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SCREEN SLANT LIGHTLIKE SUBMERSIONS†

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Abstract. We introduce two new classes of lightlike submersions, namely, screen slant and screen semi-slant lightlike submersions from an indefinite Kaehler manifold to a lightlike manifold giving characterization theorems with non trivial examples for both classes. Integrability conditions of all distributions related to the definitions of these submersions have been obtained.

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1. Introduction

In [6], Sahin and G*ü*nd*ü*zalp gave the definition of a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold. In [3, 4], Sahin introduced the notions of slant and screen-slant lightlike submanifolds of an indefinite Hermitian manifold. Following this, Shukla and Yadav defined a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold in [13]. From [12], we conclude that, contrary to the Riemannian slant submersions [5], slant lightlike submersions do not include invariant and anti-invariant subcases. To address this gap, we define screen slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold, which includes invariant and anti-invariant lightlike submersions. The paper is arranged as:

Section 2 is devoted to the basic geometry related to this study. In section 3, we define a screen slant lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold with a non-trivial example. In this section, we also give a characterization theorem and obtain a necessary and sufficient condition for the screen distribution to define a totally geodesic foliation. In the last section, we define a screen semi-slant lightlike submersion from an indefinite

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Kaehler manifold onto a lightlike manifold with non-trivial examples and obtain the integrability conditions of distributions involved in the definition of these submersions.

2. Preliminaries

A complex manifold M with a semi-Riemannian metric g of index r, where $0 < r \leq 2m$ and an almost complex structure $\mathcal{J}$ is called an indefinite Hermitian manifold, if

$$g(U_1, U_2) = g(\mathcal{J}U_1, \mathcal{J}U_2), \quad \forall\, U_1, U_2 \in \Gamma(TM). \tag{1}$$

Further, if $(M, \mathcal{J}, g)$ is an indefinite Hermitian manifold with the Levi-Civita connection ∇ on M, then we call M an indefinite Kaehler manifold if

$$(\nabla_{U_1}\mathcal{J})U_2 = 0, \quad \forall\, U_1, U_2 \in \Gamma(TM). \tag{2}$$

Null (or radical) space $RadT_pM$ of T_pM is defined as $RadT_pM = \{\xi \in T_pM : g(U, \xi) = 0,\ \forall\ U \in T_pM\}$. If $RadTM : p \in M \to RadT_pM$ gives a C^∞ distribution of rank $(r > 0)$ on M such that $0 < r \leq m$, then $RadTM$ is called a radical distribution on M. In this case, we say that manifold M is an r-lightlike manifold.

Let $\phi : M_1 \to M_2$ be a smooth submersion from a semi-Riemannian manifold M_1 to a lightlike manifold M_2. Then, $Ker\ \phi_{*p} = \{U \in T_pM_1 : \phi_{*p}U = 0\}$. It follows that $(Ker\ \phi_{*p})^\perp = \{V \in T_pM_1 : g(U, V) = 0,\ \forall\ U \in Ker\ \phi_{*p}\}$ and $Ker\ \phi_{*p}\ \cap\ (Ker\ \phi_{*p})^\perp = \Delta_p \neq \{0\}$. In this case $\Delta : p \to \Delta_p$ is said to be a radical distribution on M_1 at $p \in M_1$. As Δ is a lightlike distribution, we have $Ker\ \phi_* = \Delta \perp S(Ker\ \phi_*)$. Similarly $(Ker\ \phi_*)^\perp = \Delta \perp S(Ker\ \phi_*)^\perp$. Assume that $dim(\Delta) = r(> 0)$. As $\Delta \subset (S(ker\ \phi_*)^\perp)^\perp$ and $(S(ker\ \phi_*)^\perp)^\perp$ is non-degenerate, so there exists $N_1, N_2..., N_r$, such that $g(N_i, N_j) = 0,\ g(\xi_i, N_j) = \delta_{ij}$. Here $\{N_i\}$ are null vector fields of $(S(Ker\ \phi_*)^\perp)^\perp$ and $\{\xi_i\}$ is the lightlike basis of Δ. The distribution generated by vector fields $N_1, N_2..., N_r$ is denoted by $ltr(ker\ \phi_*)$. Then $tr(ker\ \phi_*) = ltr(ker\ \phi_*) \perp S(ker\ \phi_*)^\perp$. Moreover, we have the following decomposition

$$TM = S(Ker\ \phi_*) \perp (\Delta \oplus ltr(Ker\ \phi_*)) \perp S(Ker\phi_*)^\perp. \tag{3}$$

Let $\phi : M_1 \to M_2$ be a Riemannian submersion, then ϕ is called an r-lightlike submersion if

$$dim\ \Delta = dim\{(Ker\ \phi_*)^\perp \cap (Ker\ \phi_*)\} = r,$$

where $0 < r < min\{dim(ker\ \phi_*), dim(ker\ \phi_*)^\perp\}$.

The geometry of lightlike submersions is pictured by tensors A and T given by

$$A_{U_1}U_2 = h\nabla_{hU_1}\nu U_2 + \nu\nabla_{hU_1}hU_2, \tag{4}$$

$$T_{U_1}U_2 = \nu\nabla_{\nu U_1}hU_2 + h\nabla_{\nu U_1}\nu U_2. \tag{5}$$

Tensors A and T are horizontal and vertical tensors, respectively. Moreover, T has symmetric property for vertical vector fields U_1 and U_2, that is, $T_{U_1}U_2 = T_{U_2}U_1$.

