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EXISTENCE, UNIQUENESS AND HYERS-ULAM-RASSIAS STABILITY OF IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH BOUNDARY CONDITION

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ABSTRACT. This paper focuses on the existence and uniqueness outcome for fractional integro-differential equation (FIDE) among impulsive edge condition and Hyers-Ulam-Rassias Stability (HURS) by using fractional calculus and some fixed point theorem in some weak conditions. The outcome procured in this paper upgrade and perpetuate some studied solutions.

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1. Introduction

Finite difference issues for non - linear fractional differential equations have recently begun with the study of modeling of viscoelasticity, control, electrodynamics, and so on *etc.* [22, 12, 17]. In recent years, the investigation of such challenges has gained traction from both basic and empirical perspectives. The fractional calculus theory and applications, one can see the monographs of Kilbas et. al.[17], Lakshmikantham et. al., [20], Miller and Ross [23], Anguraj et. al., [4] , Chalishajar et. al., [10] , Vinodkumar et. al., [35] , Podlubny and Baleanu et. al., [28, 5, 6, 7]. Many disciplines of physics and technological sciences use FIDE and control issues. In particular, FIDE are measured as a nonlinear differential equation option model [8]. Samko and Kilbas [32] have done substantial research on the conjecture of incomplete integrals and derivatives. Agarwal *et al.* [1] investigate the analytical solutions for several kinds of starting and edge rate issues of incomplete differential equations, as well as

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inclusions related the Caputo conformable fractional open spaces. In some Banach spaces, many integro-differential equations may be written as FIDE and IDE [15]. Recently, the existence of solutions for fractional semilinear differential or integrodifferential equations is one of the theoretical fields investigated by Zhou.Y, Jiao.F [38, 37] and Tamizharasan et. al., [33]. Many authors have recently demonstrated the necessary condition of nonlinear fractional systems with the commands in infinite higher dimension.

On the other hand, In 1940 Ulam [34] discovered the first stability problem "less than what criteria do an preservative modeling exist close to an essentially additive map?". Next year, the first answer derived by Hyers [13] for additive functions in Banach spaces. In 1978, The generalization of Hyers result done by Rassias [30]. Cadariu and Radu [9] derived the Hyers Ulam stability using fixed point approach. Motivated by this result, S.M. Jung [14] initiated the application of these concepts in differential equation and integral equation through fixed point methods. Following this, many authors proved the constancy of differential equations, integral equation and IDE (see [2], [[24]-[26]] etc..) using fixed point approach in Banach spaces.

Recently, Trujillo et. al. [16] used a parameterization on contracting maps to show the analytical solutions to the following impulsive fractional integro-differential boundary value issue in discrete domain fields.

$$\begin{aligned} {}^cD^\rho x_2(t_1) &= g(t_1, x_2(t_1), (Sx_2)(t_1)), 0 < \rho < 1, t_1 \in J_1 = [0, T], \\ & a_1 x_2(0) + b_1 x_2(T) = c_1, \end{aligned}$$

, Where ${}^{c_1}D^\rho$ is the CFD (Caputo fractional derivative) of order ρ, the function g: $J \times X_1 \times X_1 \to X_1$ is continuous, X_1 is a Banach space (B_s) and a_1, b_1, c_1 are constants with $a_1 + b_1 \neq 0$, and S is a non-linear integral operator given by

$$(Sx_2)(t) = \int_0^{t_1} k_1(t_1, s_1) y(s_1) ds_1,$$

with $\gamma_0 = max \left\{ \int_0^t k(t_1, s_1) ds_1 : (t_1, s_1) \in J \times J \right\}$ where $k_1 \in C(J_1 \times J, R^+)$.

In this article, we extend the above work to cram the existence and uniqueness results of nonlinear FID system with impulsive boundary condition by using fractional calculus and a few constant factor approaches under a few susceptible situations. At last we established HURS of the given boundary value problem. Now consider the subsequent machine represented by the fractional integro-differential equation with control of the form,

$$\begin{gathered} {}^cD^\rho x_2(t_1) = g(t_1, x_2(t_1), (Sx_2)(t_1)), 0 < \rho < 1, t \in J = [0, T], \\ a_1 x_2(0) + b_1 x_2(T) = c_1, \\ \Delta x_2(t_i) = I_i(x_2(t_i)), i = 1, 2, ...n \end{gathered} \tag{1}$$

Let X_1be a Banach space of the function g: $J_1 \times X_1 \times X_1 \to X_1$ is continuous, X_1 is a B_s , and B is a enclosed linear operator and a_1, b_1, c_1 are constants with $a_1 + b_1 \neq 0$, and S is a nonlinear integral equation given by,

$$(Sx_2)(t_1) = \int_0^{t_1} k_1(t_1, s_1)x_2(s_1)ds_1,$$

with $\gamma_0 = max\left\{\int_0^{t_1} k_1(t_1, s_1)ds_1 : (t_1, s_1) \in J_1 \times J_1\right\}$ where $k_1 \in C_1(J_1 \times J_1, R^+)$.

