SCREEN SLANT LIGHTLIKE SUBMERSIONS†

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ABSTRACT. We introduce two new classes of lightlike submersions, namely, screen slant and screen semi-slant lightlike submersions from an indefinite Kaehler manifold to a lightlike manifold giving characterization theorems with non trivial examples for both classes. Integrability conditions of all distributions related to the definitions of these submersions have been obtained.

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1. Introduction

In [6], Sahin and Gündüzalp gave the definition of a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold. In [3, 4], Sahin introduced the notions of slant and screen-slant lightlike submanifolds of an indefinite Hermitian manifold. Following this, Shukla and Yadav defined a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold in [13]. From [12], we conclude that, contrary to the Riemannian slant submersions [5], slant lightlike submersions do not include invariant and anti-invariant subcases. To address this gap, we define screen slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold, which includes invariant and anti-invariant lightlike submersions. The paper is arranged as:

Section 2 is devoted to the basic geometry related to this study. In section 3, we define a screen slant lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold with a non-trivial example. In this section, we also give a characterization theorem and obtain a necessary and sufficient condition for the screen distribution to define a totally geodesic foliation. In the last section, we define a screen semi-slant lightlike submersion from an indefinite

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Kaehler manifold onto a lightlike manifold with non-trivial examples and obtain
the integrability conditions of distributions involved in the definition of these
submersions.

2. Preliminaries

A complex manifold $M$ with a semi-Riemannian metric $g$ of index $r$, where
$0 < r \leq 2m$ and an almost complex structure $\mathcal{J}$ is called an indefinite Hermitian
manifold, if

$$
g(U_1, U_2) = g(\mathcal{J}U_1, \mathcal{J}U_2), \quad \forall \ U_1, U_2 \in \Gamma(TM).
$$

Further, if $(M, \mathcal{J}, g)$ is an indefinite Hermitian manifold with the Levi-Civita
connection $\nabla$ on $M$, then we call $M$ an indefinite Kaehler manifold if

$$
(\nabla_U \mathcal{J})U_2 = 0, \quad \forall \ U_1, U_2 \in \Gamma(TM).
$$

Null (or radical) space $\text{Rad}_M M$ of $T_p M$ is defined as $\text{Rad}_M M = \{\xi \in T_p M : \ g(U, \xi) = 0, \ \forall \ U \in T_p M\}$. If $\text{Rad}_M M : p \rightarrow \text{Rad}_M M$ gives a $C^\infty$
distribution of rank $(r > 0)$ on $M$ such that $0 < r \leq m$, then $\text{Rad}_M M$ is called a
radical distribution on $M$. In this case, we say that manifold $M$ is an $r$-lightlike
manifold.

Let $\phi : M_1 \rightarrow M_2$ be a smooth submersion from a semi-Riemannian manifold
$M_1$ to a lightlike manifold $M_2$. Then, $\text{Ker}_p \phi = \{U \in T_p M_1 : \phi_p U = 0\}$. It
follows that $(\text{Ker}_p \phi)^\perp = \{V \in T_p M_1 : g(U, V) = 0, \ \forall \ U \in \ker \phi\}$ and
$\ker \phi \cap (\ker \phi)^\perp = 0$. In this case $\Delta : p \rightarrow \ker \phi$ is said to be a
radical distribution on $M_1$ at $p \in M_1$. As $\Delta$ is a lightlike distribution, we have
$\ker \phi = \Delta \perp S(\ker \phi)$. Similarly $(\ker \phi)^\perp = \Delta \perp S(\ker \phi)^\perp$. Assume
that $\dim(\Delta) = r$. As $\Delta \subset (\ker \phi)^\perp$ and $(\ker \phi)^\perp$ is non-
degenerate, so there exists $N_1, N_2, ..., N_r$, such that $g(N_i, N_j) = 0, \ g(\xi_i, N_j) = \delta_{ij}$. Here
$\{N_i\}$ are null vector fields of $(\ker \phi)^\perp$ and $\{\xi_i\}$ is the lightlike basis of $\Delta$. The distribution generated by vector fields $N_1, N_2, ..., N_r$ is denoted by $\text{ltr}(\ker \phi)$. Then $\text{ltr}(\ker \phi) = \text{ltr}(\ker \phi) \perp S(\ker \phi)^\perp$. Moreover, we have the following decomposition

$$
TM = S(\ker \phi) \perp (\Delta \oplus \text{ltr}(\ker \phi)) \perp S(\ker \phi)^\perp. \quad (3)
$$

Let $\phi : M_1 \rightarrow M_2$ be a Riemannian submersion, then $\phi$ is called an $r$-lightlike
submersion if

$$
\dim \Delta = \dim\{\ker \phi \perp \cap (\ker \phi)\} = r,
$$

where $0 < r < \min\{\dim(\ker \phi), \dim(\ker \phi)^\perp\}$.

The geometry of lightlike submersions is pictured by tensors $A$ and $T$ given by

$$
A_{U_1} U_2 = h\nabla_{hU_1} U_2 + h\nabla_{hU_1} hU_2, \quad (4)
$$

$$
T_{U_1} U_2 = \nu\nabla_{\nu U_1} U_2 + h\nabla_{\nu U_1} hU_2, \quad (5)
$$

Tensors $A$ and $T$ are horizontal and vertical tensors, respectively. Moreover, $T$
has symmetric property for vertical vector fields $U_1$ and $U_2$, that is, $T_{U_1} U_2 =
T_{U_2} U_1$. 


