\sum_{n=0}^{\infty} E^{(h)}_{n,p,q} \frac{t^n}{n!} = [2]^q \sum_{l=0}^{\infty} (-1)^l p^h q^l e^{[l]_{p,q} t}
\]

and

\[
\sum_{n=0}^{\infty} E^{(h)}_{n,p,q}(x) \frac{t^n}{n!} = [2]^q \sum_{l=0}^{\infty} (-1)^l p^h q^l e^{[l+x]_{p,q} t}
\]

respectively.
2. Higher-order \((h, p, q)\)-Euler numbers and polynomials

In this section, we consider the higher-order \((h, p, q)\)-Euler numbers and polynomials as follows:

**Definition 2.1.** For \(0 < q < p \leq 1\), \(h \in \mathbb{Z}\), and \(r \in \mathbb{N}\), higher-order \((h, p, q)\)-Euler numbers \(E_{n,p,q}^{(r,h)}\) and higher-order \((h, p, q)\)-Euler polynomials \(E_{n,p,q}^{(r,h)}(x)\) are defined by the following generating functions

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)}(x) \frac{t^n}{n!} = [2]_{q}^{r} \sum_{k_1, \ldots, k_r = 0}^\infty (-q)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} e^{[k_1 + \cdots + k_r + x]_{p,q} t},
\]

and

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)} t^n = [2]_{q}^{r} \sum_{k_1, \ldots, k_r = 0}^\infty (-q)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} e^{[k_1 + \cdots + k_r + x]_{p,q} t},
\]

respectively.

Note that if \(r = 1\), then \(E_{n,p,q}^{(r,h)} = E_{n,p,q}^{(h)}\) and \(E_{n,p,q}^{(r,h)}(x) = E_{n,p,q}^{(h)}(x)\). Observe that if \(p = 1, q \rightarrow 1\), then \(E_{n,p,q}^{(r,h)} \rightarrow E_{n}^{(r)}\) and \(E_{n,p,q}^{(r,h)}(x) ightarrow E_{n}^{(r)}(x)\).

From (1) and (2), we note that

**Theorem 2.2.** For \(0 < q < p \leq 1\), \(h \in \mathbb{Z}\), and \(r \in \mathbb{N}\), we have

\[
E_{n,p,q}^{(r,h)}(x + y) = \sum_{l=0}^{n} \binom{n}{l} p^l q^{y(n-l)} [y]_{p,q}^l E_{n-l,p,q}^{(r,h+l)}(x),
\]

\[
E_{n,p,q}^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^l x^l [x]_{p,q}^l E_{n-l,p,q}^{(r)},
\]

**Theorem 2.3.** For \(r \in \mathbb{N}\) and \(h \in \mathbb{Z}\), we have

\[
E_{n,p,q}^{(r,h)}(x) = [2]_{q}^{r} \sum_{k_1, \ldots, k_r = 0}^\infty (-q)^{k_1 + \cdots + k_r} p^{h k_1 + h k_r} e^{[k_1 + \cdots + k_r + x]_{p,q} n}
\]

\[
= \frac{[2]_{q}^{r}}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l+1} q^l x^{p(n-l)} \left( \frac{1}{1 + q^{l+1} p^{h+n-l}} \right)^r.
\]

**Proof.** By the Taylor series expansion of \(e^{[x]_{p,q} t}\), we have

\[
\sum_{l=0}^{\infty} E_{l,p,q}^{(r)}(x) \frac{t^l}{l!}
\]

\[
= \frac{[2]_{q}^{r}}{n} \sum_{k_1, \ldots, k_r = 0}^\infty (-1)^{k_1 + \cdots + k_r} p^{h k_1 + \cdots + h k_r} q^{k_1 + \cdots + k_r} e^{[k_1 + \cdots + k_r + x]_{p,q} t}
\]

\[
= \sum_{l=0}^{\infty} \binom{[2]_{q}^{r}}{n} \sum_{k_1, \ldots, k_r = 0}^\infty (-q)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} [k_1 + \cdots + k_r + x]_{p,q}^l \frac{t^l}{l!}.
\]
The first part of the theorem follows when we compare the coefficients of $t^r$ in
the above equation. By $(p, q)$-numbers and binomial expansion, we also note
that

$$E^{(r,h)}_{n,p,q}(x) = \left[\frac{2}{p-q}\right] \sum_{k_1, \ldots, k_r = 0}^{\infty} (-q)^{k_1 + \cdots + k_r} p^{h k_1 + \cdots + k_r} \left[ (p^{k_1 + \cdots + k_r} + x - q^{k_1 + \cdots + k_r} + x) \right]$$