To carry out the next part concept. Section 2 provides ideas and consequences from the groundwork. A sweeping interesting Gronwall kind of variation is used to demonstrate the existence and uniqueness of discoveries. In section 3, we utilize the fixed point theorem to demonstrate FIDE's existence given boundary conditions. In section 4, we establish the uniqueness findings for FIDE boundary conditions using the fixed point theorem (FPT). In this connection, several options are studied and exploited in the exhibition of uniqueness. The conclusion is based on Schaefer's FPT, but the uniqueness result is based on the Banach contraction principle. Section 5 demonstrates the FIDE's HURS using an impulsive and boundary condition.

2. Preliminaries

Thereby, we propound a few primary notations, concepts and preliminary outcomes, which are intend to employ all through the research. Let $C_1(J_1, X_1)$ contribute the B_s of all continuous functions from J_1 into X_1 of the form $\|x_2\|_\infty := sup\{\|x_2(t_1)\| : t_1 \in J_1\}$. In the function of measurable, and it is denoted $m_1 : J_1 \to R$ in the form of $\|m_1\|_{L^p(J_1,R)} = \left(\int_{J_1} |m_1(t_1)|^p dt\right)^{\frac{1}{p}}, 1 \leq p < \infty$. Let $L^p(J_1, R)$ contribute the B_s of all L_s functions with m_1 such that $\|m_1\|_{L^p(J_1,R)} < \infty$.

Definition 2.1. The Riemann-Liouville fractional integral ($RLFI$) operator in the order of $\rho > 0$ of a function g is defined as

$$I^\rho_{a_1+}g(t_1) = \int_{a_1}^{t_1} \frac{(t - s_1)^{\rho-1}}{\Gamma(\rho)} g(s_1)ds_1,$$

where $a_1 \in R$.

Here we will consider $n_1 = -[-\rho]$, where $[\cdot]$ denote the integral part of the argument, and $\rho > 0$.

Definition 2.2. For a function g_1 given on the interval $[a_1, b_1]$, the RLF of order derivative of order ρ of g, is defined by,

$$(D^\rho_{a_1+}g)(t_1) = \frac{1}{\Gamma(n_1 - \rho)} \left(\frac{d}{dt_1}\right)^{n_1} \int_{a_1}^{t} (t - s_1)^{n_1-\rho-1} g(s_1)ds_1.$$

Definition 2.3. For a function g given on the interval $[a_1, b_1]$, the Caputo fractional order derivative of order ρ of g, is defined by,

$$({}^cD^{\rho}_{a_1+}g)(t_1) = \frac{1}{\Gamma(n-\rho)}\int_{a_1}^{t_1}(t-s_1)^{n_1-\rho-1}g^{(n_1)}(s_1)ds_1.$$

Definition 2.4. A function $x(\cdot) : X_1 \to E$ is said to be strongly measurable on X_1 if there exists a sequence of simple functions $x_n(\cdot) : X_1 \to E$ such that $\lim_{n\to\infty} x_n(t_1) = x(t_1)$ for a.e. $t_1 \in X_1$.

Now, let us initiate the definition of a solution of the fractional boundary value problem (1).

Definition 2.5. A function $x_2 \in C^1(J_1, X_1)$ is said to be a solution of (1) if x_2 satisfies the equation ${}^cD^{\rho}x_2(t_1) = g(t_1, x_2(t_1), (Sx_2)(t_1))$ a.e. on J_1, and the boundary condition $a_1x_2(0) + b_1x_2(T) = c_1$, and the impulse $\Delta x_2(t_i) = I_i(x_2(t_i)), i = 1, 2, 3....n$.

For the existence of solutions for the fractional boundary value problem (1), we require the following supplementary lemma.

Lemma 2.6. *(Lemma 3.2,* [1]*) Let $x_2 \in C_1(J_1, X_1)$ be a function such that*

$$\begin{aligned} x_2(t_1) &= \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}\overline{g}(s_1)ds_1 \\ &\quad -\frac{1}{a_1+b_1}\Big[\frac{b_1}{\Gamma(\rho)}\int_0^{T}(T-s_1)^{\rho-1}\overline{f}(s_1)ds_1 - c_1\Big], \end{aligned}$$

if and only if x_2 is a solution of the following fractional Boundary value problem

$$\begin{cases} {}^cD^{\rho}x_2(t_1) = \overline{f}(t_1), \ 0 < \rho < 1, \ t \in J_1, \\ a_1x_2(0) + b_1x_2(T) = c_1. \end{cases} \tag{2}$$

As a consequence of Lemma 2.6, we have the following result which is useful in what follows.