Let $M_1$ and $M_2$ be semi-Riemannian and lightlike manifolds, respectively. Next, we assume that $\phi : M_1 \to M_2$ be a lightlike submersion with lightlike distribution $\text{Ker} \phi_*$ on $M_1$. Further, suppose that $\text{tr}(\text{Ker} \phi_*)$ is the complementary distribution to $\text{Ker} \phi_*$ in $M_1$. Let $\hat{g}$ and $\nabla$ stands for induced metric on $\text{Ker} \phi_*$ of $g$ and Levi-Civita connection on $M_1$, respectively. Using (5), $\forall U_1, U_2 \in \Gamma(\text{Ker} \phi_*)$ and $V \in \Gamma(\text{Ker} \phi_*)^\perp$, we have

$$\nabla_{U_1} U_2 = \nabla_{U_1} U_2 + T_{U_1} U_2, \quad (6)$$

$$\nabla_{U_1} V = T_{U_1} V + \nabla_{U_1}^\perp V, \quad (7)$$

where $\nabla_{U_1} U_2 = \nu \nabla_{U_1} U_2$ and $\nabla_{U_1}^\perp V = h \nabla_{U_1} V$. Here $\{\nabla_{U_1} U_2, T_{U_1} V\}$ and $\{T_{U_1} U_2, \nabla_{U_1}^\perp V\}$ belong to $\Gamma(\text{Ker} \phi_*)$ and $\Gamma(\text{tr}(\text{Ker} \phi_*))$, respectively. Let $S(\text{Ker} \phi_*)^\perp \neq \{0\}$. Denote by $L$ and $S$ the projections of $\text{tr}(\text{Ker} \phi_*)$ on $\text{ltr}(\text{Ker} \phi_*)$ and $S(\text{Ker} \phi_*)^\perp$, respectively. Then, from (6) and (7), we have

$$\nabla_{U_1} U_2 = \hat{\nabla}_{U_1} U_2 + T_{U_1}^l U_2 + T_{U_1} U_2, \quad (8)$$

$$\nabla_{U_1} N = T_{U_1} N + \nabla_{U_1}^l N + D^{ls}(U_1, N), \quad (9)$$

$$\nabla_{U_1} W = T_{U_1} W + D^{ll}(U_1, W) + \nabla_{U_1}^l W, \quad (10)$$

$\forall U_1, U_2 \in \Gamma(\text{Ker} \phi_*)$, $V \in \Gamma(S(\text{Ker} \phi_*)^\perp)$ and $N \in \Gamma(\text{ltr}(\text{Ker} \phi_*))$. Using (8)-(10), we obtain

$$g(T_{U_1}^l U_2, W) + g(U_2, D^{ll}(U_1, W)) = -\hat{g}(U_2, T_{U_1} W), \quad (11)$$

$$g(D^{ls}(U_1, N), W) = -g(N, T_{U_1} W). \quad (12)$$

For an r-lightlike or co-isotropic submersion $\phi$ and if $\psi : \text{Ker} \phi_* \to S(\text{Ker} \phi_*)$, then $\forall U_1, U_2 \in \Gamma(\text{Ker} \phi_*)$ and $\xi \in \Gamma \Delta$, we put

$$\hat{\nabla}_{U_1} \psi U_2 = \nabla_{U_1}^l \psi U_2 + T_{U_1} \psi U_2, \quad (13)$$

$$\hat{\nabla}_{U_1} \xi = T_{U_1} \xi + \nabla_{U_1}^l \xi, \quad (14)$$

where $\hat{\nabla}_{U_1} \psi U_2$, $T_{U_1} \xi \in \Gamma(S(\text{Ker} \phi_*))$ and $T_{U_1} \psi U_2$, $\nabla_{U_1}^l \xi \in \Gamma \Delta$.

3. Screen Slant Lightlike Submersions

**Lemma 3.1.** Let $\phi : M_1 \to M_2$ be a 2r-lightlike submersion from an indefinite Kaehler manifold $M_1$ onto a lightlike manifold $M_2$. Assume that $\text{Ker} \phi_*$ is a lightlike distribution on $M_1$. Then $S(\text{Ker} \phi_*)$ is Riemannian.

**Proof.** Let $	ext{Ker} f_*$ be a lightlike distribution of dimension $m$ on $M_1$. Then there exists

$$\{\xi_i, N_i, U_\alpha, Z_\alpha\}, i \in \{1, \ldots, 2r\}, \alpha \in \{2r + 1, \ldots, m\}, a \in \{2r + 1, \ldots, n\},$$

where $\{\xi_i\}$, $\{N_i\}$ are lightlike basis of $\Delta$, $\text{ltr}(\text{Ker} \phi_*)$ and $U_\alpha$, $Z_\alpha$ are orthonormal basis of $S(\text{Ker} \phi_*)$, $S(\text{Ker} \phi_*)^\perp$, respectively. With the help of basis
\{\xi_1, \ldots, \xi_{2r}, N_1, \ldots, N_{2r}\} of \Delta \oplus \text{ltr}(\text{Ker } \phi_\ast), we set up the following orthonormal basis \{U_1, \ldots, U_{4r}\}

\[
U_1 = \frac{(\xi_1 + N_1)}{\sqrt{2}}, \quad U_2 = \frac{(\xi_1 - N_1)}{\sqrt{2}},
\]
\[
U_3 = \frac{(\xi_2 + N_2)}{\sqrt{2}}, \quad U_4 = \frac{(\xi_2 - N_2)}{\sqrt{2}},
\]
\[
\ldots
\]
\[
U_{4r-1} = \frac{(\xi_{2r} + N_{2r})}{\sqrt{2}}, \quad U_{4r} = \frac{(\xi_{2r} - N_{2r})}{\sqrt{2}}.
\]

Thus, \(\text{span}\{\xi_i, N_i\}\) is a non-degenerate space with constant index 2r, which enables us to conclude that \(\Delta \oplus \text{ltr}(\text{Ker } \phi_\ast)\) is non-degenerate with index 2r on \(M\). Moreover,

\[
\text{index}(TM)
= \text{index}(\Delta \oplus \text{ltr}(\text{Ker } \phi_\ast)) + \text{index}(S(\text{Ker } \phi_\ast) \perp (S(\text{Ker } \phi_\ast))^\perp),
\]
implies \(S(\text{Ker } \phi_\ast) \perp S(\text{Ker } \phi_\ast)^\perp\) has a constant index zero. Hence, \(S(\text{Ker } \phi_\ast)\) and \(S(\text{Ker } \phi_\ast)^\perp\) are Riemannian distributions. \(\square\)

Using this lemma, we give the following definition:

**Definition 3.2.** Let \(\phi: M_1 \rightarrow M_2\) be a lightlike submersion from a real 2m-dimensional indefinite Kaehler manifold \(M_1\) onto a lightlike manifold \(M_2\). We say that \(\phi\) is a screen slant lightlike submersion if \(J \Delta = \Delta\) and screen distribution \(S(\text{Ker } \phi_\ast)\) is slant.

From the definition it is clear \(\text{Ker } \phi_\ast\) is invariant (respectively anti invariant) iff \(\theta = 0\) (respectively \(\theta = \frac{\pi}{2}\)). Thus, a screen slant lightlike submersion is a natural generalization of invariant and anti-invariant lightlike submersions. If a screen slant lightlike submersion is neither invariant nor anti-invariant, then it is called a proper screen slant lightlike submersion.