Let M_1 and M_2 be semi-Riemannian and lightlike manifolds, respectively. Next, we assume that $\phi : M_1 \to M_2$ be a lightlike submersion with lightlike distribution $Ker\ \phi_*$ on M_1. Further, suppose that $tr(Ker\ \phi_*)$ is the complementary distribution to $Ker\ \phi_*$ in M_1. Let $\hat{g}$ and ∇ stands for induced metric on $Ker\ \phi_*$ of g and Levi-Civita connection on M_1, respectively. Using (5), $\forall\ U_1, U_2 \in \Gamma(Ker\ \phi_*)$ and $V \in \Gamma(Ker\ \phi_*)^\perp$, we have

$$\nabla_{U_1} U_2 = \hat{\nabla}_{U_1} U_2 + T_{U_1} U_2, \tag{6}$$

$$\nabla_{U_1} V = T_{U_1} V + \nabla^\perp_{U_1} V, \tag{7}$$

where $\hat{\nabla}_{U_1} U_2 = \nu \nabla_{U_1} U_2$ and $\nabla^\perp_{U_1} V = h \nabla_{U_1} V$. Here $\{\hat{\nabla}_{U_1} U_2, T_{U_1} V\}$ and $\{T_{U_1} U_2, \nabla^\perp_{U_1} V\}$ belong to $\Gamma(Ker\ \phi_*)$ and $\Gamma(tr(Ker\ \phi_*))$, respectively. Let $S(Ker\ \phi_*)^\perp \neq \{0\}$. Denote by L and S the projections of $tr(Ker\ \phi_*)$ on $ltr(Ker\ \phi_*)$ and $S(Ker\ \phi_*)^\perp$, respectively. Then, from (6) and (7) , we have

$$\nabla_{U_1} U_2 = \hat{\nabla}_{U_1} U_2 + T^l_{U_1} U_2 + T^s_{U_1} U_2, \tag{8}$$

$$\nabla_{U_1} N = T_{U_1} N + \nabla^{\perp l}_{U_1} N + D^{\perp s}(U_1, N), \tag{9}$$

$$\nabla_{U_1} W = T_{U_1} W + D^{\perp l}(U_1, W) + \nabla^{\perp s}_{U_1} W, \tag{10}$$

$\forall\ U_1, U_2 \in \Gamma(Ker\ \phi_*),\ V \in \Gamma(S(Ker\ \phi_*)^\perp)$ and $N \in \Gamma(ltr(Ker\ \phi_*))$. Using (8)-(10), we obtain

$$g(T^s_{U_1} U_2, W) + g(U_2, D^{\perp l}(U_1, W)) = -\hat{g}(U_2, T_{U_1} W), \tag{11}$$

$$g(D^{\perp s}(U_1, N), W) = -g(N, T_{U_1} W). \tag{12}$$

For an r-lightlike or co-isotropic submersion ϕ and if $\psi : Ker\ \phi_* \to S(Ker\ \phi_*)$, then $\forall\ U_1, U_2 \in \Gamma(Ker\ \phi_*)$ and $\xi \in \Gamma\Delta$, we put

$$\hat{\nabla}_{U_1} \psi U_2 = \hat{\nabla}^*_{U_1} \psi U_2 + T^*_{U_1} \psi U_2, \tag{13}$$

$$\hat{\nabla}_{U_1} \xi = T^*_{U_1} \xi + \nabla^{*\perp}_{U_1} \xi, \tag{14}$$

where $\hat{\nabla}^*_{U_1} \psi U_2,\ T^*_{U_1} \xi \in \Gamma(S(Ker\ \phi_*))$ and $T^*_{U_1} \psi U_2,\ \nabla^{*\perp}_{U_1} \xi \in \Gamma\Delta$.

3. Screen Slant Lightlike Submersions

Lemma 3.1. *Let $\phi : M_1 \to M_2$ be a $2r$-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Assume that $Ker\ \phi_*$ is a lightlike distribution on M_1. Then $S(Ker\ \phi_*)$ is Riemannian.*

Proof. Let $Ker\ f_*$ be a lightlike distribution of dimension m on M_1. Then there exists

$$\{\xi_i, N_i, U_\alpha, Z_a\}, i \in \{1, ..., 2r\}, \alpha \in \{2r+1, ..., m\}, a \in \{2r+1, ..., n\},$$

where $\{\xi_i\}$, $\{N_i\}$ are lightlike basis of Δ, $ltr(Ker\ \phi_*)$ and U_α, Z_a are orthonormal basis of $S(Ker\ \phi_*)$, $S(Ker\ \phi_*)^\perp$, respectively. With the help of basis

$\{\xi_1, ..., \xi_{2r}, N_1, ..., N_{2r}\}$ of $\Delta \oplus ltr(Ker\ \phi_*)$, we set up the following orthonormal basis $\{U_1, ..., U_{4r}\}$

$$\begin{aligned} U_1 &= \frac{(\xi_1 + N_1)}{\sqrt{2}}, & U_2 &= \frac{(\xi_1 - N_1)}{\sqrt{2}}, \\ U_3 &= \frac{(\xi_2 + N_2)}{\sqrt{2}}, & U_4 &= \frac{(\xi_2 - N_2)}{\sqrt{2}}, \\ &... & &... \\ &... & &... \\ U_{4r-1} &= \frac{(\xi_{2r} + N_{2r})}{\sqrt{2}}, & U_{4r} &= \frac{(\xi_{2r} - N_{2r})}{\sqrt{2}}. \end{aligned}$$

Thus, span$\{\xi_i, N_i\}$ is a non-degenerate space with constant index $2r$, which enables us to conclude that $\Delta \oplus ltr(Ker\ _*)$ is non-degenerate with index $2r$ on M. Moreover,

$$\begin{aligned} &index(TM) \\ &\qquad = index(\Delta \oplus ltr(Ker\ f_*)) + index(S(Ker\ \phi_*) \perp (S(Ker\ \phi_*))^{\perp}), \end{aligned}$$

implies $S(Ker\ \phi_*) \perp S(Ker\ \phi_*)^{\perp}$ has a constant index zero. Hence, $S(Ker\ \phi_*)$ and $S(Ker\ \phi_*)^{\perp}$ are Riemannian distributions. □

Using this lemma, we give the following definition:

Definition 3.2. Let $\phi : M_1 \to M_2$ be a lightlike submersion from a real 2m-dimensional indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. We say that ϕ is a screen slant lightlike submersion if $\mathcal{J}\Delta = \Delta$ and screen distribution $S(Ker\ \phi_*)$ is slant.

From the definition it is clear $Ker\ \phi_*$ is invariant (respectively anti invariant) iff $\theta = 0$ (respectively $\theta = \frac{\pi}{2}$). Thus, a screen slant lightlike submersion is a natural generalization of invariant and anti-invariant lightlike submersions. If a screen slant lightlike submersion is neither invariant nor anti-invariant, then it is called a proper screen slant lightlike submersion.

In the remaining part of this section we consider that $Ker\ \phi_*$ is a 2r-lightlike distribution of indefinite Kaehler manifold M.