This completes the proof of Theorem 2.3. □

**Theorem 2.4.** For $r \in \mathbb{N}$, we have

$$E^{(r,h)}_{n,p,q}(x) = \left[\frac{2}{p-q}\right] \sum_{m=0}^{\infty} \binom{r + m - 1}{m} (-1)^m q^m [m + x]_{p,q}^n.$$

**Proof.** By Taylor-Maclaurin series expansion of $(1 - a)^{-n}$, we have

$$\left(\frac{1}{1 + q^{l+1} p^{n-l+h}}\right)^r = \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (-1)^m (q^{l+1} p^{n-l+h})^m.$$

Also, by (4) and binomial expansion, one can obtain the desired result immediately. □

For $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod} 2)$, by Theorem 2.3, we can show

$$E^{(r,h)}_{n,p,q}(x) = \frac{[2]}{p-q} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l p^{(n-l)x} \times \sum_{a_1, \ldots, a_r = 0}^{d-1} \sum_{k_1, \ldots, k_r = 0}^{\infty} (-1)^{a_1 + \cdots + a_r} (-1)^{k_1 + \cdots + k_r} \times q^{l+1} ((a_1 + dk_1 + \cdots + a_r + dk_r) p^{(n-l+h)(a_1 + dk_1 + \cdots + a_r + dk_r)}.$$
Theorem 2.5. (Distribution relation of higher-order \((h, p, q)\)-Euler polynomials). For \(d \in \mathbb{N}\) with \(d \equiv 1 \mod 2\), we have
\[
E_{n,p,q}^{(r,h)}(x) = \frac{[2]^r_{q_d}d}{{[2]^r_{q^d}d}_{p,q}} \sum_{a_1,\ldots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E_{n,p^d,q^d}^{(r,h)} \left( \frac{a_1+\cdots+a_r+x}{d} \right).
\]

Proof. Since
\[
E_{n,p^d,q^d}^{(r,h)} \left( \frac{a_1+\cdots+a_r+x}{d} \right)
= \frac{[2]_{q^d}^r}{(p^d-q^d)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l p^{(n-l)(a_1+\cdots+a_r+x)} \left( \frac{1}{1+q^{d(l+1)p^{d(n-l+h)}}} \right)^r,
\]
we have
\[
\sum_{a_1,\ldots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E_{n,p^d,q^d}^{(r,h)} \left( \frac{a_1+\cdots+a_r+x}{d} \right)
= \frac{[2]_{q^d}^r}{(p^d-q^d)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l p^{(n-l)(a_1+\cdots+a_r)} \left( \frac{1}{1+q^{d(l+1)p^{d(n-l+h)}}} \right)^r.
\]
Hence, by \((5)\) and Theorem 2.3, we have
\[
\frac{[2]_{q^d}^r}{[2]_{q^d}^r} d p,q \sum_{a_1,\ldots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E_{n,p^d,q^d}^{(r,h)} \left( \frac{a_1+\cdots+a_r+x}{d} \right)
= \frac{[2]_{q^d}^r}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l p^{(n-l)x} \left( \frac{1}{1+q^{d(l+1)p^{n-l+h}}} \right)^r.
\]
This completes the proof of Theorem 2.5. \(\square\)

3. Multiple \((h, p, q)\)-Hurwitz-Euler eta function

In this section, we define multiple \((h, p, q)\)-Hurwitz-Euler eta function. This function interpolates the higher-order \((h, p, q)\)-Euler polynomials at negative integers.