Lemma 2.7. *Let $x_2 \in C_1(J_1, X_1)$ be defined as*

$$\begin{aligned} x_2(t_1) &= \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}[g(s_1, x_2(s_1), (Sx_2)(s_1))]ds_1 \\ &\quad -\frac{b_1}{a_1+b_1}\left(\frac{1}{\Gamma(\rho)}\int_0^{T}(T-s_1)^{\rho-1}[g(s_1, x_2(s_1), (Sx_2)(s_1))]ds_1\right) \\ &\quad +\frac{c_1}{a_1+b_1} + \frac{a_1}{a_1+b_1}\sum_{0<t_i<t_1} T(t-t_i)I_i[x_2(t_i)], \ t_1 \in J_1. \end{aligned}$$

if and only if x_2 is a solution to the problem (1).

Lemma 2.8. *(Bochner theorem,* [3]*) A measurable function g: $J_1 \to X_1$ is Bochner integrable if $\|g\|$ is Lebesgue integrable (L_I).*

Lemma 2.9. *(Arzela-Ascoli theorem, [28]) Let X_1 be a B_s and $F \subset C_1(J_1, X_1)$. If the requirements listed below are met:*

(i) *F is uniformly bounded subset of $C_1(J_1, X_1)$*
(ii) *F is equi-continuous in $(t_i, t_{i+1}), i = 0, 1, 2, \cdots m$, where $t_0 = 0$, $t_{m+1} = T$*

Then F is a relatively compact subset of $C_1(J_1, X_1)$.

Theorem 2.10. *(Schaefer's fixed point theorem, [28]) Let $G : X_1 \to X_1$ completely continuous operator and it is denoted by,*

$$E(G) = \{x \in X_1 : x = \lambda G x_1 \ for \ some \ \lambda \in [0, 1]\}$$

is bounded, then G has fixed points.

On the other hand to apply theorem (2.10) We need the following generalized singular Gronwall type inequality with mixed type singular integral operator to establish the existence of solutions.

Lemma 2.11. *([16]) Let us consider $x_2 \in C_1(J_1, X_1)$ satisfy the below condition,*

$$\begin{aligned} \|x_2(t_1)\| \leq a_1 + b_1 \int_0^{t_1} (t_1 - s_1)^{\rho-1} \|x_2(s_1)\|^{\lambda} ds_1 \\ + c_1 \int_0^{t_1} (T - s_1)^{\rho-1} \|x_2(s_1)\|^{\lambda} ds_1, \end{aligned} \tag{3}$$

where $\rho \in (0, 1)$,$\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\rho}$, and $a_1, b_1, c_1 \geq 0$. Then it exists $M^* > 0$ such that $\|x_2(t)\| \leq M^*$.

Our outcome of this theorem (4.1) is based on Banach contraction principle. Let $M = \|\xi_1 + \phi_0 \xi_2\|_{L^{\frac{1}{\beta_1}}(J_1, X_1)}$, $H = \|\alpha\|_{L^{\frac{1}{\beta_2}}(J_1, X_1)}$.

Theorem 2.12. *([11]) Assume that (X_1, s_1) is a generalized complete metric space. Assume that $\Theta : X_1 \to X_1$ is a Lipschitz constant strictly contractive operator $Q < 1$, yet there occurs a non - negative real integer C such that $s_1(\Theta^{c+1} x, \Theta^{c_1} x) < \infty$ for some $x_1 \in X_1$, then the following statement are correct:*

(a) The sequence $\{\Theta^n x\}$ reaches a defined terminal point x^* of Θ
(b) x^* is the one and only fixed point of Θ in $Y^* = \{y \in X_1 | s_1(\Theta^c x, y) < \infty\}$;
(c) If $X_1 \in X^*$, then $d(X_2, x^*) \leq \frac{1}{1-P} d(\Theta X_2, X_1)$.

3. Existence results

Theorem 3.1. *Let us assume that (H4)-(H6) hold. Then the impulsive fractional boundary value problem (1) has at least one solution on J_1.*

Proof. Convert the (1) into a fixed point problem. Let us Consider the operator $G : C_1(J_1, X_1) \to C_1(J_1, X_1)$ defined as equation (5). It is clear that G is well defined due to (H5).

Step 1. To prove G is continuous.

Let $\{x_n\}$ be a convergent sequence in $C_1(J_1, X_1)$. By LCT (Lebesgue's convergence theorem), $Sx_n \to Sx_2$ as $x_n \to x_2$. For all $t_1 \in J_1$, we have

$$\begin{aligned}
& |(Gx_n)(t_1) - (Gx_2)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\rho)} \int_0^{t_1} ||t_1 - s_1||^{\rho-1} \\
& ||g(s_1, x_n(s_1), (Sx_{n_1})(s_1)) - g_1(s, x_2(s_1), (Sx_2)(s_1))||ds_1 \\
& + \frac{b_1}{a_1 + b_1\Gamma(\rho)} \int_0^{T} ||T - s_1||^{\rho-1} \\
& ||g(s_1, x_n(s_1), (Sx_{n_1})(s_1)) - g(s_1, x_2(s_1), (Sx_2)(s_1))||ds_1 \\
& + \frac{a_1}{a_1 + b_1} \sum_{i=1}^{n} ||T(t_1 - t_i)|| ||I_i[x_n(t_i)] - I_i[x_2(t_i)]|| \\
\leq\ & \left[\frac{1}{\Gamma(\rho)} \frac{MT^{\rho-\beta_1}}{\left(\frac{\rho-\beta_1}{1-\rho_1}\right)^1 - \beta_1} \left[1 + \frac{b_1}{a_1 + b_1}\right] + \frac{aM_1}{a_1 + b_1} \sum_{i=1}^{n} L_{I_i} \right] \|x_n - x_2\| \\
\leq\ & \Omega \|x_n - x_2\|
\end{aligned}$$

Since G is continuous. We get $\|Gx_n - Gx_2\|_\infty \to 0$ as $n \to \infty$.