Now, for any $U \in \Gamma(S(Ker\ \phi_*))$, consider

$$\mathcal{J}U = \tau U + \omega U. \tag{15}$$

Here $\tau U \in \Gamma(Ker\ \phi_*)$ and $\omega U \in \Gamma(tr(Ker\ \phi_*))$.

Corollary 3.3. *Let ϕ be a screen slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, $\forall\ U \in \Gamma(Ker\ \phi_*)$, we have*

(i) *$U \in \Gamma(S(Ker\ \phi_*))$ implies $\omega U \in \Gamma(S(Ker\ f\phi_*)^{\perp})$,*
(ii) *$U \in \Gamma(\Delta)$ implies $\omega U = 0$.*

Proof. Invariance of Δ with respect to $\mathcal{J}$ implies that $\mathcal{J}(ltr(Ker\ \phi_*))$ $= ltr(Ker\ \phi_*)$, which implies (i). Other assertion is clear from definition 3.2. $\square$

Now, assume that χ and Q are the projection morphisms on the distributions $S(Ker\ f_*)$ and Δ, respectively. Then, for any $U \in \Gamma(Ker\ \phi_*)$, we put

$$U = \chi U + QU, \tag{16}$$

$\chi U \in \Gamma(S(Ker\ \phi_*))$ and $QU \in \Gamma(\Delta)$. From (16), we have

$$\mathcal{J}U = JQU + J\chi U = \tau QU + \tau\chi U + \omega\chi U, \tag{17}$$

where

$$\mathcal{J}QU = \tau QU, \quad \omega QU = 0, \tag{18}$$

and

$$\tau\phi U \in \Gamma(S(Ker\ \phi_*)).$$

Also, let us decompose $S(Ker\ \phi_*)^{\perp}$ as

$$S(Ker\ \phi_*)^{\perp} = \nu \perp \omega\chi(S(Ker\ \phi_*)). \tag{19}$$

So, for $Z \in \Gamma(S(Ker\ \phi_*)^{\perp})$, we write

$$\mathcal{J}Z = \mathcal{C}Z + \beta Z, \tag{20}$$

Here $\mathcal{C}Z \in \Gamma(\nu)$ and $\beta Z \in \Gamma(S(Ker\ \phi_*))$.

From definition (3.2) it is clear that any proper screen slant lightlike submersion must be r-lightlike, that is a proper screen slant lightlike submersion must not be screen slant isotropic or co-isotropic or totally lightlike submersion. We follow [6] for the notations used in examples.

Example 3.4. Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ endowed with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$, where $u_1, ..., u_8$ and $v_1, ..., v_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $\phi : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as

$$(u_1, ..., u_8) \longmapsto \Big(u_1 + u_3, u_2 + u_4, (u_5 - u_7)/\sqrt{2}, u_8\Big).$$

Then

$$Ker\ \phi_* = Span\Big\{U_1 = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial u_2} - \frac{\partial}{\partial x_4}, U_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_7}\Big), U_4 = \frac{\partial}{\partial u_6}\Big\}$$

and

$$(Ker\ \phi_*)^{\perp} = Span\Big\{U_1, U_2, X = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}\Big), Y = \frac{\partial}{\partial u_8}\Big\}.$$

So, $\Delta = Span\{U_1, U_2\}$. By easy computation we can see that $\mathcal{J}U_1 = U_2$. Thus Δ is invariant. Further, $S(Ker\ \phi_*) = Span\{U_3, U_4\}$ is a slant with slant angle $\theta = \dfrac{\pi}{4}$. Hence, ϕ is a proper screen slant lightlike submersion.

In the remaining part of this section we assume that $\phi : (M_1, g, \mathcal{J}) \to (M_2, g')$ be a $2r$-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2.

Theorem 3.5. *Let $\phi : M_1 \to M_2$ be a lightlike submersion and $Ker\ \phi_*$ is a lightlike distribution of M_1. Then ϕ is a screen slant lightlike submersion if and only if*

(i) $\mathcal{J}(ltr(Ker\ \phi_*)) = ltr(Ker\ \phi_*)$,

(ii) *For any $U \in \Gamma(S(Ker\ \phi_*))$, there exists a constant $\lambda \in [-1, 0]$, such that*

$$(\chi \circ \tau)^2 U = \lambda U, \tag{21}$$

where $\lambda = -cos^2\theta|_{S(Ker\ f_*)}$.

Proof. Lemma (3.1) implies that $S(Ker\ f_*)$ is a Riemannian. If ϕ is a screen slant lightlike submersion, then $\mathcal{J}\Delta = \Delta$. Using (1), (17) and Corollary 3.3, we have

$$g(JN, U) = -g(N, \tau QU) - g(N, \tau\chi U) - g(N, \omega\chi U) = 0,$$

for $U \in \Gamma(S(Ker\ \phi_*))$ and $N \in \Gamma(ltr(Ker\ \phi_*))$. Also, for $Z \in \Gamma(S(Ker\ f_*)^{\perp})$, using (1) and (20), we derive

$$g(\mathcal{J}N, Z) = -g(N, \mathcal{C}Z) - g(N, \beta Z) = 0.$$

Further, if $\mathcal{J}N \in \Gamma(\Delta)$, then $\mathcal{J}\mathcal{J}N = J^2N = -N \in \Gamma(ltr(Ker\ \phi_*))$. Therefore, we arrive at a contradiction, as Δ is invariant with respect to $\mathcal{J}$. Thus, the proof of (i) is completed. For the (ii) part, as f is a screen slant lightlike submersion, there exists a constant angle θ, independent of $U \in S(Ker\ f_*)$ and $p \in M$, such that

$$cos\theta(U) = \frac{g(\tau\phi U, \mathcal{J}U)}{|\tau\phi U||\mathcal{J}U|} = -\frac{g(\mathcal{J}\tau\phi U, U),}{|\tau\phi U||\mathcal{J}U|} = -\frac{g((\phi \circ \tau)^2U, U)}{|\tau\phi U||\mathcal{J}U|}. \tag{22}$$

Also, we have

$$cos\theta(U) = \frac{|\tau\phi U|}{|\mathcal{J}U|}. \tag{23}$$

Using, (22) and (23) we get

$$cos^2\theta(U) = -\frac{\hat{g}(U, (\phi \circ \tau)^2U)}{|U|^2}.$$

As $\theta(U)$ is constant, we obtain $(\phi \circ \tau)^2U = \lambda U, \lambda \in [-1, 0]$. Thus, we have (ii). Similarly converse part can be obtained. □

Following corollary is the immediate consequence of Theorem 3.5:

Corollary 3.6. *If $\phi : M_1 \to M_2$ be a lightlike submersion, then*

$$\hat{g}(\tau\chi U_1, \tau\chi U_2) = cos^2\theta|_{S(Ker\ \phi_*)}\hat{g}(U_1, U_2), \tag{24}$$

and

$$\hat{g}(\omega\chi U_1, \omega\chi U_2) = sin^2\theta|_{S(Ker\ \phi_*)}\hat{g}(U_1, U_2), \tag{25}$$

where $U_1, U_2 \in \Gamma(Ker\ \phi_*)$.