Choi and Srivastava [5] defined the multiple Hurwitz-Euler eta function \(\eta_r(s, a)\) by means of
\[
\eta_r(s, a) = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{(k_1+\cdots+k_r+a)^s}, \quad (\text{Re}(s) > 0; a > 0; r \in \mathbb{N}).
\]
It is known that \( \eta_r(s, a) \) can be continued analytically to be whole complex \( s \)-plane (see [5]). The \((h, p, q)\)-extension of the multiple Hurwitz-Euler eta function can be defined as follows:

**Definition 3.1.** For \( s, x \in \mathbb{C} \) with \( \text{Re}(x) > 0 \), the multiple \((h, p, q)\)-Hurwitz-Euler eta function \( \eta_{p,q}^{(r,h)}(s, x) \) is define by

\[
\eta_{p,q}^{(r,h)}(s, x) = [2]^r_q \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} q^{k_1 + \cdots + k_r}}{[k_1 + \cdots + k_r + x]^s_{p,q}}.
\]

Observe that if \( p = 1, q \to 1 \), then \( 2^r \eta_{p,q}^{(r,h)}(s, a) = \eta_r(s, a) \). Let

\[
F_{p,q}^{(r,h)}(t, x) = \sum_{n=0}^{\infty} E_{n, p,q}^{(r,h)}(x) \frac{t^n}{n!} = [2]^r_q \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} q^{k_1 + \cdots + k_r} e^{[k_1 + \cdots + k_r + x]_{p,q} t}.
\]

**Theorem 3.2.** For \( r \in \mathbb{N} \), we have

\[
\eta_{p,q}^{(r,h)}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r,h)}(x, -t) t^{s-1} dt,
\]

where \( \Gamma(s) = \int_0^\infty z^{s-1} e^{-z} dz \).

**Proof.** From (7) and Definition 3.1, we get

\[
\eta_{p,q}^{(r,h)}(s, x) = [2]^r_q \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} q^{k_1 + \cdots + k_r}}{[k_1 + \cdots + k_r + x]^s_{p,q}}
\]

\[
= [2]^r_q \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} q^{k_1 + \cdots + k_r}}{[k_1 + \cdots + k_r + x]^s_{p,q}} \int_0^\infty z^{s-1} e^{-z} dz
\]

\[
= [2]^r_q \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(-1)^{k_1 + \cdots + k_r} p^{h(k_1 + \cdots + k_r)} q^{k_1 + \cdots + k_r}}{[k_1 + \cdots + k_r + x]^s_{p,q}} \int_0^\infty e^{[k_1 + \cdots + k_r + x]_{p,q} t} t^{s-1} dt
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r,h)}(x, -t) t^{s-1} dt.
\]

This completes the proof of Theorem 3.2. \( \Box \)

The value of multiple \((h, p, q)\)-Hurwitz-Euler eta function \( \eta_{p,q}^{(r,h)}(s, x) \) at negative integers is given explicitly by the following theorem:

**Theorem 3.3.** Let \( n \in \mathbb{N} \). Then we obtain

\[
\eta_{p,q}^{(r,h)}(-n, x) = E_{n, p,q}^{(r,h)}(x).
\]
Proof. Again, by (7) and (8), we have
\[
\eta_{p,q}^{(r,h)}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r,h)}(x, -t) t^{s-1} dt
\]
(9)
\[
= \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,p,q}^{(r,h)}(x) \frac{(-1)^m}{m!} \int_0^\infty t^{m+s-1} dt.
\]
We note that
\[
\Gamma(-n) = \int_0^\infty e^{-z} z^{-n-1} dz
\]
\[
= \lim_{z \to 0} 2\pi i \frac{1}{n!} \left( \frac{d}{dz} \right)^n (z^{n+1} e^{-z} z^{-n-1})
\]
(10)
\[
= 2\pi i \frac{(-1)^n}{n!}.
\]
For \( n \in \mathbb{N} \), let us take \( s = -n \) in (8). Then, by (9), (10), and Cauchy residue theorem, we have
\[
\eta_{p,q}^{(r,h)}(-n, x) = \lim_{s \to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,p,q}^{(r,h)}(x) \frac{(-1)^m}{m!} \int_0^\infty t^{m+n-1} dt
\]
\[
= 2\pi i \left( \lim_{s \to -n} \frac{1}{\Gamma(s)} \right) \left( E_{n,p,q}^{(r,h)}(x) \frac{(-1)^n}{n!} \right)
\]
\[
= 2\pi i \left( \frac{1}{2\pi i \frac{(-1)^n}{n!}} \right) \left( E_{n,p,q}^{(r,h)}(x) \frac{(-1)^n}{n!} \right)
\]
\[
= E_{n,p,q}^{(r,h)}(x).
\]
This completes the proof of Theorem 3.3. \( \square \)

