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ON AMR-ALGEBRA†

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Abstract. The main objective of this paper is to introduce the notion of AMR-algebra and its generalization, and to compare them with other algebras such as BCK, BCI, BCH, $\cdots$, etc. We show moreover that the K-part of AMR-algebra is an abelian group, and the weak AMR-algebra is also an abelian group and generalizes many known algebras like BCI, BCH, and G.

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1. Introduction

In 1966, Y. Imai and K. Iseki formulated the notions of BCK-algebras and BCI-algebras first, in [2]. These algebras have been extensively studied since their introduction. In 1983, Q. P. Hu and X. Li introduced the notion of a BCH-algebra, which is a generalization of the notion of BCK and BCI-algebras, and studied a few properties of these algebras. In 1998, Y. B. Jun, E. H. Roh, and H. S. Kim,[3] introduced a new notion, called a BH-algebra, which is a generalization of BCK, BCI, and BCH-algebras. In 2001, J. Neggers, S. S. Ahn, and H. S. Kim, [7] introduced a new notion, called a Q-algebra, and generalized some theorems discussed in BCI and BCK-algebras. In 2002, J. Neggers and H. S. Kim, [8] introduced a new notion, called a B-algebra, and obtained several results. In 2006, C. B. Kim and H. S. Kim, [4] introduced the notion of a BM-algebra, which is a specialization of the notion of B-algebras. In 2007, A. Walendziak, [10] introduced a new notion, called a BF-algebra, which is a generalization of the B-algebra. In 2008, C. B. Kim and H. S. Kim [5] introduced BG-algebra as a generalization of B-algebra. In 2012, C. B. Kim and H. S. Kim, [14] introduced BO-algebra as a generalization of BG and BH-algebra, Ravi Kumar Bandru and N. Rafi, in [13], introduced the notion of G-algebra, which

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is a generalization of QS algebras, and Ravi Kumar Bandru, in [12], introduced the notion of BRK-algebra, which is a generalization of BCK, BCI, BCH, Q, QS, and BM-algebras. In 2014, R. A. K. Omar introduced the notion of RG-algebra, which is a generalization of BCI-algebras [9].

Definition 1.1. [2] A BCI-algebra is a non empty set X endowed with binary operation $*$ and a constant 0 such that for all $x, y, z \in X$, we have

(a) $((x * y) * (x * z)) * (z * y) = 0$,
(b) $(x * (x * y)) * y = 0$,
(c) $x * x = 0$,
(d) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

If a BCI-algebra satisfies the identity $0 * x = 0$ for all $x \in X$, then it is called a BCK-algebra.

Example 1.2. [2] The set $X = \{0, 1, 2, 3, 4\}$ with a binary operation $*$ defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	2	1	0

is a BCK-algebra.

Definition 1.3. [3] A BH-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$,
(b) $x * 0 = x$,
(c) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Example 1.4. [3] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X, *, 0)$ is BH-algebra.

Definition 1.5. [7] A Q-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$,

(b) $x * 0 = x$,
(c) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$..

A Q-algebra is said to be a QS-algebra if it satisfies the additional relation: $(x * y) * (x * z) = z * y$.

Example 1.6. [7] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X, *, 0)$ is Q-algebra.

Definition 1.7. [8] A B-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$,
(b) $x * 0 = x$,
(c) $(x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$..

Example 1.8. [8] Let $X = \{0, 1, 2\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is B-algebra.

Definition 1.9. [4] A BM-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * 0 = x$
(b) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Example 1.10. [4] Let $X = \{0, 1, 2\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is BM-algebra.

Definition 1.11. [10] A BF-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$,
(b) $x * 0 = x$,
(c) $0 * (x * y) = y * x$ for all $x, y \in X$.

Example 1.12. [10] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(X, *, 0)$ is BF-algebra.

Definition 1.13. [5] A BG-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$,
(b) $x * 0 = x$,
(c) $(x * y) * (0 * y) = x$ for all $x, y \in X$..

Example 1.14. [5] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	1	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is BG-algebra.

Definition 1.15. [14] A BO-algebra is a non empty set X endowed with binary operation $*$ and a constant 0 such that for all $x, y, z \in X$, we have

(a) $x * x = 0$
(b) $x * 0 = x$
(c) $x * (y * z) = (x * y) * (0 * z)$ for all $x, y, z \in X$.