Now, for any $U_1, U_2 \in \Gamma(Ker\ \phi_*)$, using (2), (8), (10) and (17)-(20), we obtain

$$(\hat{\nabla}_{U_1}\tau)U_2 = -T_{U_1}\omega\chi U_2 + BT^s_{U_1}U_2, \tag{26}$$

$$\mathcal{J}T^l_{U_1}U_2 = T^l_{U_1}JQU_2 + T^l_{U_1}\tau\chi U_2 + D^{\perp l}(U_1, \omega\chi U_2), \tag{27}$$

$$(\nabla_{U_1}\omega)U_2 = -T^s_{U_1}JQU_2 - T^s_{U_1}\tau\chi U_2 + CT^s_{U_1}U_2. \tag{28}$$

Theorem 3.7. *Let $\phi : M_1 \to M_2$ be a screen slant lightlike submersion. Then*

(i) *Δ is integrable iff $T^s_{U_1}\mathcal{J}U_2 = T^s_{\mathcal{J}U_1}U_2,\ \forall\ U_1, U_2 \in \Gamma(\Delta)$.*

(ii) *$S(Ker\ \phi_*)$ is integrable iff*

$$Q(\hat{\nabla}_{U_1}\tau\chi U_2 - \hat{\nabla}_{U_2}\tau\chi U_1) = Q(T_{U_2}\omega\chi U_1 - T_{U_1}\omega\chi U_2),$$

for any $U_1, U_2 \in \Gamma(S(Ker\ \phi_*))$.

Proof. Let $U_1, U_2 \in \Gamma\Delta$. From (29) and corollary (3.6), we have $\omega\nabla_{U_1}U_2 = T^s_{U_1}\mathcal{J}U_2 - CT^s_{U_1}U_2$. Above equation gives $T^s_{U_1}\mathcal{J}U_2 - T^s_{\mathcal{J}U_1}U_2 = \omega[U_1, U_2]$, which implies (i). Also, using (17), (18) and (27) we arrived at $\hat{\nabla}_{U_1}\tau\chi U_2 + T_{U_1}\omega\chi U_2 = \mathcal{J}Q\hat{\nabla}_{U_1}U_2 - \tau\chi\hat{\nabla}_{U_1}U_2 + BT^s_{U_1}U_2, \forall\ U_1, U_2 \in \Gamma(S(Ker\ \phi_*))$. Then, we have $\hat{\nabla}_{U_1}\tau\chi U_2 - \hat{\nabla}_{U_2}\tau\chi U_1 + T_{U_1}\omega\chi U_2 - T_{U_2}\omega\chi U_1 = \mathcal{J}Q[U_1, U_2] - \tau\chi[U_1, U_2]$. So, $Q(\hat{\nabla}_{U_1}\tau\chi U_2 - \hat{\nabla}_{U_2}\tau\chi U_1) + Q(T_{U_1}\omega\chi U_2 - T_{U_2}\omega\chi U_1) = \mathcal{J}Q[U_1, U_2]$, which implies (ii). □

Theorem 3.8. *Let $\phi : M_1 \to M_2$ be a screen slant lightlike submersion. Then $S(Ker\ \phi_*)$ defines a totally geodesic foliation if and only if $-\mathcal{J}T_{U_1}\omega\chi U_2 + T_{U_1}\omega\chi\tau\chi U_2$ has no component in the radical distribution Δ, for any $U_1, U_2 \in \Gamma(S(Ker\ \phi_*))$.*

Proof. If $U_1, U_2 \in \Gamma(S(Ker\ \phi_*)), N \in \Gamma(ltr(Ker\ \chi_*))$, then using (1), (2) and (8), we have $g(\hat{\nabla}_{U_1}U_2, N) = g(\nabla_{U_1}\mathcal{J}U_2, \mathcal{J}N)$. Using (9) and (17), last equation implies $g(\hat{\nabla}_{U_1}U_2, N) = g(\nabla_{U_1}\tau\chi U_2, \mathcal{J}N) + g(T_{U_1}\omega\chi U_2, \mathcal{J}N)$. Then, using (8)-(10) and (17), this equation gives $g(\hat{\nabla}_{U_2}U_2, N) = g(\hat{\nabla}_{U_1}(\chi\circ\tau)^2U_2, N) + g(T_{U_1}\omega\chi\tau\chi U_2, N) + g(T_{U_1}\omega\chi U_2,$
$\mathcal{J}N)$. Then using Theorem 3.5, we arrive at

$$g(\hat{\nabla}_{U_1}U_2, N) = -cos^2\theta g(\hat{\nabla}_{U_1}U_2, N) + g(T_{U_1}\omega\chi\tau\chi U_2, N) + g(T_{U_1}\omega\chi U_2, \mathcal{J}N).$$

Thus, we get

$$(1 + cos^2\theta)g(\hat{\nabla}_{U_1}U_2, N) = g(T_{U_1}\omega\chi\tau\chi U_2, N) + g(T_{U_1}\omega\chi U_2, \mathcal{J}N),$$

which completes the proof. □

Theorem 3.9. *Let $\phi : M_1 \to M_2$ be a screen slant lightlike submersion. Then τ is parallel if and only if $\forall\ U_1 \in \Gamma(Ker\ \phi_*),\ U_2, Z \in \Gamma(S(Ker\ \phi_*))$ and $N \in \Gamma(ltr(Ker\ \phi_*))$, we have*
$D^{\perp s}(U_1, N) \in \Gamma(\nu)$,
$g(T^s_{U_1}Z, \omega\chi U_2) = \hat{g}(T^s_{U_1}U_2, \omega\chi Z)$.