Step 2. G maps bounded sets into bounded sets in $C_1(J_1, X_1)$.

In fact, it is shows that for every $\eta > 0$,then there exists a $\delta > 0$, such that for each $x_2 \in B_\eta = \{x_2 \in C_1(J_1, X_1) : \|x_2\|_\infty \leq \eta\}$, we have $\|Gx_2\|_\infty \leq \delta$.

For all $t_1 \in J_1$, we get

$$\begin{aligned}
||(Gx_2)(t_1)|| \leq\ & \frac{N}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s_1)^{\rho-1}(1 + \gamma_0\|x_2(s_1)\|^\lambda)ds_1 \\
& + \frac{b_1N}{a_1 + b_1\Gamma(\rho)} \int_0^{T} (T - s_1)^{\rho-1}(1 + \gamma_0\|x_2(s_1)\|^\lambda)ds_1 \\
& + \frac{a_1M_1}{a_1 + b_1} \sum_{i=1}^{n} \|I_i[x_2(t_i)]\| + \frac{c_1}{a_1 + b_1} \\
\leq\ & \frac{NT^\rho}{\Gamma(\rho + 1)} \left(1 + \frac{b_1}{a_1 + b_1}\right) + \frac{N\gamma_0T^\rho(\eta)^\lambda}{\Gamma(\rho + 1)} \left(1 + \frac{b_1}{a_1 + b_1}\right) \\
& + \frac{a_1M_1}{a_1 + b_1} \sum_{i=1}^{n} \|I_i[x_2(t_i)]\| + \frac{c_1}{a_1 + b_1}
\end{aligned}$$

which is given that $\|Gx_2\|_\infty \leq l$.

Step 3. G maps bounded sets into equi-continuous sets of $C_1(J_1, X_1)$.

Let $0 \leq t_2 \leq t_3 \leq T, x_2 \in B_\eta$. Using (H6), we have

$$\begin{aligned}
|(Gx_2)(t_3) - (Gx_2)(t_2)| \leq\ & \frac{1}{\Gamma(\rho)} \int_0^{t_2} [(t_2 - s_1)^{\rho-1} - (t_3 - s_1)^{\rho-1}] \\
& |g(s_1, x_2(s_1), (Sx_2)(s_1))|ds_1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\rho)}\int_{t_2}^{t_3}(t_3-s_1)^{\rho-1}|g(s_1,x_2(s_1),(Sx_2)(s_1))|ds_1\\
&+\frac{a_1}{a_1+b_1}\sum_{i=1}^{n}\|T(t_3-t_2)-T(t_2-t_i)\|I_i[x_2(t_i)]\\
\leq\ &\frac{N[1+\gamma_0(\eta)^\lambda]}{\Gamma(\rho+1)}(t_2^\rho-t_3^\rho)+\frac{(t_3-t_2)^\rho}{\Gamma(\rho+1)}
\end{aligned}
$$

Therefore, $||(Gx_2)(t_3)-(Gx_2)(t_2)||\to 0$ as $t_3\to t_2$. Thus, G is equicontinuous. By the result of Steps 1-3 together with the lemma(2.9) , we can conclude that G is completely continuous.

Step 4. Now it leftovers to prove that

$$E(G)=x_2\in C_1(J_1,X_1): x_2=\lambda Gx_2,\ for\ some\ \lambda\in(0,1)$$

is bounded.

Let $x_2\in E(G)$, then $x_2=\lambda Gx_2$ for some $\lambda\in(0,1)$. Therefore, for all $t_1\in J_1$, we have

$$
\begin{aligned}
x_2(t_1) &= \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}[g(s_1,x_2(s_1),(Sx_2)(s_1))]ds_1\\
&\quad-\frac{b_1}{(a_1+b_1)\Gamma(\rho)}\int_0^{T}(T-s)^{\rho-1}[g(s_1,x_2(s_1),(Sx_2)(s_1))]ds_1\\
&\quad+\frac{c_1}{a_1+b_1}+\frac{a_1}{a_1+b_1}\sum_{0<t_i<t_1}T(t_1-t_i)I_i||x_L(t_i)||
\end{aligned}
$$