Let
\[
F_{p,q}^{(r,h)}(t) = \sum_{l=0}^{\infty} E_{l,p,q}^{(r,h)} \frac{t^l}{l!}
\]
(11)
\[
= [2]^r \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r} e^{k_1+\cdots+k_r} |k_1+\cdots+k_r|_{p,q} t.
\]
By the \( l \)-th differentiation on both side of (11) at \( t = 0 \), we obtain the following
\[
\frac{d^l}{dt^l} F_{p,q}^{(r,h)}(t) \bigg|_{t=0}
\]
\[
= [2]^r \sum_{k_1, \ldots, k_r=0}^{\infty} (-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r} |k_1+\cdots+k_r|_{p,q}^l
\]
(12)
\[
= E_{l,p,q}^{(r,h)}, (l \in \mathbb{N}).
\]
By using the above equation, we are now ready to define multiple \((h, p, q)\)-Euler eta function. We define multiple \((h, p, q)\)-Euler eta function as follows:

**Definition 3.4.** For \(s \in \mathbb{C}\), we define

\[
\eta_{p,q}^{(r,h)}(s) = [2]^r q \sum_{k_1, \ldots, k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r}}{|k_1+\cdots+k_r|_{p,q}^s}.
\]

Relation between \(\zeta_{p,q}^{(r,h)}(s)\) and \(E_{n,p,q}^{(r,h)}\) is given by the following theorem.

**Theorem 3.5.** Let \(n \in \mathbb{N}\), we have

\[
\zeta_{p,q}^{(r,h)}(-n) = E_{n,p,q}^{(r,h)}.
\]

By (4), we have

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)} \frac{t^n}{n!} = [2]^r q \sum_{m=0}^{\infty} \frac{(-1)^m p^m q^m e^{m[r,p,q]t}}{m!}.
\]

By using Taylor series of \(e^{m[r,p,q]t}\) in the above, we have

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( [2]^r q \sum_{m=0}^{\infty} \frac{(-1)^m p^m q^m [m]_{p,q}^n}{m!} \right) \frac{t^n}{n!}.
\]

By comparing coefficients \(\frac{t^n}{n!}\) in the above equation, we have

\[
E_{n,p,q}^{(r,h)} = [2]^r q \sum_{n=0}^{\infty} \left( [2]^r q \sum_{m=0}^{\infty} \frac{(-1)^m p^m q^m [m]_{p,q}^n}{m!} \right) \frac{t^n}{n!}.
\]

By using (13), another type of multiple \((h, p, q)\)-Euler zeta function can be defined as follows.

**Definition 3.6.** For \(s \in \mathbb{C}\), we define

\[
\zeta_{p,q}^{(h)}(s) = [2]^r q \sum_{m=1}^{\infty} \frac{(-1)^m p^m q^m [m]_{p,q}^n}{m!}.
\]

The function \(\zeta_{p,q}^{(h)}(s)\) interpolates the number \(E_{n,p,q}^{(r,h)}\) at negative integers. Substituting \(s = -n\) with \(n \in \mathbb{N}\) into (14), and using (13), we obtain the following theorem and corollary:

**Theorem 3.7.** Let \(l \in \mathbb{N}\), we have

\[
\eta_{p,q}^{(r,h)}(-l) = \zeta_{p,q}^{(r,h)}(-l) = E_{l,p,q}^{(r,h)}.
\]

**Corollary 3.8.** For \(0 < q < p \leq 1\), \(h \in \mathbb{Z}\), \(r \in \mathbb{N}\), and \(n \in \mathbb{N}\), we have

\[
\sum_{m=1}^{\infty} \frac{(-1)^m p^m q^m [m]_{p,q}^n}{m!} = \sum_{k_1, \ldots, k_r=1}^{\infty} (-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r} [k_1+\cdots+k_r]_{p,q}^n.
\]
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