In the remaining part of this section we consider that \(\text{Ker } \phi_\ast\) is a 2r-lightlike distribution of indefinite Kaehler manifold \(M\).

Now, for any \(U \in \Gamma(S(\text{Ker } \phi_\ast))\), consider

\[
\mathcal{J}U = \tau U + \omega U.
\]

Here \(\tau U \in \Gamma(\text{Ker } \phi_\ast)\) and \(\omega U \in \Gamma(\text{tr}(\text{Ker } \phi_\ast))\).

**Corollary 3.3.** Let \(\phi\) be a screen slant lightlike submersion from an indefinite Kaehler manifold \(M_1\) onto a lightlike manifold \(M_2\). Then, \(\forall U \in \Gamma(\text{Ker } \phi_\ast)\), we have

(i) \(U \in \Gamma(S(\text{Ker } \phi_\ast))\) implies \(\omega U \in \Gamma(S(\text{Ker } f\phi_\ast)^\perp)\),

(ii) \(U \in \Gamma(\Delta)\) implies \(\omega U = 0\).
Proof. Invariance of $\Delta$ with respect to $\mathcal{J}$ implies that $\mathcal{J}(ltr(Ker \phi_*)) = ltr(Ker \phi_*)$, which implies (i). Other assertion is clear from definition 3.2. \hfill \Box

Now, assume that $\chi$ and $Q$ are the projection morphisms on the distributions $S(Ker f_*)$ and $\Delta$, respectively. Then, for any $U \in \Gamma(Ker \phi_*)$, we put

$$U = \chi U + QU,$$

(16)

$\chi U \in \Gamma(S(Ker \phi_*))$ and $QU \in \Gamma(\Delta)$. From (16), we have

$$\mathcal{J}U = JQU + J\chi U = \tau QU + \tau \chi U + \omega \chi U,$$

(17)

where

$$\tau \phi U \in \Gamma(S(Ker \phi_ *)),$$

$$\omega QU = 0,$$

(18)

and

$$\tau \phi U \in \Gamma(S(Ker \phi_ *)).$$

Also, let us decompose $S(Ker \phi_*)^\perp$ as

$$S(Ker \phi_*)^\perp = \nu \perp \omega \chi(S(Ker \phi_*)).$$

(19)

So, for $Z \in \Gamma(S(Ker \phi_*)^\perp)$, we write

$$JZ = CZ + \beta Z,$$

(20)

Here $CZ \in \Gamma(\nu)$ and $\beta Z \in \Gamma(S(Ker \phi_*)).$

From definition (3.2) it is clear that any proper screen slant lightlike submersion must be $r$-lightlike, that is a proper screen slant lightlike submersion must not be screen slant isotropic or co-isotropic or totally lightlike submersion. We follow [6] for the notations used in examples.

**Example 3.4.** Let $\mathbb{R}^{8}_{0,2,6}$ and $\mathbb{R}^{4}_{2,0,2}$ endowed with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$, where $u_1, ..., u_8$ and $v_1, ..., v_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $\phi : (\mathbb{R}^8, g) \rightarrow (\mathbb{R}^4, g')$ as

$$(u_1, ..., u_8) \mapsto (u_1 + u_3, u_2 + u_4, (u_5 - u_7)/\sqrt{2}, u_8).$$

Then

$$Ker \phi_* = Span \{U_1 = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial u_2} - \frac{\partial}{\partial x_4},$$

$$U_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_7}), U_4 = \frac{\partial}{\partial u_6} \}$$

and

$$(Ker \phi_*)^\perp = Span \{U_3, U_4, X = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}), Y = \frac{\partial}{\partial u_6} \}.$$n

So, $\Delta = Span\{U_1, U_2\}$. By easy computation we can see that $\mathcal{J}U_1 = U_2$. Thus $\Delta$ is invariant. Further, $S(Ker \phi_*) = Span\{U_3, U_4\}$ is a slant with slant angle $\theta = \frac{\pi}{4}$. Hence, $\phi$ is a proper screen slant lightlike submersion.
In the remaining part of this section we assume that \( \phi: (M_1, g, J) \rightarrow (M_2, g') \) be a 2r-lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \).

**Theorem 3.5.** Let \( \phi: M_1 \rightarrow M_2 \) be a lightlike submersion and \( \text{Ker} \phi_* \) is a lightlike distribution of \( M_1 \). Then \( \phi \) is a screen slant lightlike submersion if and only if

(i) \( J(\text{ltr}(\text{Ker} \phi_*)) = \text{ltr}(\text{Ker} \phi_*) \),

(ii) For any \( U \in \Gamma(S(\text{Ker} \phi_*)) \), there exists a constant \( \lambda \in [-1, 0] \), such that

\[
(\chi \circ \tau)^2 U = \lambda U, \tag{21}
\]

where \( \lambda = -\cos^2 \theta_{|S(\text{Ker} f_*)} \).

**Proof.** Lemma (3.1) implies that \( S(\text{Ker} f_*) \) is a Riemannian. If \( \phi \) is a screen slant lightlike submersion, then \( J\Delta = \Delta \). Using (1), (17) and Corollary 3.3, we have

\[
g(JN, U) = -g(N, \tau QU) - g(N, \tau \chi U) - g(N, \omega \chi U) = 0,
\]

for \( U \in \Gamma(S(\text{Ker} \phi_*)) \) and \( N \in \Gamma(\text{ltr}(\text{Ker} \phi_*)) \). Also, for \( Z \in \Gamma(S(\text{Ker} f_*)) \), using (1) and (20), we derive

\[
g(JN, Z) = -g(N, CZ) - g(N, \beta Z) = 0.
\]

Further, if \( JN \in \Gamma(\Delta) \), then \( J^2 N = J^2 N = N \in \Gamma(\text{ltr}(\text{Ker} \phi_*)) \). Therefore, we arrive at a contradiction, as \( \Delta \) is invariant with respect to \( J \). Thus, the proof of (i) is completed. For the (ii) part, as \( f \) is a screen slant lightlike submersion, there exists a constant angle \( \theta \), independent of \( U \in S(\text{Ker} f_* p) \) and \( p \in M \), such that

\[
\cos \theta(U) = \frac{g(\tau \phi U, JU)}{|\tau \phi U||JU|} = -\frac{g(J\phi U, U)}{|\tau \phi U||JU|} = -\frac{g((\phi \circ \tau)^2 U, U)}{|\tau \phi U||JU|}. \tag{22}
\]

