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APPLICATIONS OF SIMILARITY MEASURES FOR PYTHAGOREAN FUZZY SETS BASED ON SINE FUNCTION IN DECISION-MAKING PROBLEMS

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Abstract. Pythagorean fuzzy sets (PFSs) are capable of modelling information with more uncertainties in decision-making problems. The essential feature of PFSs is that they are described by three parameters: membership function, non-membership function and hesitant margin, with the total of the squares of each parameter equal to one. The purpose of this article is to suggest some new similarity measures and weighted similarity measures for PFSs. Numerical computations have been carried out to validate our proposed measures. Applications of these measures have been applied to some real-life decision-making problems of pattern detection and medicinal investigations. Moreover, a descriptive illustration is employed to compare the results of the proposed measures with the existing analogous similarity measures to show their effectiveness.

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1. Introduction

Decision making is considered as a process of perception that is used to solve problems we face in daily life. Because of the complexities of current socio-economic environment, decision making is one of the most important idea aimed at achieving the best possible solution, or at least achieving satisfaction by detecting and selecting alternatives. Decision making is complicated when there is uncertainty associated with it. Despite dealing with ambiguity, Zadeh's [49] fuzzy set is a good instrument, but over time it was found to be inadequate. To overcome the problem that has arisen, the intuitionistic fuzzy sets (IFSs) suggested by Atanassov [1, 2] has demonstrated its effectiveness. To deal with the

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information provided without loss and ultimately to get a better option various operator are utilized. Atanassov [1, 2] studied a non-membership degree ζ of an element along with membership degree δ, with hesitation margin η such that they satisfy the linear inequality $\delta+\zeta \leq 1$ and $\delta+\zeta+\eta = 1$. Unlike the scenario captured in IFSs, there exists possibility when $\delta + \zeta \geq 1$. This inadequacy in IFSs surely preceded to a configuration, called PFSs. PFSs proposed by Yager [41, 42] is a novel tool to deal with vagueness considering membership degree δ and non-membership ζ satisfying the requirements $\delta + \zeta \leq 1$ or $\delta + \zeta \geq 1$, and hence it sees that $\delta^2 + \zeta^2 + \eta^2 = 1$, where η is the PFS index.

Various researchers have presumably exploited Yager's [43] Pythagorean fuzzy sets concept and applied it to dynamic, clinical discovery, design acknowledgment, and a variety of other plausible issues. Zhang and Xu [54] presented a similarity methodology based on a scoring capacity to organise the Pythagorean fuzzy positive ideal arrangement (PIS) and the Pythagorean fuzzy negative ideal arrangement (PFNA) to cope with the dynamic issue with PFSs (NIS). They used the order preference by similarity to ideal solution methodology to determine the distances between each option using PIS and NIS, separately. Peng and Yang [24] suggested essential tasks for PFSs and furnished fuzzy aggregation operators alongside their significant possessions. Additionally, they fostered a Pythagorean prevalence and mediocrity positioning calculation over tackle collective choice making issues considering vulnerability. Strategy for dynamic issues with the assistance of accumulation administrators and distance measures has been created by Zeng *et al.* [52]. They proposed PFSs weighted averaging distance operator and fostered TOPSIS strategy. Further, Yager [43] presented a portion of the fundamental set tasks for PFSs and set up the connection between Pythagorean membership grade and complex grade. Likewise, the arrangements of multicriteria dynamic with fulfillment through Pythagorean membership grade have been done.