Proof. From (27), we have $g((\hat{\nabla}_{U_1}\tau)U_2, N) = 0$, for $U_2 \in \Gamma(\Delta), U_1 \in \Gamma(Ker\ \phi_*)$ and $N \in \Gamma(ltr(Ker\ \phi_*))$. Also, for $U_1, U_2 \in \Gamma(S(Ker\ \phi_*))$, we obtain $g((\hat{\nabla}_{U_1}\tau)U_2, N) = -g(T_{U_1}\omega\chi U_2, N)$. Using (12), last equation gives

$$g((\hat{\nabla}_{U_1}\tau)U_2, N) = g(D^{\perp s}(U_1, N), \omega\chi U_2). \tag{29}$$

From (1), (15), (16) and (27), we have

$$\hat{g}((\hat{\nabla}_{U_1}\tau)U_2, Z) = -\hat{g}(T_{U_1}\omega\chi U_2, Z) - \hat{g}(T^s_{U_1}U_2, \omega\chi Z),$$

for $U_1, U_2 \in \Gamma(Ker\ \phi_*)$ and $Z \in \Gamma(S(Ker\ \phi_*))$. Finally, using (11) above equation gives

$$\hat{g}((\hat{\nabla}_{U_1}\tau)U_2, Z) = g(T^s_{U_1}Z, \omega\chi U_2) - \hat{g}(T^s_{U_1}U_2, \omega\chi Z), \tag{30}$$

for $U_1, U_2 \in \Gamma(Ker\ \phi_*)$ and $Z \in \Gamma(S(Ker\phi_*))$. Using (29) and (30), we get our assertion. □

4. Screen Semi-Slant Lightlike Submersions

Lemma 3.1 motivates us to give the following definition:

Definition 4.1. Let M_1 be an indefinite Kaehler manifold and M_2 be a lightlike manifold. Also, let $\phi : (M_1, g, \mathcal{J}) \to (M_2, g')$ be a $2r$-lightlike submersion such that $2r < dim(Ker\ \phi_*)$. We say that ϕ is a screen semi-slant lightlike submersion if the lightlike distribution Δ is invariant with respect to $\mathcal{J}$ and screen distribution $S(Ker\ \phi_*)$ contains two non-null orthogonal distributions D_1 and D_2 such that $S(Ker\ \phi_*) = D_1 \oplus D_2$, where D_1 is invariant and D_2 is slant.

A screen semi-slant lightlike submersion is called proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \pi/2$. From definition (4.1) following cases may arise:
If $D_1 = 0$, ϕ is a screen-slant lightlike submersion.
If $D_2 = 0$, ϕ is a invariant lightlike submersion.
If $D_1 = 0$ and $\theta = \pi/2$, ϕ is a anti-invariant lightlike submersion.
If $D_1 \neq 0$ and $\theta = \pi/2$, ϕ is a SCR lightlike submersion.

So, we can say that above defined class of lightlike submersions includes invariant, anti-invariant, screen slant and SCR lightlike submersions as its subcases.

Example 4.2. Let $\mathbb{R}^{12}_{0,2,10}$ and $\mathbb{R}^6_{4,0,2}$ equipped with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2,$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$. Consider the map $\phi : (\mathbb{R}^{12}, g) \to (\mathbb{R}^6, g')$ as

$$(u_1, ..., u_{12}) \longmapsto \Big(u_1 - x_7, u_2 - u_8, \frac{u_3 + u_6}{\sqrt{2}}, u_5, u_{11}, u_{12}\Big).$$

Then $\Delta = Span\Big\{\xi_1 = \dfrac{\partial}{\partial u_1} + \dfrac{\partial}{\partial u_7}, \xi_2 = \dfrac{\partial}{\partial u_2} + \dfrac{\partial}{\partial u_8}\Big\}$. Clearly $\mathcal{J}\xi_1 = \xi_2$, so Δ is invariant. By easy calculation we can see that

$$D_2 = Span\Big\{\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_6}\Big), \frac{\partial}{\partial u_4}\Big\}$$

is slant distribution with slant angle $\theta = \dfrac{\pi}{4}$. Further, we see that

$$D_1 = Span\Big\{\frac{\partial}{\partial u_9}, \frac{\partial}{\partial u_{10}}\Big\}$$

is invariant with respect to $\mathcal{J}$. Hence, ϕ is a proper screen semi-slant lightlike submersion.

Example 4.3. Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be endowed with

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2,$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$. Taking the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(u_1, ..., u_8) \longmapsto \Big((u_1 - u_5)/\sqrt{2}, (u_2 - u_6)/\sqrt{2}, u_3 + u_7, u_4 + u_8\Big)$. It gives

$$Ker\ \phi_* = Span\Big\{U_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_5}\Big), U_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_6}\Big), U_3 = \frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_7},\ U_4 = \frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_8}\Big\},$$

which implies

$$(Ker\ \phi_*)^{\perp} = Span\Big\{U_1, U_2,\ X = \frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_7},\ Y = \frac{\partial}{\partial u_4} + \frac{\partial}{\partial u_8}\Big\}.$$

Thus f is a 2-lightlike submersion with $\Delta = Span\{U_1, U_2\}$, which is cleary seen to be invariant. Since $\mathcal{J}U_3 = U_4$ we have $S(Ker\ f_*) = D_1 = Span\{U_3, U_4\}$ and $D_2 = 0$. Hence, ϕ is a invariant lightlike submersion.

Example 4.4. Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be equipped with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (dx_7)^2 + (dx_8)^2,$$

and degenerate metric $g' = (dv_3)^2 + (dv_4)^2$, respectively. Taking the map $\phi : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(u_1, ..., u_8) \longmapsto \Big(u_1 - u_3,\ u_2 - u_4,\ u_6, \dfrac{u_5 - u_8}{2}\Big)$. Then, we obtain

$$Kerf_* = Span\Big\{U_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}, U_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4}, U_3 = \frac{1}{2}\Big(\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_8}\Big), U_4 = \frac{\partial}{\partial u_7}\Big\},$$

and

$$(Kerf_*)^{\perp} = Span\Big\{U_1, U_2,\ X = \frac{1}{2}\Big(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_8}\Big), Y = \frac{\partial}{\partial u_6}\Big\}.$$

Then, $\Delta = Span\{U_1, U_2\}$, which is clearly seen to be invariant. Further, $S(Ker\ \phi_*) = D_2 = Span\{X, Y\}$ is slant with angle $\frac{\pi}{4}$. Thus, $D_1 = 0$. Hence ϕ is a screen slant lightlike submersion.