Example 1.16. [14] The set $X = \{0, 1, 2, 3, 4\}$ with a binary operation $*$ defined by

$*$	0	1	2	3	4
0	0	2	1	4	3
1	1	0	3	2	4
2	2	4	0	3	1
3	3	1	4	0	2
4	4	3	2	1	0

is a BO-algebra.

Definition 1.17. [12] A G-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * x = 0$
(b) $x * (x * y) = y$ for all $x, y \in X$.

Example 1.18. [12] Let $X = \{0, 1, 2\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is G-algebra.

Definition 1.19. [12] A BRK-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * 0 = x$,
(b) $(x * y) * x = 0 * y$, for all $x, y \in X$

Example 1.20. [12] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is BRK–algebra.

Definition 1.21. [9] A RG-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * 0 = x$,
(b) $x * y = (x * z) * (y * z) \ \ \forall\ x, y, z \in X$.
(c) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Example 1.22. [9] Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

Then $(X, *, 0)$ is RG-algebra.

Definition 1.23. [11] A Z-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(a) $x * 0 = 0$,
(b) $0 * x = x$
(c) $x * x = x$
(d) $x * y = y * x$ when $x \neq 0$ and $y \neq 0$ for all $x, y \in X$.

Example 1.24. [11] Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then $(X, *, 0)$ is Z-algebra.

In this paper, we define new extensions of BCK/BCI-algebras called AMR-algebra and general AMR-algebra. We moreover derive some basic algebraic properties of these new algebras. The connection between these new algebras and the other algebras is established and some examples are in addition provided to distinguish them from other algebras. We introduce the so-called K-part of AMR-algebra and we prove that it is an abelian group. Furthermore, we show that the weak AMR-algebra is a generalization of BCK, BCI, BCH, Q, B, BF, BG, BH, BRK, BO, BM, QS, RG, and G-algebras is also abelian group.

2. AMR-Algebras

In this section, we introduce the notion of AMR-algebra with some important results related to it and we will give the relation between AMR-algebra and others algebras.

Definition 2.1. An AMR-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(i) $x * 0 = x$
(ii) $(x * y) * z = y * (z * x)$ for all $x, y, z \in X$.

We can define a binary relation $x \leq y$ if and only if $x * y = 0$.

Example 2.2. Let $X = \{0, 1, 2\}$ be a set with a binary operation $*$ defined by:

$*$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	2

Then $(X, *, 0)$ is AMR-algebra.

Proposition 2.3. If $(X, *, 0)$ is an AMR-algebra, then, for all $x, y, z, t \in X$, the following conditions hold:

(1) $0 * x = x$,
(2) $x * y = y * x$,
(3) $(x * y) * (z * t) = (x * z) * (y * t)$.

Proof. Let us prove (1). From Definition 2.1 (i), we get

$$0 * x = 0 * (x * 0) = (0 * x) * 0$$

and from (ii), we have

$$0 * x = (0 * x) * 0 = x * (0 * 0) = (0 * 0) = x * 0 = x.$$

For (2), it follows from assertion (1) and Definition 2.1 (ii) that

$$x * y = 0 * (x * y) = (y * 0) * x = y \ * x.$$

Assertion (3) follows finally on using (2) and Definition 2.1 (ii) as

$$\begin{aligned}(x * y) * (z * t) &= y * ((z * t) * x) = y * (x * (t * z)) \\ &= y * ((z * x) * t) = y * ((x * z) * t) \\ &= (t * y) * (x * z) = (x * z) * (y * t).\end{aligned}$$

□

Proposition 2.4. *If $(X, *, 0)$ is an AMR-algebra and satisfy the identity $x*x = 0$ then, for any $x, y, z, t \in X$, the following properties hold.*

(a) $(x * y) * (x * z) = z * y$.
(b) $(x * (x * y)) * (y * x) = x$.
(c) *If $x * y = 0$, then $x = y$.*
(d) *If $x * y = x * z$, then $y = z$.*