Also, we have

\[
\cos \theta(U) = \frac{||\tau \phi U||}{|JU|}. \tag{23}
\]

Using, (22) and (23) we get

\[
\cos^2 \theta(U) = -\frac{\tilde{g}(U, (\phi \circ \tau)^2 U)}{|U|^2}.
\]

As \( \theta(U) \) is constant, we obtain \( (\phi \circ \tau)^2 U = \lambda U, \lambda \in [-1, 0] \). Thus, we have (ii). Similarly converse part can be obtained. \( \square \)

Following corollary is the immediate consequence of Theorem 3.5:

**Corollary 3.6.** If \( \phi: M_1 \rightarrow M_2 \) be a lightlike submersion, then

\[
\tilde{g}(\tau \chi U_1, \tau \chi U_2) = \cos^2 \theta_{|S(\text{Ker} \phi_*)}\tilde{g}(U_1, U_2), \tag{24}
\]

and

\[
\tilde{g}(\omega \chi U_1, \omega \chi U_2) = \sin^2 \theta_{|S(\text{Ker} \phi_*)}\tilde{g}(U_1, U_2), \tag{25}
\]

where \( U_1, U_2 \in \Gamma(\text{Ker} \phi_*). \)
Now, for any $U_1, U_2 \in \Gamma(\text{Ker } \phi_*)$, using (2), (8), (10) and (17)-(20), we obtain
\begin{align*}
(\nabla_{U_1} \tau)U_2 &= -T_{U_1} \omega \chi U_2 + BT_{U_1}^1 U_2, \\
\mathcal{J}T_{U_1}^2 U_2 &= T_{U_1}^1 \mathcal{J}QU_2 + T_{U_1}^1 \tau \chi U_2 + D_{\tau \chi}(U_1, \omega \chi U_2), \\
(\nabla_{U_1} \omega)U_2 &= -T_{U_1}^2 \mathcal{J}QU_2 - U^{\mathcal{J}} U_1^2 \tau \chi U_2 + CT_{U_1}^1 U_2.
\end{align*}

\textbf{Theorem 3.7.} Let $\phi: M_1 \rightarrow M_2$ be a screen slant lightlike submersion. Then
\begin{enumerate}[(i)]
  \item $\Delta$ is integrable iff $T_{U_1}^3 U_2 = T_{U_1}^3 U_2, \forall U_1, U_2 \in \Gamma(\Delta)$.
  \item $S(\text{Ker } \phi_*)$ is integrable iff $Q(\nabla_{U_1} \tau \chi U_2 - \nabla_{U_2} \tau \chi U_1) = Q(T_{U_2} \omega \chi U_1 - T_{U_1} \omega \chi U_2)$, for any $U_1, U_2 \in \Gamma(S(\text{Ker } \phi_*)$).
\end{enumerate}

\textbf{Proof.} Let $U_1, U_2 \in \Gamma\Delta$. From (29) and corollary (3.6), we have $\omega \nabla_{U_1} U_2 = T_{U_1}^1 \mathcal{J}QU_2 - CT_{U_1}^1 U_2$. Above equation gives $T_{U_1}^1 \mathcal{J}QU_2 = T_{U_1}^1 \omega \chi U_2$, which implies (i). Also, using (17), (18) and (27) we arrived at $\nabla_{U_1} \tau \chi U_2 + T_{U_1} \omega \chi U_2 = \mathcal{J}Q\nabla_{U_1} U_2 - \tau \chi \nabla_{U_1} U_2 + BT_{U_1}^1 U_2, \forall U_1, U_2 \in \Gamma(S(\text{Ker } \phi_*)$. Then, we have $\nabla_{U_1} \tau \chi U_2 - \nabla_{U_2} \tau \chi U_1 + T_{U_1} \omega \chi U_2 - T_{U_2} \omega \chi U_1 = \mathcal{J}Q(U_1, U_2) - \tau \chi |U_1, U_2|$. So, $Q(\nabla_{U_1} \tau \chi U_2 - \nabla_{U_2} \tau \chi U_1) + Q(T_{U_1} \omega \chi U_2 - T_{U_2} \omega \chi U_1) = \mathcal{J}Q(U_1, U_2)$, which implies (ii). \hfill $\Box$

\textbf{Theorem 3.8.} Let $\phi: M_1 \rightarrow M_2$ be a screen slant lightlike submersion. Then $S(\text{Ker } \phi_*)$ defines a totally geodesic foliation if and only if $-\mathcal{J}T_{U_1} \omega \chi U_2 + T_{U_1} \omega \chi \tau \chi U_2$ has no component in the radical distribution $\Delta$, for any $U_1, U_2 \in \Gamma(S(\text{Ker } \phi_*)$).

\textbf{Proof.} If $U_1, U_2 \in \Gamma(S(\text{Ker } \phi_*)$, $N \in \Gamma(ltr(\text{Ker } \chi_*))$, then using (1), (2) and (8), we have $g(\nabla_{U_1} U_2, N) = g(\nabla_{U_1} \mathcal{J}QU_2, \mathcal{J}N)$. Using (9) and (17), last equation implies $g(\nabla_{U_1} U_2, N) = g(\nabla_{U_1} \tau \chi U_2, \mathcal{J}N) + g(T_{U_1} \omega \chi U_2, \mathcal{J}N).$ Then, using (8)-(10) and (17), this equation gives $g(\nabla_{U_1} U_2, N) = g(\nabla_{U_1} (\chi \circ \tau)^2 U_2, N) + g(T_{U_1} \omega \chi \tau \chi U_2, \mathcal{J}N).$ Then using Theorem 3.5, we arrive at
\begin{align*}
g(\nabla_{U_1} U_2, N) &= -\cos^2 \theta g(\nabla_{U_1} U_2, N) + g(T_{U_1} \omega \chi \tau \chi U_2, N) + g(T_{U_1} \omega \chi U_2, \mathcal{J}N).
\end{align*}
Thus, we get
\begin{align*}(1 + \cos^2 \theta)g(\nabla_{U_1} U_2, N) &= g(T_{U_1} \omega \chi \tau \chi U_2, N) + g(T_{U_1} \omega \chi U_2, \mathcal{J}N),
\end{align*}
which completes the proof. \hfill $\Box$