Example 4.5. Let $\mathbb{R}^{12}_{0,4,8}$ and $\mathbb{R}^{6}_{4,0,2}$ be equipped with

$$g = -(du_1)^2 - (du_2)^2 - (du_3)^2 - (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2,$$

and degenerate metric $g' = (dv_5)^2 + (dv_6)^2$, respectively. Consider the map $\phi : (\mathbb{R}^{12}, g) \to (\mathbb{R}^6, g')$, such that

$$(u_1, ..., u_{12}) \longmapsto \Big(u_1 - u_5, u_2 - u_6, (u_3 + u_7)/\sqrt{2}, (u_4 + u_8)/\sqrt{2}, u_9, u_{11}\Big).$$

Then, we obtain

$$Kerf_* = Span\Big\{U_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_5},\ U_2 = \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_6},\ U_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_7}\Big), U_4 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_8}\Big),\ U_5 = \frac{\partial}{\partial u_{10}},\ U_6 = \frac{\partial}{\partial u_{12}}\Big\},$$

and

$$(Kerf_*)^\perp = Span\Big\{U_1,\ U_2,\ U_3,\ U_4,\ X = \frac{\partial}{\partial u_9},\ Y = \frac{\partial}{\partial u_{11}}\Big\}.$$

Thus, f is a 4-lightlike submersion with $\Delta = Span\{U_1, U_2, U_3, U_4\}$. Further, we can see easily that $\mathcal{J}U_1 = U_2$ and $\mathcal{J}U_3 = U_4$. Therefore, Δ is invariant with respect to $\mathcal{J}$. Also $\mathcal{J}U_5 = X$ and $\mathcal{J}U_6 = Y$, implies that $S(Ker\ \phi_*) = D_2 = Span\{U_5, U_6\}$. Finally, since $\mathcal{J}D_2 = S(Ker\ \phi_*)^\perp$, ϕ is a anti-invariant lightlike submersion.

Example 4.6. Let $\mathbb{R}^{16}_{0,2,14}$ and $\mathbb{R}^{8}_{6,0,2}$ be equipped with the metric

$$g = -(du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2 + (du_{13})^2 + (du_{14})^2 + (du_{15})^2 + (du_{16})^2,$$

and degenerate metric $g' = (dv_5)^2 + (dv_6)^2$, respectively. Consider the map $\phi : (\mathbb{R}^{16}, g) \to (\mathbb{R}^8, g')$ as

$$(u_1, ..., u_{16}) \longmapsto \Big((u_1 - u_3)/\sqrt{3}, (u_2 - u_4)/\sqrt{3}, u_5 + u_7, u_6 + u_8, u_{10}, u_{12}, u_{14}, u_{16}\Big).$$

Then, we obtain

$$Ker\phi_* = Span\Big\{U_1 = \frac{1}{\sqrt{3}}\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}\Big),\ U_2 = \frac{1}{\sqrt{3}}\Big(\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_4}\Big), U_3 = \frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}, U_4 = \frac{\partial}{\partial u_6} - \frac{\partial}{\partial u_8},$$

$$U_5=\frac{\partial}{\partial u_9},\ U_6=\frac{\partial}{\partial u_{11}},\ U_7=\frac{\partial}{\partial u_{13}},\ U_8=\frac{\partial}{\partial u_{15}}\Big\},$$

and

$$(Ker\phi_*)^{\perp}=Span\Big\{U_1,\ U_2,\ X_1=\frac{\partial}{\partial u_5}+\frac{\partial}{\partial u_7}, X_2=\frac{\partial}{\partial u_6}+\frac{\partial}{\partial u_8},$$
$$X_3=\frac{\partial}{\partial u_{10}},\ X_4=\frac{\partial}{\partial u_{12}},\ X_5=\frac{\partial}{\partial u_{14}},\ X_6=\frac{\partial}{\partial u_{16}}\Big\},$$

It follows that $\Delta = Span\{U_1, U_2\}$, which is clearly invariant. Now, as $\mathcal{J}U_3 = U_4$, therefore $D_1 = Span\{U_3, U_4\}$ is an invariant distribution. Further, since $\mathcal{J}U_5 = X_3,\ \mathcal{J}U_6 = X_4,\ \mathcal{J}U_7 = X_5$ and $\mathcal{J}U_8 = X_6$, we conclude that $\mathcal{J}D_2 = S(Ker\ \phi_*)^{\perp}$. Hence, ϕ is a proper-SCR lightlike submersion.

Now, for all $U \in \Gamma(Ker\ \phi_*)$, we write

$$\mathcal{J}U = \chi U + FU.$$

Here χU (resp. FU) is tangential (resp. normal) component of $\mathcal{J}U$. Next, we denote the projections of $Ker\ \phi_*$ on Δ, D_1 and D_2 by χ_1, χ_2 and χ_3, respectively. Then, $U = \chi_1 U + \chi_2 U + \chi_3 U$, for $U \in \Gamma(Ker\ \phi_*)$, which implies $\mathcal{J}U = \mathcal{J}\chi_1 U + \mathcal{J}\chi_2 U + \mathcal{J}\chi_3 U$. Then,

$$\mathcal{J}U = \mathcal{J}\chi_1 U + \mathcal{J}\chi_2 U + \xi\chi_3 U + F\chi_3 U, \tag{31}$$

where $\xi\chi_3 U$ (resp. $F\chi_3 U$) denotes the tangential (resp. transversal) component of $J\phi_3 U$. So, we have $\mathcal{J}\chi_1 U \in \Gamma(\Delta)$, $\mathcal{J}\chi_2 U \in \Gamma(D_1)$, $\xi\chi_3 U \in \Gamma(D_2)$ and $F\chi_3 U \in \Gamma(S(Ker\ \phi_*)^{\perp})$. In the same way, denote the projections of $tr(Ker\ \phi_*)$ on $ltr(Ker\ \phi_*)$ and $S(Ker\ \phi_*)^{\perp}$ by Q_1 and Q_2, respectively. So, $\forall\ W \in \Gamma(tr(Ker\ \phi_*))$, we put $W = Q_1 W + Q_2 W$, which gives $JW = JQ_1 W + \mathcal{J}Q_2 W$. Then, we have