Proof. Assertions (a), (b), and (c) follows immediately from Definition 2.1 and the identity $x * x = 0$ in the following way:

$$(x * y)*(x*z) = y*((x*z)*x) = y*(z*(x * x)) = y*(z*(x * x)) = y*(z*0) = y*z.$$

This proves (a). For (b), we have

$$(x * (x * y)) * (y * x) = ((y * x) * x) * (y * x) = x * ((y * x) * (y * x)) = x * 0 = x.$$

Assertion (c) follows obviously as

$$x = x * 0 = x * (y * x) = (x * x) * y = 0 * y = y.$$

Assume that $x*y = x*z$, then $x*(x*y) = x*(x*z)$ and hence $x*(y*x) = x*(z*x)$, by Proposition 2.3 (2). We moreover have $(x * x) * y = (x * x) * z$, by (ii) in Definition 2.1, so $0 * y = 0 * z$ because $x * x = 0$. Therefore, we deduce that $y = z$ by Proposition 2.3 (1). Assertion (d) is now proved. □

There is no relation between AMR-algebra and BCK, BCI, BCH, Q, B, BF, BG, BH, BRK, BO, RG, G, Z, and QS-algebras. In Example 2.2 we have $1 * 1 = 2$, this shows that AMR-algebra is not one of BCK, BCI, BCH, Q, B, BF, BG, BH, BRK, BO, G, Z, or QS-algebras. Further, neither of these algebras is AMR-algebra, this is due to known examples given in the literature. Motivated by the problem, we introduce the following result.

Theorem 2.5. *If $(X, *, 0)$ is an AMR-algebra and satisfies in addition the identity $x * x = 0$, then $(X, *, 0)$ is a $BCI-$, $BF-$, $BG-$, $BH-$, $B-$, $BRK-$, $Q-$, $BO-$, $BCH-$, $BM-$, $QS-$, $RG-$, and $G-$algebra.*

Proof. By using Proposition 2.4 (a) and the hypotheses, we have

$$((x * y) * (x * z)) * (z * y) = (z * y) * (z * y) = 0.$$

Furthermore, one can easily check that

$$(x * (x * y)) * y = ((x * 0) * (x * y)) * y = (y * 0) * y = y * y = 0.$$

From the assumption, we have $x * x = 0$.

Now, if $y * x = 0 = x * y$, we then obtain $x = y$, by using Proposition 2.4 (c). Finally, if $x * 0 = 0$, but we know $x * 0 = x$, this consequently implies $x = 0$. Then $(X, *, 0)$ is a BCI-algebra. □

3. *K*-Part of AMR-Algebras

In this section, we define *t*he K-part of an AMR-algebra. The major property of K-part in AMR-algebras is that it is an abelian group. We give a necessary condition for AMR-algebra to being an abelian group.

Definition 3.1. Let X be an AMR-algebra. For any subset S of X, we define $K(S) = \{x \in S | x * x = 0\}$. In particular, if $S = X$, then we say that $K(X)$ is the K-part of an AMR-algebra.

Definition 3.2. A nonempty subset I of an AMR-algebra X is called a sub-algebra of X if $x * y \in I$, whenever $x, y \in I$.

Definition 3.3. A nonempty subset I of an AMR-algebra X is called an ideal of X if for any $x, y, z \in X$. (a_1) . $0 \in I$. (a_2). $(x * y) * z \in I$, $(z * x) \in I$ implies $y \in I$.

$\{0\}$ and X are ideals of X. We call $\{0\}$ and X the zero ideal and the trivial ideal of X, respectively. An ideal I is said to be proper if $I \neq X$. In Example 2.2, the set $(X, *, 0)$ is an AMR-algebra and the set $I = \{0, 2\}$ is a sub-algebra, of X and $S = \{0, 1\}$ is not sub-algebra of X.

Theorem 3.4. *Every AMR-algebra $(X, *, 0)$ satisfying the condition $x * x = 0$, for all $x \in X$ is an abelian group under the operation $*$.*

Proof. By Definition 2.1 and Proposition 2.4, for all $x, y, z \in X$, we have $(x * y) * z = y * (z * x) = (z * x) * y = (x * z) * y = z * (y * x) = z * (x * y) = (y * z) * x = x * (y * z)$.