\textbf{Theorem 3.9.} Let $\phi: M_1 \rightarrow M_2$ be a screen slant lightlike submersion. Then $\tau$ is parallel if and only if $\forall U_1 \in \Gamma(\text{Ker } \phi_*)$, $U_2, Z \in \Gamma(S(\text{Ker } \phi_*)$ and $N \in \Gamma(ltr(\text{Ker } \phi_*)$, we have $D^{\perp \nu}(U_1, N) \in \Gamma(\nu)$,
\begin{align*}
g(T_{U_1}^2 U_2, \omega \chi U_2) = g(T_{U_1}^2 U_2, \omega \chi Z).
\end{align*}
Proof. From (27), we have 
\[ \hat{g}((\hat{\nabla} U_1 \tau)U_2, N) = 0, \] 
for \( U_2 \in \Gamma(\Delta), U_1 \in \Gamma(Ker \phi) \) and \( N \in \Gamma(\text{it}r(Ker \phi_*)) \). Also, for \( U_1, U_2 \in \Gamma(S(Ker \phi_*)) \), we obtain 
\[ \hat{g}((\hat{\nabla} U_1 \tau)U_2, N) = -\hat{g}(T_{U_1}, \omega \chi U_2, N). \] 
Using (12), last equation gives 
\[ \hat{g}((\hat{\nabla} U_1 \tau)U_2, N) = g(D_{U_1}^\perp, \omega \chi U_2). \] 

From (1), (15), (16) and (27), we have 
\[ \hat{g}((\hat{\nabla} U_1 \tau)U_2, Z) = -\hat{g}(T_{U_1}, \omega \chi U_2, Z) - \hat{g}(T_{U_1}, \omega \chi Z), \] 
for \( U_1, U_2 \in \Gamma(Ker \phi) \) and \( Z \in \Gamma(S(Ker \phi_*)) \). Finally, using (11) above equation gives 
\[ \hat{g}((\hat{\nabla} U_1 \tau)U_2, Z) = g(T_{U_1}^\perp, \omega \chi U_2) - \hat{g}(T_{U_1}^\perp, \omega \chi Z), \] 
for \( U_1, U_2 \in \Gamma(Ker \phi) \) and \( Z \in \Gamma(S(Ker \phi_*)) \). Using (29) and (30), we get our assertion. \[ \square \]

4. Screen Semi-Slant Lightlike Submersions

Lemma 3.1 motivates us to give the following definition:

**Definition 4.1.** Let \( M_1 \) be an indefinite Kaehler manifold and \( M_2 \) be a lightlike manifold. Also, let \( \phi : (M_1, g, \mathcal{J}) \to (M_2, g') \) be a \( 2r \)-lightlike submersion such that \( 2r < \text{dim}(Ker \phi_*) \). We say that \( \phi \) is a screen semi-slant lightlike submersion if the lightlike distribution \( \Delta \) is invariant with respect to \( \mathcal{J} \) and screen distribution \( S(Ker \phi_*) \) contains two non-null orthogonal distributions \( D_1 \) and \( D_2 \) such that \( S(Ker \phi_*) = D_1 \oplus D_2 \), where \( D_1 \) is invariant and \( D_2 \) is slant.

A screen semi-slant lightlike submersion is called proper if \( D_1 \neq \{0\}, D_2 \neq \{0\} \) and \( \theta \neq \pi/2 \). From definition (4.1) following cases may arise:

- If \( D_1 = 0 \), \( \phi \) is a screen-slant lightlike submersion.
- If \( D_2 = 0 \), \( \phi \) is a invariant lightlike submersion.
- If \( D_1 = 0 \) and \( \theta = \pi/2 \), \( \phi \) is a anti-invariant lightlike submersion.
- If \( D_1 \neq 0 \) and \( \theta = \pi/2 \), \( \phi \) is a SCR lightlike submersion.

So, we can say that above defined class of lightlike submersions includes invariant, anti-invariant, screen slant and SCR lightlike submersions as its subcases.

**Example 4.2.** Let \( \mathbb{R}_{12}^{1,0,2} \) and \( \mathbb{R}_{4,0,2}^{6} \) equipped with the metric 
\[ g = -(du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 \] 
\[ + (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2; \] 
and degenerate metric \( g' = (dv_3)^2 + (dv_4)^2 \). Consider the map \( \phi : (\mathbb{R}^{12}, g) \to (\mathbb{R}^{6}, g') \) as 
\[ (u_1, ..., u_{12}) \mapsto \left( u_1 - x_7, u_2 - u_8, u_3 + \frac{u_6}{\sqrt{2}}, u_5, u_{11}, u_{12}\right). \]
Let $\Delta = \text{Span}\{\xi_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_7}, \xi_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_8}\}$. Clearly $J \xi_1 = \xi_2$, so $\Delta$ is invariant. By easy calculation we can see that

$$D_2 = \text{Span}\left\{\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_6}\right), \frac{\partial}{\partial u_4}\right\}$$

is slant distribution with slant angle $\theta = \frac{\pi}{4}$. Further, we see that

$$D_1 = \text{Span}\left\{\frac{\partial}{\partial u_9}, \frac{\partial}{\partial u_{10}}\right\}$$

is invariant with respect to $J$. Hence, $\phi$ is a proper screen semi-slant lightlike submersion.

**Example 4.3.** Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be endowed with

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2,$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$. Taking the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(u_1, ..., u_8) \mapsto \left((u_1 - u_5)/\sqrt{2}, (u_2 - u_6)/\sqrt{2}, u_3 + u_7, u_4 + u_8\right)$. It gives

$$\text{Ker} \phi_* = \text{Span}\left\{U_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_7}\right), U_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_6}\right), U_3 = \frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_7}, U_4 = \frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_8}\right\},$$

which implies

$$(\text{Ker} \phi_*)^\perp = \text{Span}\left\{U_1, U_2, X = \frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_7}, Y = \frac{\partial}{\partial u_4} + \frac{\partial}{\partial u_8}\right\}.$$ 

Thus $f$ is a 2-lightlike submersion with $\Delta = \text{Span}\{U_1, U_2\}$, which is clearly seen to be invariant. Since $JU_3 = U_4$ we have $S(\text{Ker} f_*) = D_1 = \text{Span}\{U_3, U_4\}$ and $D_2 = 0$. Hence, $\phi$ is an invariant lightlike submersion.