$$\mathcal{J}W = \mathcal{J}Q_1 W + BQ_2 W + CQ_2 W, \tag{32}$$

where $BQ_2 W$ (resp. $CQ_2 W$) denotes the tangential (resp. transversal) component of $JQ_2 W$. Thus, we have $\mathcal{J}Q_1 W \in \Gamma(ltr(Ker\ f_*))$, $BQ_2 W \in \Gamma(D_2)$ and $CQ_2 W \in \Gamma(S(Ker\ \phi_*)^{\perp})$. Using (2), (9), (11), (31) and (32) and identifying the components of $\Delta, D_1, D_2, ltr(Ker\ \phi_*)$ and $S(Ker\ \phi_*)^{\perp}$, we get

$$\chi_1(\hat{\nabla}_U \mathcal{J}\chi_1 V) + \chi_1(\hat{\nabla}_U \mathcal{J}\chi_2 V) + \chi_1(\hat{\nabla}_U \xi\chi_3 V)$$
$$= -\chi_1(T_U F\chi_3 V) + \mathcal{J}\chi_1 \hat{\nabla}_U V, \tag{33}$$

$$\chi_2(\hat{\nabla}_U \mathcal{J}\chi_1 V) + \chi_2(\hat{\nabla}_U \mathcal{J}\chi_2 V) + \chi_2(\hat{\nabla}_U \xi\chi_3 V)$$
$$= -\chi_2(T_U F\chi_3 V) + \mathcal{J}\chi_2 \hat{\nabla}_U V, \tag{34}$$

$$\chi_3(\hat{\nabla}_U \mathcal{J}\chi_1 V) + \chi_3(\hat{\nabla}_U \mathcal{J}\chi_2 V) + \chi_3(\hat{\nabla}_U \xi\chi_3 V)$$
$$= -\chi_3(T_U F\chi_3 V) + \xi\chi_3 \hat{\nabla}_U V + B(T^s_U V), \tag{35}$$

$$T^l_U \mathcal{J}\chi_1 V + T^l_U \mathcal{J}\chi_2 V + T^l_U \xi\chi_3 V = \mathcal{J}T^l_U V - D^{\perp l}(U, F\chi_3 V), \tag{36}$$

$$T^s_U \mathcal{J}\chi_1 V + T^s_U \mathcal{J}\chi_2 V + T^s_U \xi\chi_3 V = CT^s_U V - \nabla^{\perp s}_U F\chi_3 V + F\chi_3 \hat{\nabla}_U V. \quad (37)$$

Theorem 4.7. *Let ϕ be a 2r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, ϕ is a screen semi-slant lightlike submersion if and only if*

(i) $\mathcal{J}(ltr(Ker\ \phi_*)) = ltr(Ker\ \phi_*)$ *and* $\mathcal{J}(D_1) = D_1$,

(ii) $\exists$ *a constant* $\lambda \in [0, 1)$, *in such a way, that* $(\chi_3 \circ \xi)^2 U = -\lambda U$, $\forall U \in \Gamma(D_2)$, *where* D_1 *and* D_2 *are orthogonal distributions, such that* $S(Ker\ \phi_*) = D_1 \oplus D_2$ *and* $\lambda = cos^2\theta$, θ *is a slant angle of* D_2.

Proof. Using (1) and (31), we get

$$g(\mathcal{J}N, U) = -g(N, \mathcal{J}U) = -g(N, \mathcal{J}\chi_1 U + \mathcal{J}\chi_2 U + \xi\chi_3 U + F\chi_3 U) = 0,$$

for any $N \in \Gamma(ltr(Ker\ \phi_*))$ and $U \in \Gamma(S(Ker\ \phi_*))$. Therefore $\mathcal{J}N$ does not belong to $S(Ker\ \phi_*)$. Now, if $W \in \Gamma(S(Ker\ \phi_*)^\perp)$, using (1) and (32), we derive

$$g(\mathcal{J}N, W) = -g(N, \mathcal{J}W) = -g(N, BW + CW) = 0,$$

which implies that $\mathcal{J}N$ does not belongs to $\Gamma(S(Ker\ \phi_*)^\perp)$. Also, if $\mathcal{J}N \in \Gamma(\Delta)$, then $\mathcal{J}(\mathcal{J}N) = \mathcal{J}^2 N = -N \in \Gamma(ltr(Ker\ \phi_*))$, which is absurd as Δ is invariant with respect to $\mathcal{J}$. Thus $ltr(Ker\ \phi_*)$ is invariant with respect to $\mathcal{J}$. Now, if $U \in \Gamma(D_2)$, we get

$$cos(\theta)(U) = \frac{g(\mathcal{J}U, \xi\chi_3(U))}{|\mathcal{J}(U)||\xi\chi_3(U)|} = -\frac{g(U, \mathcal{J}\xi\chi_3 U)}{|\mathcal{J}U||\xi\chi_3 U|} = -\frac{g(U, (\chi_3 \circ \xi)^2 U)}{|\mathcal{J}U||\xi\chi_3 U|}, \quad (38)$$

where θ ia constant angle independent of point $p \in M_1$. Moreover,

$$cos(\theta)(U) = \frac{|\xi\chi_3(U)|}{|\mathcal{J}(U)|}. \quad (39)$$

Using (38) and (39), we obtain

$$cos^2\theta(U) = -\frac{\hat{g}(U, (\chi_3 \circ \xi)^2 U)}{|U|^2}.$$

Now, since $\theta(U)$ is constant, we have $(\chi_3 \circ \xi)^2 U = -\lambda U, \lambda \in [0, 1)$, where $\lambda = cos^2\theta$. The converse part can be proved in a similar way. □

Theorem 4.8. *Let $\phi : M_1 \to M_2$ be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the null distribution Δ is integrable if and only if $\forall\ U, V \in \Gamma(\Delta)$, we have*

(i) $\chi_2(\hat{\nabla}_U \mathcal{J}\chi_1 V) = \chi_2(\hat{\nabla}_V \mathcal{J}\chi_1 U)$,

(ii) $\chi_3(\hat{\nabla}_U J\chi_1 V) = \chi_3(\hat{\nabla}_V \mathcal{J}\chi_1 U)$,

(iii) $T^s_U \mathcal{J}\chi_1 V = T^s_V \mathcal{J}\chi_1 U$.