For all $t_1\in J_1$, we have

$$
\begin{aligned}
||x_2(t_1)|| &\leq \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}N\left[1+\gamma_0\|x_2(s_1)\|^\lambda\right]ds_1\\
&\quad+\frac{b_1}{a_1+b_1}\frac{1}{\Gamma(\rho)}\int_0^{T}(T-s_1)^{\rho-1}N\left[1+\gamma_0\|x_2(s_1)\|^\lambda\right]ds_1\\
&\quad+\frac{c_1}{a_1+b_1}+\frac{aM_1}{a_1+b_1}\sum_{i=1}^{n}\|I_i[x_2(t_i)]\|\\
&\leq M*
\end{aligned}
$$

From lemma (2.11), to exists a $M^*>0$, such that

$$\|x_2(t_1)\|\leq M^*, t_1\in J_1$$

Thus for all $t_1\in J_1$, we have

$$\|x_2\|_\infty\leq M^*.$$

It is give the set $E(G)$ is bounded. By the result of theorem (2.10), we conclude that g_1 has a fixed point. Which is a solution of (1). □

4. Uniqueness results

It will provide the below possibilities for our consideration.

(H1) Let $g : J_1 \times X_1 \times X_1 \to X_1$ is measurable function.

(H2) Let the function $g : J_1 \times X_1 \times X_1 \to X_1$ is continuous and get a constant $\beta_1 \in (0, \rho)$ and the function is denoted by $\xi_1(t_1), \xi_2(t_1) \in L^{\frac{1}{\beta_1}}(J_1, X_1)$ such that
$\|g(t_1, x(t_1), (Sx)(t_1)) - g(t_1, y(t_1), (Sy)(t_1))\| \leq \xi_1(t_1)\|x-y\| + \xi_2(t_1)\|S_x - S_y\|$, for each $t_1 \in J_1$ and all $x, y \in X_1$.

(H3) The another constant $\beta_2 \in (0, \rho)$ and the function is $\alpha(t_1) \in L^{\frac{1}{\beta_2}}(J_1, X_1)$. such that

$$\|g(t_1, x_2, Sx_2)\| \leq \alpha(t_1),$$

for each $t_1 \in J_1$ and all $x_2 \in X_1$.

(H4) The constant $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\rho}$ and $N > 0$. such that,

$$\|g(t_1, u, Su)\| \leq N(1 + \gamma_0 \|u\|^{\lambda})$$

for all $t_1 \in J_1$ and all $u \in X_1$.

(H5) For each of $t_1 \in J_1$, the set

$$K_1 = \left\{(t_1 - s_1)^{\rho-1} g(s_1, x_2(s_1), (Sx_2)(s_1)) : x_2 \in C_1(J_1, X_1), s_1 \in [0, t_1])\right\}$$

is relatively compact.

(H6) Let, $\Omega = \frac{MT^{\rho-\beta_1}}{\Gamma(\rho)(\frac{\rho-\beta_1}{1-\beta_1})^{1-\beta_1}} \left(1 + \frac{|b_1|}{|a_1+b_1|}\right) + \frac{|a_1|M_1}{|a_1+b_1|} \sum_{i=1}^{n} L_i < 1$
ie., $\Omega < 1$

In this section, we use the well-known BFPT and contraction property to show the uniqueness of the problem (1).

Theorem 4.1. *Assume that (H1)-(H6)hold. If*

$$\Omega = \frac{MT^{\rho-\beta_1}}{\Gamma(\rho)(\frac{\rho-\beta_1}{1-\beta_1})^{1-\beta_1}} \left(1 + \frac{|b_1|}{|a_1+b_1|}\right) + \frac{|a_1|M_1}{|a_1+b_1|} \sum_{i=1}^{n} L_i < 1 \qquad (4)$$

In equation(1) to get the unique solution on J_1.

Proof. Since we get, $\|(t_1 - s_1)^{\rho-1} g(s_1, x_2(s_1), (Sx_2)(s_1))\|$ is L_I with respect to $s_1 \in [0, t_1]$ for all $t_1 \in J_1$ and $x_1 \in C_1(J_1, X_1)$. Then $(t_1 - s_1)^{\rho-1} g(s_1, x_2(s_1), Sx_2(s_1))$ is Bochner integrable with respect to $s_1 \in [0, t_1]$ for all $t_1 \in J_1$ due to Lemma (2.8). Hence, the IFBVP (Impulsive fractional boundary value problem) (1) is below equation,

$$\begin{aligned} x_2(t_1) &= \frac{1}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s_1)^{\rho-1} [g_1(s_1, x_2(s_1), (Sx_2)(s_1))] ds_1 \\ &\quad - \frac{b_1}{a_1 + b_1} \left(\frac{1}{\Gamma(\rho)} \int_0^{T} (T - s_1)^{\rho-1} [g_1(s_1, x_2(s_1), (Sx_2)(s_1))] ds_1 \right) \end{aligned}$$

$$+\frac{c}{a_1+b_1}+\frac{a_1}{a_1+b_1}\sum_{0<t_i<t} T(t_1-t_i)I_i[x_2(t_i)],\ t\in J.$$