We also have $x * 0 = x$ and $0 * x = x$, which means that 0 is the zero element of X. By $x * x = 0$, every element x in X has as its inverse the element x itself. Therefore, $(X, *, 0)$ is an abelian group. □

Corollary 3.5. *The K-part of an AMR-algebra is an abelian group.*

Definition 3.6. A weak AMR-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1 $x * x = 0$,

2 $x * 0 = x$,

3 $(x * y) * z = y * (z * x)$ for all $x, y, z \in X$.

Example 3.7. Let $X = \{0, 1\}$ be a set with a binary operation $*$ defined by

$*$	0	1
0	0	1
1	1	0

Then $(X, *, 0)$ is weak AMR-algebra.

Corollary 3.8. *Every weak AMR-algebra is an abelian group.*

4. Algorithm for AMR-algebras and Algorithm for AMR- ideal in AMR-algebras

As an application, we give the following Algorithm.

Input (X: set, $*$: binary operation)
Output("X is an AMR-algebras or not")
Begin
If $X = \emptyset$ then go to (1.);
End If
If $0 \notin X$ then go to (1.);
End If
Stop:= false;
$i := 1$
While $i \leq X$ and not (Stop) do
If $x_i * 0 \neq x_i$ then
Stop: = true;
End If
$j := 1$
While $j \leq X$ and not (Stop) do
If $x_i * y_j \neq y_j * x_i$ then

Stop: = true;
End If
End If
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $(x_i * y_j) * z_k \neq y_j * (z_k * x_i$ then
Stop: = true;
End If
End While
End While
End While
If Stop then
(1.) Output("X is not an AMR-algebras ")
Else
Output("X is an AMR-algebras ")
End If
End.

Input (X: AMR-algebra, I: subset of X)
Output("I is an AMR-ideal of X or not")
If $I = \emptyset$ then
Go to (1.);
End If
Stop:= false;
$i := 1$
While $i \leq |X|$ and not (Stop) do
$j := 1$
While $j \leq |X|$ and not (Stop) do
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $(x_i * y_j) * z_k \in I$, and $z_k * x_i \in I$ then
If $y_i \notin I$ then
Stop: = false
End If
End While
End While
End While
If Stop then
Output("I is an AMR-ideal of X ")
Else
(1.) Output("I is not an AMR-ideal of X ")
End If
End.

References

1. Q.P. Hu and X. Li, *On BCH-algebras*, Mathematics Seminar Notes **11** (1983), 313-320.
2. Y. Imai and K. Iseki, *On axiom systems of propositional calculi*, Proc. Japan acad. **42** (1996), 19-22.
3. Y.B. Jun, E.H. Roh, and H.S. Kim, *On BH-algebras*, Sci. Math. Japonica **1** (1998), 347-354.
4. C.B. Kim and H.S. Kim, *On BM-algebras*, Sci. Math. Japonicae **63** (2006), 421-427.
5. C.B. Kim and H.S. Kim, *On BG-algebras*, Demonstr. Math. **41** (2008), 497-505.
6. Y.L. Liu, Y. Xu and J. Meng, *BCI-implicative ideals of BCI-algebras*, Inform. Sci. **177** (2007), 4987-4996.
7. J. Neggers, S.S. Ahn, and H.S. Kim, *On Q-algebras*, Int. J. Math. and Math. Sci. **27** (2001), 749-757.
8. J. Neggers and H.S. Kim, *On B-algebras*, Math. Vensik **54** (2002), 21-29.
9. R.A.K. Omar, *On RG-algebras*, Pure Math. Sci. **3** (2014), 59-70.
10. A. Walendziak, *On BF-algebras*, Math. Slovaca **57** (2007), 119-128.
11. M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha and S. Sabarinathan, *A note on Z-algebra*, Ital J. Pure Appl. Math. **38** (2017), 707-714.
12. Ravi Kumar Bandru, *On BRK-algebras*, International Journal of Mathematics and Mathematical Sciences **2012** (2012), 1-12.
13. M. Kondo, *On the class of QS-algebras*, Int. J. Math. and Math. Sc. **2004** (2004), 2629-2639.
14. C.B. Kim and H.S. Kim, *On BO-algebras*, Math. Slovaca **62** (2012), 855-864.

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