**Example 4.4.** Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be equipped with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2,$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$, respectively. Taking the map $\phi : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(u_1, ..., u_8) \mapsto \left(u_1 - u_3, u_2 - u_4, u_6, \frac{u_5 - u_8}{2}\right)$. Then, we obtain

$$\text{Ker} f_* = \text{Span}\left\{U_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}, U_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4}, U_3 = \frac{1}{2}\left(\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_8}\right), U_4 = \frac{\partial}{\partial u_7}\right\},$$

and

$$(\text{Ker} f_*)^\perp = \text{Span}\left\{U_1, U_2, X = \frac{1}{2}\left(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_8}\right), Y = \frac{\partial}{\partial u_6}\right\}.$$
Let \( \Delta = \text{Span}\{U_1, U_2\} \), which is clearly seen to be invariant. Further, \( S(\ker \phi_*) = D_2 = \text{Span}\{X, Y\} \) is slant with angle \( \pi/4 \). Thus, \( D_1 = 0 \). Hence \( \phi \) is a screen slant lightlike submersion.

**Example 4.5.** Let \( \mathbb{R}^{12}_{0,4,8} \) and \( \mathbb{R}^{6}_{4,0,2} \) be equipped with

\[
g = -(du_1)^2 - (du_2)^2 - (du_3)^2 - (du_4)^2 + (du_5)^2 + (du_6)^2
+ (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2,
\]
and degenerate metric \( g' = (du_5)^2 + (du_6)^2 \), respectively. Consider the map \( \phi : (\mathbb{R}^{12}, g) \rightarrow (\mathbb{R}^{6}, g') \), such that

\[
(u_1, ..., u_{12}) \rightarrow (u_1 - u_5, u_2 - u_6, (u_3 + u_7)/\sqrt{2}, (u_4 + u_8)/\sqrt{2}, u_9, u_{11}).
\]

Then, we obtain

\[
\ker f_* = \text{Span}\left\{U_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_5}, U_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_6}, U_3 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_7} \right), U_4 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_8} \right), U_5 = \frac{\partial}{\partial u_{10}}, U_6 = \frac{\partial}{\partial u_{12}} \right\},
\]

and

\[
(\ker f_*)^\perp = \text{Span}\left\{U_1, U_2, U_3, U_4, X = \frac{\partial}{\partial u_9}, Y = \frac{\partial}{\partial u_{11}} \right\}.
\]

Thus, \( f \) is a 4-lightlike submersion with \( \Delta = \text{Span}\{U_1, U_2, U_3, U_4\} \). Further, we can see easily that \( \mathcal{J}U_1 = U_2 \) and \( \mathcal{J}U_3 = U_4 \). Therefore, \( \Delta \) is invariant with respect to \( \mathcal{J} \). Also \( \mathcal{J}U_5 = X \) and \( \mathcal{J}U_6 = Y \), implies that \( S(\ker \phi_*) = D_2 = \text{Span}\{U_5, U_6\} \). Finally, since \( \mathcal{J}D_2 = S(\ker \phi_*)^{\perp}, \phi \) is an anti-invariant lightlike submersion.

**Example 4.6.** Let \( \mathbb{R}^{16}_{0,2,14} \) and \( \mathbb{R}^{8}_{0,2} \) be equipped with the metric

\[
g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2
+ (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2
+ (du_{13})^2 + (du_{14})^2 + (du_{15})^2 + (du_{16})^2,
\]
and degenerate metric \( g' = (du_5)^2 + (du_6)^2 \), respectively. Consider the map \( \phi : (\mathbb{R}^{16}, g) \rightarrow (\mathbb{R}^{8}, g') \) as

\[
(u_1, ..., u_{16}) \rightarrow \left( (u_1 - u_3)/\sqrt{3}, (u_2 - u_4)/\sqrt{3}, u_5 + u_7, u_6 + u_8, u_{10}, u_{12}, u_{14}, u_{16} \right).
\]

Then, we obtain

\[
\ker \phi_* = \text{Span}\left\{U_1 = \frac{1}{\sqrt{3}} \left( \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3} \right), U_2 = \frac{1}{\sqrt{3}} \left( \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4} \right), U_3 = \frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}, U_4 = \frac{\partial}{\partial u_6} - \frac{\partial}{\partial u_8} \right\}.
\]
\[ U_5 = \frac{\partial}{\partial u_9}, \quad U_6 = \frac{\partial}{\partial u_{11}}, \quad U_7 = \frac{\partial}{\partial u_{13}}, \quad U_8 = \frac{\partial}{\partial u_{15}} \] 

and

\[(\ker \phi^*)^\perp = \text{Span}\{U_1, U_2, X_1 = \frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_7}, X_2 = \frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_8}, \]
\[X_3 = \frac{\partial}{\partial u_{10}}, \quad X_4 = \frac{\partial}{\partial u_{12}}, \quad X_5 = \frac{\partial}{\partial u_{14}}, \quad X_6 = \frac{\partial}{\partial u_{16}} \} \]

It follows that \( \Delta = \text{Span}\{U_3, U_4\} \), which is clearly invariant. Now, as \( JU_3 = U_4 \), therefore \( D_1 = \text{Span}\{U_3, U_4\} \) is an invariant distribution. Further, since \( JU_3 = X_3 \), \( JU_6 = X_4 \), \( JU_7 = X_5 \) and \( JU_8 = X_6 \), we conclude that \( JD_2 = S(\ker \phi^*)^\perp \). Hence, \( \phi \) is a proper-SCR lightlike submersion.