Let

$$r \geq \frac{T^{\rho-\beta_2}H}{\Gamma(\rho)(\frac{\rho-\beta_2}{1-\beta_2})^{1-\beta_2}}\left(1+\frac{|b_1|}{|a_1+b_1|}\right)+\frac{|a_1|M_1}{|a_1+b_1|} \sum_{i=1}^{n}|I_i[x_2(t_i)]|+\frac{|c_1|}{|a_1+b_1|}.$$

Let us consider the operator G on $B_r := \{x_2 \in C_1(J_1, X_1) : \|x_2\| \leq r\}$ as follows

$$\begin{aligned}(Gx_2)(t_1) = {} & \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}g(s_1,x_2(s_1),(Sx_2)(s_1))ds_1 \\ & -\frac{b_1}{a_1+b_1}\frac{1}{\Gamma(\rho)}\int_0^{T}(T-s_1)^{\rho-1}g(s_1,x_2(s_1),(Sx_2)(s_1))ds_1 \\ & +\frac{a_1}{a_1+b_1}\sum_{0<t_i<t_1}T(t_1-t_i)I_i[x_2(t_i)]+\frac{c_1}{a_1+b_1},\ t_1\in J_1.\end{aligned} \tag{5}$$

The solution is exists of (1) is same to the operator g has a fixed point on B_r. We using the Banach contraction principle to demonstrate that g has a fixed point.

Step 1. For all $x_2 \in B_r$. For all $x_2 \in B_r$, by (H3) and Hölder inequality, we get

$$\begin{aligned}\|(Gx_2)(t_1)\| \leq {} & \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}||g(s_1,x_2(s_1),(Sx_2)(s_1))||ds_1 \\ & +\frac{b_1}{(a_1+b_1)\Gamma(\rho)}\int_0^{T}(T-s_1)^{\rho-1}||g(s_1,x_2(s_1),(Sx_2)(s_1))||ds_1 \\ & +\frac{a_1}{(a_1+b_1)}\sum_{i=1}^{n}||T(t_1-t_i)|||I_i[x_2(t_i)]|+\frac{c_1}{a_1+b_1} \\ \leq {} & \frac{1}{\Gamma(\rho)}\frac{T^{\rho-\beta_2}H}{(\frac{\rho-\beta_2}{1-\beta_2})^{1-\beta_2}}+\frac{b_1}{a_1+b_1}\frac{1}{\Gamma(\rho)}\frac{T^{\rho-\beta_2}}{(\frac{\rho-\beta_2}{1-\beta_2})^{1-\beta_2}}H \\ & +\frac{aM_1}{a_1+b_1}\sum_{i=1}^{n}|I_i[x_2(t_i)]|+\frac{c_1}{a_1+b_1} \\ \leq {} & r.\end{aligned}$$

which implies that $\|Gx_2\|_\infty \leq r$. Thus, $Gx_2 \in B_r$, for all $x_2 \in B_r$. i.e., $G : B_r \to B_r$ is well defined.

Step 2. G is a contraction mapping on B_r. For $x_1, x_2 \in B_r$ and any $t_1 \in J_1$, using (H2) and Hölder inequality, we get

$$\begin{aligned}& ||(Gx_1)(t_1)-(Gx_2)(t_1)|| \\ \leq {} & \frac{1}{\Gamma(\rho)}\int_0^{t_1}||t_1-s_1|^{\rho-1}||g(s_1,x_1(s_1),(Sx_1)(s_1))\end{aligned}$$

$$
\begin{aligned}
&-g(s_1, x_2(s_1), (Sx_2)(s_1))||ds_1\\
&+\frac{b_1}{a_1+b_1\Gamma(\rho)}\int_0^T ||T-s_1||^{\rho-1}||g(s_1, x_1(s_1), (Sx_1)(s_1))\\
&-g(s_1, x_2(s_1), (Sx_2)(s_1))||ds_1\\
&+\frac{a_1}{a_1+b_1}\sum_{0<t_i<t}||T(t_1-t_i)|||I_i[x_1(t_i)]-I_i[x_2(t_i)]||\\
\leq\ & \left[\frac{1}{\Gamma(\rho)}\frac{MT^{\rho-\beta_1}}{\left(\frac{\rho-\beta_1}{1-\beta_1}\right)^1-\rho_1}\left[1+\frac{b_1}{a_1+b_1}\right]+\frac{a_1M_1}{a_1+b_1}\sum_{i=1}^{n}L_{I_i}\right]\|x_1-x_2\|\\
\leq\ & \Omega\|x_1-x_2\|
\end{aligned}
$$

Since $C_1(J_1, B_r)$ is continuous, we have $|x_1(0)-x_2(0)| \to 0$ as $n \to \infty$.So we get,

$$|(Gx_1)(t_1)-(Gx_2)(t_1)| \leq \Omega(t_1)\|x_1-x_2\|_\infty.$$

Thus G is a contraction due to equation (3).