Proof. Let $U, V \in \Gamma(\Delta)$. From (34), we have $\chi_2(\hat{\nabla}_U \mathcal{J}\chi_1 V) = \mathcal{J}\chi_2 \hat{\nabla}_U V$. It follows that

$$\chi_2(\hat{\nabla}_U \mathcal{J}\chi_1 V) - \chi_2(\hat{\nabla}_V \mathcal{J}\chi_1 U) = \mathcal{J}\chi_2[U,\ V]. \quad (40)$$

Finally, in view of (35), we have $\chi_3(\hat{\nabla}_U J\chi_1 V) = \psi\chi_3\hat{\nabla}_U V + BT^s_U V$, which implies

$$\chi_3(\hat{\nabla}_U \mathcal{J}\chi_1 V) - \chi_3(\hat{\nabla}_V \mathcal{J}\chi_1 U) = \xi\chi_3[U,\ V]. \tag{41}$$

Using (37), we obtain $T^s_U \mathcal{J}\chi_1 V = CT^s_U V + F\chi_3\hat{\nabla}_U V$, which gives

$$T^s_U \mathcal{J}\chi_1 V - T^s_V \mathcal{J}\chi_1 U = F\chi_3[U,\ V]. \tag{42}$$

Using (40), (41) and (42), the proof follows. □

Theorem 4.9. *Let $\phi : M_1 \to M_2$ be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the non-null distribution D_1 is integrable if and only if $\forall\ U, V \in \Gamma(D_1)$, we have*

(i) $T^s_U \mathcal{J}\chi_2 V = T^s_V \mathcal{J}\chi_2 U$,
(ii) $\chi_1(\hat{\nabla}_U \mathcal{J}\chi_2 V) = \chi_1(\hat{\nabla}_V \mathcal{J}\chi_2 U)$,
(iii) $\chi_3(\hat{\nabla}_U \mathcal{J}\chi_2 V) = \chi_3(\hat{\nabla}_V \mathcal{J}\chi_2 U)$.

Proof. Let $U, V \in \Gamma(D_1)$. Now, from (33), we have $\chi_1(\hat{\nabla}_U J\phi_2 V) = J\phi_1\hat{\nabla}_U V$, which gives

$$\chi_1(\hat{\nabla}_U \mathcal{J}\chi_2 V) - \chi_1(\hat{\nabla}_V \mathcal{J}\chi_2 U) = \mathcal{J}\chi_1[U,\ V]. \tag{43}$$

In view of (35), we have $\chi_3(\hat{\nabla}_U \mathcal{J}\chi_2 V) = \xi\chi_3\hat{\nabla}_U V + BT^s_U V$. It follows that

$$\chi_3(\hat{\nabla}_U \mathcal{J}\chi_2 V) - \chi_3(\hat{\nabla}_V \mathcal{J}\chi_2 U) = \xi\chi_3[U,\ V]. \tag{44}$$

Using (37), we obtain $T^s_U \mathcal{J}\chi_2 V = CT^s_U V + F\chi_3\hat{\nabla}_U V$. It gives

$$T^s_U \mathcal{J}\chi_2 V - T^s_V \mathcal{J}\chi_2 U = F\chi_3[U,\ V]. \tag{45}$$

Thus, the proof is completed by using (43), (44) and (45). □

Theorem 4.10. *Let $\phi : M_1 \to M_2$ be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then the non-null distribution D_2 is integrable if and only if for any $U, V \in \Gamma(D_2)$, we have*
$\chi_1(\hat{\nabla}_U\xi\chi_3 V - \hat{\nabla}_V\xi\chi_3 U) = \chi_1(T_V F\chi_3 U - T_U F\chi_3 V)$,
$\chi_2(\hat{\nabla}_U\xi\chi_3 V - \hat{\nabla}_V\xi\chi_3 U) = \chi_2(T_V F\chi_3 U - T_U F\chi_3 V)$.

Proof. Let $U, V \in \Gamma(D_2)$. Using (33), we have $\chi_1(\hat{\nabla}_U\xi\chi_3 V) + \chi_1(T_U F\chi_3 V) = \mathcal{J}\chi_1\hat{\nabla}_U V$, which implies

$$\chi_1(\hat{\nabla}_U\xi\chi_3 V) - \chi_1(\hat{\nabla}_V\xi\chi_3 U) + \chi_1(T_U F\chi_3 V) - \chi_1(T_V F\chi_3 U) = \mathcal{J}\chi_1[U,\ V]. \tag{46}$$

Using (34), we drive $\chi_2(\hat{\nabla}_U\xi\chi_3 V) + \chi_2(T_U F\chi_3 V) = \mathcal{J}\chi_2\hat{\nabla}_U V$, which gives

$$\chi_2(\hat{\nabla}_U\xi\chi_3 V) - \chi_2(\hat{\nabla}_V\xi\chi_3 U) + \chi_2(T_U F\chi_3 V) - \chi_2(T_V F\chi_3 U) = \mathcal{J}\chi_2[U,\ V]. \tag{47}$$

Thus, the proof follows from (46) and (47). □

Theorem 4.11. *Let $\phi : M_1 \to M_2$ be a screen semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, induced connection $\hat{\nabla}$ on $S(Ker\ \phi_*)$ is a metric connection if and only if $BT^s_V\xi = 0$ and $T^*_V\xi = 0$ on $\Gamma(Ker\ \phi_*)$, $\forall\ V \in \Gamma(Ker\ \phi_*)$ and $\xi \in \Gamma(\Delta)$.*

Proof. Connection $\hat{\nabla}$ on $S(Ker\ \phi_*)$ is a metric connection if and only if Δ is a parallel distribution with respect to $\hat{\nabla}$. Using (2), (8) and (14), we obtain $\nabla_V \mathcal{J}\xi = \mathcal{J}\hat{\nabla}^{*\perp}_V \xi + \mathcal{J}T^*_V \xi + \mathcal{J}T^l_V \xi + \mathcal{J}T^s_V \xi$, for any $V \in \Gamma(Ker\ \phi_*)$ and $\xi \in \Gamma(\Delta)$. Comparing the tangential components, we get $\hat{\nabla}_V \mathcal{J}\xi = \mathcal{J}\hat{\nabla}^{*\perp}_V \xi + \mathcal{J}T^*_\xi V + BT^s_V \xi$, which completes the proof. □

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