Now, for all \( U \in \Gamma(\ker \phi^*) \), we write

\[ JU = \chi U + FU. \]

Here \( \chi U \) (resp. \( FU \)) is tangential (resp. normal) component of \( JU \). Next, we denote the projections of \( \ker \phi^* \) on \( \Delta \), \( D_1 \) and \( D_2 \) by \( \chi_1, \chi_2 \) and \( \chi_3 \), respectively. Then, \( U = \chi_1 U + \chi_2 U + \chi_3 U \), for \( U \in \Gamma(\ker \phi^*) \), which implies \( JU = J\chi_1 U + J\chi_2 U + J\chi_3 U \). Then,

\[ JU = J\chi_1 U + J\chi_2 U + \xi\chi_3 U + F\chi_3 U, \quad (31) \]

where \( \xi\chi_3 U \) (resp. \( F\chi_3 U \)) denotes the tangential (resp. transversal) component of \( J\phi^* U \). So, we have \( J\chi_1 U \in \Gamma(\Delta) \), \( J\chi_2 U \in \Gamma(D_1) \), \( \xi\chi_3 U \in \Gamma(D_2) \) and \( F\chi_3 U \in \Gamma(S(\ker \phi^*)^\perp) \). In the same way, denote the projections of \( \text{tr}(\ker \phi^*) \) on \( \text{tr}(\ker \phi^*) \) by \( Q_1 \) and \( Q_2 \), respectively. So, \( \forall W \in \Gamma(\text{tr}(\ker \phi^*)) \), we put \( W = Q_1 W + Q_2 W \), which gives \( JW = JQ_1 W + JQ_2 W \). Then, we have

\[ JW = JQ_1 W + BQ_2 W + CQ_2 W, \quad (32) \]

where \( BQ_2 W \) (resp. \( CQ_2 W \)) denotes the tangential (resp. transversal) component of \( JQ_2 W \). Thus, we have \( JQ_1 W \in \Gamma(\text{tr}(\ker \phi^*)) \), \( BQ_2 W \in \Gamma(D_2) \) and \( CQ_2 W \in \Gamma(S(\ker \phi^*)^\perp) \). Using (2), (9), (11), (31) and (32) and identifying the components of \( \Delta, D_1, D_2, \text{tr}(\ker \phi^*) \) and \( S(\ker \phi^*)^\perp \), we get

\[ \chi_1(\nabla_U J\chi_1 V) + \chi_1(\nabla_U J\chi_2 V) + \chi_1(\nabla_U \xi\chi_3 V) = -\chi_1(T_U F\chi_3 V) + J\chi_1 \nabla_U V, \quad (33) \]

\[ \chi_2(\nabla_U J\chi_1 V) + \chi_2(\nabla_U J\chi_2 V) + \chi_2(\nabla_U \xi\chi_3 V) = -\chi_2(T_U F\chi_3 V) + J\chi_2 \nabla_U V, \quad (34) \]

\[ \chi_3(\nabla_U J\chi_1 V) + \chi_3(\nabla_U J\chi_2 V) + \chi_3(\nabla_U \xi\chi_3 V) = -\chi_3(T_U F\chi_3 V) + \xi\chi_3 \nabla_U V + B(T_U V), \quad (35) \]

\[ T_U J\chi_1 V + T_U J\chi_2 V + T_U \xi\chi_3 V = JT_U V - D^U(U, F\chi_3 V), \quad (36) \]
\[ T^*_U \mathcal{J}_1 V + T^*_U \mathcal{J}_2 V + T^*_U \xi_3 V = C T^*_U V - \nabla^P_U F \chi_3 + F \chi_3 \nabla_U V. \quad (37) \]

**Theorem 4.7.** Let \( \phi \) be a 2r-lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \). Then, \( \phi \) is a screen semi-slant lightlike submersion if and only if

(i) \( \mathcal{J}(\mathfrak{tr}(\text{Ker} \phi_1)) = \mathfrak{tr}(\text{Ker} \phi_2) \) and \( \mathcal{J}(D_1) = D_1 \),

(ii) \( \exists \) a constant \( \lambda \in [0, 1) \), in such a way, that \( (\chi_3 \circ \xi)^2 U = -\lambda U \), \( \forall U \in \Gamma(D_2) \), where \( D_1 \) and \( D_2 \) are orthogonal distributions, such that \( S(\text{Ker} \phi) = D_1 \oplus D_2 \) and \( \lambda = \cos^2 \theta \), \( \theta \) is a slant angle of \( D_2 \).

**Proof.** Using (1) and (31), we get

\[ g(\mathcal{J}_N, U) = -g(N, \mathcal{J}_U) = -g(N, \mathcal{J}_1 U + \mathcal{J}_2 U + \xi_3 U + F \chi_3 U) = 0, \]

for any \( N \in \Gamma(\mathfrak{tr}(\text{Ker} \phi_1)) \) and \( U \in \Gamma(S(\text{Ker} \phi_2)) \). Therefore \( \mathcal{J} S \) does not belong to \( S(\text{Ker} \phi_2) \). Now, if \( W \in \Gamma(S(\text{Ker} \phi_2)^\perp) \), using (1) and (32), we derive

\[ g(\mathcal{J}_N, W) = -g(N, \mathcal{J}_W) = -g(N, BW + CW) = 0, \]

which implies that \( \mathcal{J} S \) does not belong to \( \Gamma(S(\text{Ker} \phi_2)^\perp) \). Also, if \( \mathcal{J}_N \in \Gamma(\Delta) \), then \( \mathcal{J}(\mathcal{J}_N) = \mathcal{J}^2 N = -N \in \Gamma(\mathfrak{tr}(\text{Ker} \phi_2)) \), which is absurd as \( \Delta \) is invariant with respect to \( \mathcal{J} \). Thus \( \mathfrak{tr}(\text{Ker} \phi_2) \) is invariant with respect to \( \mathcal{J} \). Now, if \( U \in \Gamma(D_2) \), we get

\[ \cos(\theta)(U) = \frac{g(\mathcal{J}_2 U, \xi_3(U))}{|\mathcal{J}|(U)} = -\frac{g(U, \mathcal{J}_2 \xi_3 U)}{|\mathcal{J}|(U)} = -\frac{g(U, (\chi_3 \circ \xi)^2 U)}{|\mathcal{J}|(U)}, \quad (38) \]

where \( \theta \) is constant angle independent of point \( p \in M_1 \). Moreover,

\[ \cos(\theta)(U) = \frac{|\xi_3(U)|}{|\mathcal{J}(U)|}. \quad (39) \]

Using (38) and (39), we obtain

\[ \cos^2 \theta(U) = -\frac{g(U, (\chi_3 \circ \xi)^2 U)}{|U|^2}. \]

Now, since \( \theta(U) \) is constant, we have \( (\chi_3 \circ \xi)^2 U = -\lambda U, \lambda \in [0, 1) \), where \( \lambda = \cos^2 \theta \). The converse part can be proved in a similar way. \( \square \)

**Theorem 4.8.** Let \( \phi : M_1 \to M_2 \) be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \). Then, the null distribution \( \Delta \) is integrable if and only if \( \forall U, V \in \Gamma(\Delta) \), we have

(i) \( \chi_2(\nabla_U \mathcal{J}_1 V) = \chi_2(\nabla_U \mathcal{J}_1 U) \),

(ii) \( \chi_3(\nabla_U \mathcal{J}_2 V) = \chi_3(\nabla_U \mathcal{J}_2 U) \),

(iii) \( T^*_U \mathcal{J}_1 V = T^*_U \mathcal{J}_1 U \).