By Banach contraction principle, the G has an unique fixed point. □

5. Hyers-Ulam-Rassias Stability (HURS)

Theorem 5.1. *Set* $\Omega_1 := \left[(L_1+L_1L_2)\left(1+\frac{b_1}{a_1+b_1}\right)+\frac{a_1}{a_1+b_1}\sum_{i=1}^{n}ML_{I_i}\right] < 1$. *Suppose that* $g_1 : I\times I\times\mathbb{R}\to\mathbb{R}$ *is a continuous function which satisfies a Lipschitz condition*

$$|g_1(t_1, x_1, \bar{x_1})-g_1(t_1, x_2, \bar{x_2})| \leq L_1\left[|x_1-x_2|+|\bar{x_1}-\bar{x_2}|\right] \quad \forall t_1\in I, \tag{6}$$

$x_1, x_2, \bar{x_1}, \bar{x_2}\in\mathbb{R}$ *and* $k_1 : I\times I\times\mathbb{R}_{\not k}\to\mathbb{R}_{\not k}$ *is a continuous and it satisfies a Lipschitz condition*

$$|k_1(t_1, s_1, u)-k_1(t_1, s_1, v)| \leq L_2\left[|u-v|\right] \quad \forall t_1, s_1\in I, \quad \forall u, v\in\mathbb{R} \tag{7}$$

Which gives the function $x_2 : I\to\mathbb{R}_{\not k}$ *to satisfies,*

$$|{}^cD^\rho x_2(t_1)-g_1(t_1, x_2(t_1), (Sx_2)t_1)| \leq \varphi(t) \tag{8}$$

for all $t_1\in I$*, where* $\varphi : I\to(0,\infty)$ *is a continuous function with*

$$\left|\frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}\varphi(s_1)ds\right| \leq K_1\varphi(t) \tag{9}$$

for all $t_1\in I$*, then there exists a function* $x_2 : I\to\mathbb{R}_{\not k}$*. such that*

$$
\begin{aligned}
x_0(t_1) &= \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}g_1(s_1, x_0(s_1), (Sx_0)(s_1))ds_1\\
&-\frac{b_1}{(a_1+b_1)\Gamma(\rho)}\int_0^T(T-s_1)^{\rho-1}g_1(s_1, x_0(s_1), (Sx_0)(s_1))ds_1\\
&+\frac{c_1}{a_1+b_1}+\frac{a_1}{a_1+b_1}\sum_{0<t_i<t}T(t_1-t_i)I_i[y(t_i)]
\end{aligned}
$$

and

$$\left|x_(t_1)-x_0(t)\right| \leq \frac{K_1}{1-\Omega_1}\varphi(t) \quad \forall t_1 \in I \tag{10}$$

Proof. Let X_1 indicate the set of all continuous functions on I. We set a generalized complete metric (see [14]) on X_1 as follows

$$d(u,v) = \inf\{C \in [0,\infty] |\, |u(t_1)-v(t_1)| \leq C\varphi(t_1) \quad \forall t_1 \in I\} \tag{11}$$

Define an operator $\Lambda : X_1 \to X_1$ by

$$\begin{aligned}(\Lambda u)(t_1) = & \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}g_1(s_1,u(s_1),(Su)(s))ds \\ & -\frac{b_1}{(a_1+b_1)\Gamma(\rho)}\int_0^T(T-s)^{\rho-1}g_1(s,u(s),(Su)(s_1))ds_1 \\ & +\frac{c_1}{a_1+b_1}+\frac{a_1}{a_1+b_1}\sum_{0<t_i<t_1}T(t_1-t_i)I_i[y(t_i)]\end{aligned} \tag{12}$$

for all $t_1 \in I$ and $f \in X_1$.
Now we check that Λ is strictly contractive on X_1.
For any $u, v \in X_1$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(u,v) \leq C_{uv}$, that is, by (11) we have

$$|u(t_1)-v(t_1)| \leq C_{uv}\varphi(t_1) \tag{13}$$

for any $t_1 \in I$.
It then follows from (6), (7), (9), (12) and (13) that

$$\begin{aligned}& |(\Lambda u)t_1-(\Lambda v)t_1| \\ \leq\ & \frac{1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}\left|g(s_1,u(s_1),(Su)(s_1))-G(s_1,v(s_1),(Sv)(s_1))\right|ds_1 \\ & +\frac{b_1}{(a_1+b_1)\Gamma(\rho)}\int_0^T(T-s_1)^{\rho-1} \\ & \left|g(s_1,u(s_1),(Su)(s_1))-G(s_1,v(s_1),(Sv)(s_1))\right|ds_1 \\ & +\frac{a_1}{a_1+b_1}\sum_{i=1}^{n}|T(t_1-t_i)||I_i[g(t_i)]-I_i[g(t_i)]| \\ \leq\ & \frac{L_1}{\Gamma(\rho)}\int_0^{t_1}(t_1-s_1)^{\rho-1}\left[|u(s_1)-v(s_1)|+|(Su)(s_1)-(Sv)(s_1)|\right]ds_1 \\ & +\frac{b_1L_1}{(a_1+b_1)\Gamma(\rho)}\int_0^T(T-s_1)^{\rho-1}\left[|u(s_1)-v(s_1)|+|(Su)(s_1)-(Sv)(s)|\right]ds \\ & +\frac{a_1}{a_1+b_1}\sum_{i=1}^{n_1}ML_{I_i}|u(t_1)-v(t_1)| \\ \leq\ & \left[K_1(L_1+L_1L_2)\left(1+\frac{a_1}{a_1+b_1}\right)+\frac{a_1}{a_1+b_1}\sum_{i=1}^{n}M_1L_{I_i}\right]C_{uv}\varphi(t_1)\end{aligned}$$

$$\leq \quad \Omega_1 C_{uv}\varphi(t)$$

for all $t_1 \in I$. That is

$$d(\Lambda u, \Lambda v) \leq \Omega_1 C_{uv}\varphi(t_1).$$

Hence we can conclude that

$$d(\Lambda u, \Lambda v) \leq \Omega_1 d(u,v)$$

for any $u, v \in X_1$. It follows from (12) that for an arbitrary $v_0 \in X_1$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}
& |(\Lambda v_0)(t_1) - v_0(t_1)| \\
= \; & \Bigg| \frac{1}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s_1)^{\rho-1} g_1(s_1, v_0(s_1), (Sv_0)(s_1)) ds_1 \\
& + \frac{a_1}{a_1+b_1} \sum_{0<t_i<t} T(t_1 - t_i) I_i[x_2(t_i)] \\
& - \frac{b_1}{(a_1+b_1)\Gamma(\rho)} \int_0^T (T - s_1)^{\rho-1} g_1(s_1, v_0(s_1), (Sv_0)(s_1)) ds_1 \\
& + \frac{c_1}{a_1+b_1} - v_0(t_1) \Bigg| \\
\leq \; & C\varphi(t)
\end{aligned}$$

for all $t_1 \in I$, since $g_1(t_1, v_0(t), (Sv_0)(t))$ and $v_0(t)$ are enclosed on I and $\min\limits_{t_1 \in I} \varphi(t) > 0$.
Thus (11) gives that

$$d(\Lambda g_0, g_0) < \infty$$

Since, the according to theorem (2.2), there exists a continuous function $x_0 : I \to \mathbb{R}_{\not\Bbbk}$ such that the sequence $\{\Lambda^n v_0\}$ converges to x_0 in (X_1, d) and $\Lambda x_0 = x_0$, that is, x_0 is a solution of (1.1) for all $t \in I$.
We want to verify that

$$\{v \in X_1 | d(v_0, v) < \infty\} = X_1$$

Since v and v_0 both of them are enclosed on I, for any $v \in X_1$, and $\min\limits_{t_1 \in I} \varphi(t_1) > 0$, there exists a constant $0 < C_v < \infty$. such that

$$|v_0(t_1) - v(t_1)| \leq C_v$$

So, we get $d(v_0, v) < \infty$ for all $v \in X_1$. That is $\{g \in X_1 | d(v_0, v) < \infty\} = X_1$.
The theorem (2.2), gives the x_0 is the unique continuous function with the property (10).
From the equation (8) we show that,

$$-\varphi(t_1) \leq^c D^\rho_{a+} x_2(t_1) - g_1(t_1, x_2(t), (Sx_2)(t_1)) \leq \varphi(t_1) \tag{14}$$

for all $t_1 \in I$.
We obtain by integrating each component in the preceding inequality and substituting the boundary conditions.

$$\begin{aligned} & \Big| x_2(t_1) - \frac{1}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s_1)^{\rho-1} g(s_1, x_2(s_1), (Sx_2)(s_1)) ds_1 \\ & + \frac{b_1}{(a_1+b_1)\Gamma(\rho)} \int_0^{T} (T - s_1)^{\rho-1} g(s_1, x_2(s_1), (Sx_2)(s_1)) ds_1 \\ & - \frac{c_1}{a_1+b_1} - \frac{a_1}{a_1+b_1} \sum_{0<t_i<t} T(t_1 - t_i) I_i[x_2(t_i)] \Big| \\ \leq \; & \frac{1}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s_1)^{\rho-1} \varphi(s_1) ds_1 \end{aligned}$$

for any $t_1 \in I$.
Thus, by (9) and (12), we get

$$|x_2(t_1) - (\Lambda x_2)(t_1)| \leq K_1 \varphi(t_1)$$

for each $t_1 \in I$, which implies that

$$d(x_2, \Lambda x_2) \leq K_1 \varphi(t_1). \tag{15}$$

By using theorem (2.2), and (15), we prove that,

$$d(x_2, x_0) \leq \frac{1}{1-\Omega_1} d(x_2, \Lambda x_2) \leq \frac{K_1}{1-\Omega_1} \varphi(t) \tag{16}$$

Accordingly, this yields the inequality (10) for all $t_1 \in I$. □

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