**Proof.** Let \( U, V \in \Gamma(\Delta) \). From (34), we have \( \chi_2(\nabla_U \mathcal{J}_1 V) = \mathcal{J}_2 \nabla_U V \). It follows that

\[ \chi_2(\nabla_U \mathcal{J}_1 V) - \chi_2(\nabla_U \mathcal{J}_1 U) = \mathcal{J}_2 [U, V]. \quad (40) \]
Finally, in view of (35), we have \( \chi_3(\tilde{\nabla}_U J \chi_1 V) = \psi \chi_3 \tilde{\nabla}_U V + BT^*_U V \), which implies
\[
\chi_3(\tilde{\nabla}_U J \chi_1 V) - \chi_3(\tilde{\nabla}_V J \chi_1 U) = \xi \chi_3[U, V]. \tag{41}
\]
Using (37), we obtain \( T^*_U J \chi_1 V = CT^*_U V + F \chi_3 \tilde{\nabla}_U V \), which gives
\[
T^*_U J \chi_1 V - T^*_V J \chi_1 U = F \chi_3[U, V]. \tag{42}
\]
Using (40), (41) and (42), the proof follows.

\[ \square \]

**Theorem 4.9.** Let \( \phi : M_1 \to M_2 \) be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \). Then, the non-null distribution \( D_1 \) is integrable if and only if \( \forall U, V \in \Gamma(D_1) \), we have
(i) \( T^*_U J \chi_2 V = T^*_V J \chi_2 U \),
(ii) \( \chi_1(\tilde{\nabla}_U J \chi_2 V) = \chi_1(\tilde{\nabla}_V J \chi_2 U) \),
(iii) \( \chi_3(\tilde{\nabla}_U J \chi_2 V) = \chi_3(\tilde{\nabla}_V J \chi_2 U) \).

Proof. Let \( U, V \in \Gamma(D_1) \). Now, from (33), we have \( \chi_1(\tilde{\nabla}_U J \phi_2 V) = J \phi_1 \tilde{\nabla}_V V \), which gives
\[
\chi_1(\tilde{\nabla}_U J \chi_2 V) - \chi_1(\tilde{\nabla}_V J \chi_2 U) = J \chi_1[U, V]. \tag{43}
\]
In view of (35), we have \( \chi_3(\tilde{\nabla}_U J \chi_2 V) = \xi \chi_3 \tilde{\nabla}_V V + BT^*_U V \). It follows that
\[
\chi_3(\tilde{\nabla}_U J \chi_2 V) - \chi_3(\tilde{\nabla}_V J \chi_2 U) = \xi \chi_3[U, V]. \tag{44}
\]
Using (37), we obtain \( T^*_U J \chi_2 V = CT^*_U V + F \chi_3 \tilde{\nabla}_U V \). It gives
\[
T^*_U J \chi_2 V - T^*_V J \chi_2 U = F \chi_3[U, V]. \tag{45}
\]
Thus, the proof is completed by using (43), (44) and (45).

\[ \square \]

**Theorem 4.10.** Let \( \phi : M_1 \to M_2 \) be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \). Then, the non-null distribution \( D_2 \) is integrable if and only if for any \( U, V \in \Gamma(D_2) \), we have
\[
\chi_1(\tilde{\nabla}_U \xi \chi_3 V - \tilde{\nabla}_V \xi \chi_3 U) = \chi_1(T_U F \chi_3 U - T_U F \chi_3 V), \chi_2(\tilde{\nabla}_U \xi \chi_3 V - \tilde{\nabla}_V \xi \chi_3 U) = \chi_2(T_U F \chi_3 U - T_U F \chi_3 V).
\]

Proof. Let \( U, V \in \Gamma(D_2) \). Using (33), we have \( \chi_1(\tilde{\nabla}_U \xi \chi_3 V) + \chi_1(T_U F \chi_3 V) = J \chi_1 \tilde{\nabla}_U V \), which implies
\[
\chi_1(\tilde{\nabla}_U \xi \chi_3 V) - \chi_1(\tilde{\nabla}_V \xi \chi_3 U) + \chi_1(T_U F \chi_3 V) - \chi_1(T_V F \chi_3 U) = J \chi_1[U, V]. \tag{46}
\]
Using (34), we drive \( \chi_2(\tilde{\nabla}_U \xi \chi_3 V) + \chi_2(T_U F \chi_3 V) = J \chi_2 \tilde{\nabla}_U V \), which gives
\[
\chi_2(\tilde{\nabla}_U \xi \chi_3 V) - \chi_2(\tilde{\nabla}_V \xi \chi_3 U) + \chi_2(T_U F \chi_3 V) - \chi_2(T_V F \chi_3 U) = J \chi_2[U, V]. \tag{47}
\]
Thus, the proof follows from (46) and (47).

\[ \square \]

**Theorem 4.11.** Let \( \phi : M_1 \to M_2 \) be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold \( M_1 \) onto a lightlike manifold \( M_2 \). Then, induced connection \( \tilde{\nabla} \) on \( S(\text{Ker } \phi_*) \) is a metric connection if and only if \( BT^*_V \xi = 0 \) and \( T^*_V \xi = 0 \) on \( \Gamma(\text{Ker } \phi_*) \), \( \forall V \in \Gamma(\text{Ker } \phi_*) \) and \( \xi \in \Gamma(\Delta) \).
Proof. Connection $\nabla$ on $S(\text{Ker } \phi_\ast)$ is a metric connection if and only if $\Delta$ is a parallel distribution with respect to $\nabla$. Using (2), (8) and (14), we obtain

$$\nabla_V J\xi = J\nabla^\perp_V \xi + JT^\ast_V \xi + JT^\ast_V \xi,$$

for any $V \in \Gamma(\text{Ker } \phi_\ast)$ and $\xi \in \Gamma(\Delta)$. Comparing the tangential components, we get $\nabla_V J\xi = J\nabla^\perp_V \xi + JT^\ast_V \xi$, which completes the proof. □

References


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