The similarity measure is a significant exploration matter in the FSs and can be utilized to decide the comparability degree between two items. Measures of similarity between fuzzy sets attracted the attention of researchers to them extensive applications in areas such as machine learning, pattern recognition, decision making and in image processing are proposed and studied in the current years (Bustince *et al.* [3, 4]; Lee *et al.* [14]). Szmidt and Kacprzyk [32, 33] nurtured a similitude measure between IFSs dependent on the Hamming distance and Hung and Yang [11] decided the distance between IFSs dependent on the Hausdorff distance and produced some likeness gauges between IFSs. Li and Zeng [17] recommended some distance measures for PFSs, which ponder four boundaries and seen that the four boundaries are not the traditional highlights of PFSs. However, Taruna *et al.* [34] offered trigonometric divergence measure between fuzzy sets A and B. Li and Cheng [15] suggested a proper comparability measure among IFSs and employed it to design acknowledgment issues. Peng [26], Li *et al.* [16] and explored some new comparability measure and another uniqueness measure for PFSs by fusing four boundaries 3 or more customary parts of PFSs. The thoughts of closeness and difference of PFSs as augmentation of the task were presented by Zeng *et al.* [51] by combining boundaries and employed to multi-standards dynamic issues. Shi and Ye [29] and Ye [44, 45] proposed cosine similarity measures for ambiguous sets and IFS. Additionally, Tian [35] and Rajarajeswari and Uma [27] characterized cotangent similarity proportion of IFSs. The similarity proportion of the IFSs and PFSs are extensively exploited in various fields, similar to the pattern recognition (Hwang *et al.* [12]; Peng and Garg [23]; Song *et al.* [31]; Gong [9]; Zhang *et al.* [55]; Li and Cheng [15]; Peng *et al.* [25]), the clinical finding (Muthukumar and Krishnan [21]; Son and Phong [30]; Wei *et al.* [36]; Hung and Wang [10]; Maoying [19]), decision-making (Ye [46, 47]; Chen *et al.* [5]; Liang and Xu [18]; Xu [38, 39]; Zhang [52]; Zhang *et al.* [53]). Some formulae of Pythagorean fuzzy information measures (distance measure, similarity measure, entropy, inclusion measure) and corresponding transformation relationships were developed (Peng *et al.* [25]). Some similarity measures between PFSs dependent on the cosine work were intended (Wei and Wei [37]) by applied to design identification and clinical analysis. A similarity metric for PFSs based on a combination of cosine likeness and Euclidean distance was proposed, emphasising association and non-association degrees (Mohd and Abdullah [20]). Ejegwa [7] combined the 3 traditional boundaries of PFSs with more sensible, dependable, and effective yield, are irrefutable. Cotangent comparability for rough IFSs was proposed (Immaculate *et al.* [13]) and a clinical conclusion was additionally given to confirm the proposed similarity measure. Ejegwa [8] proposed novel similarity measures for PFSs which were used to dynamic issues.

The following is a breakup of the paper's structure: The definitions of the FS, IFS, and PFS are introduced in Section 2. The PFSs are compared using two similarity measures and two weighted similarity measures in Section 3. Our measurements have been validated using numerical computations. Section 4 deals with pattern recognition, medical diagnosis, and MADM using similarity measures and weighted similarity measures. With the help of an example, Section 5 compares the new similarity measures with the existing measures. The article is summarized in Section 6.

2. Preliminaries

We introduce some basic notions about FSs, IFSs, and PFSs in this part, which are employed in the outcome.

Definition 2.1 (Zadeh [49])**.** Let X be a nonempty set. A fuzzy set $\widetilde{A}$ in $X = \{x_1, x_2, \ldots, x_n\}$ is characterized by a membership function:

$$\widetilde{A} = \{\langle x, \delta_{\widetilde{A}}(x)\rangle \mid x \in X\} \tag{1}$$

where $\delta_{\widetilde{A}}(x) : X \to [0, 1]$ is a measure of belongingness of degree of membership of an element $x \in X$ in $\widetilde{A}$.

Definition 2.2 (Atanassov [1])**.** An IFS $\widetilde{A}$ in X is given by

$$\widetilde{A} = \{\langle x, \delta_{\widetilde{A}}(x), \zeta_{\widetilde{A}}(x)\rangle \mid x \in X\} \tag{2}$$

where $\delta_{\widetilde{A}}(x) : X \to [0,1]$ and $\zeta_{\widetilde{A}}(x) : X \to [0,1]$, where $0 \le \delta_{\widetilde{A}}(x) + \zeta_{\widetilde{A}}(x) \le 1$, $\forall\ x \in X$. The number $\delta_{\widetilde{A}}(x)$ and $\zeta_{\widetilde{A}}(x)$ represents, respectively, the MD and NMD of the element x to the set $\widetilde{A}$.
For each IFS $\widetilde{A}$ in X, if

$$\eta_{\widetilde{A}}(x) = 1 - \delta_{\widetilde{A}}(x) - \zeta_{\widetilde{A}}(x), \quad \forall\ x \in X. \tag{3}$$

Then $\eta_{\widetilde{A}}(x)$ is called the degree of indeterminacy of x to $\widetilde{A}$. For convenience, Xu [37] denoted this pair as $\widetilde{A} = (\delta_{\widetilde{A}}, \zeta_{\widetilde{A}})$.

Further, Dutta and Goala [6] proposed the novel distance measure between IFSs P and Q as following:

$$Dis^1(P,Q) = \frac{2}{n}\sum_{i=1}^{n} \frac{\sin\left\{\frac{\pi}{6}|\delta_P(x_i) - \delta_Q(x_i)|\right\} + \sin\left\{\frac{\pi}{6}|\zeta_P(x_i) - \zeta_Q(x_i)|\right\}}{1 + \sin\left\{\frac{\pi}{6}|\delta_P(x_i) - \delta_Q(x_i)|\right\} + \sin\left\{\frac{\pi}{6}|\zeta_P(x_i) - \zeta_Q(x_i)|\right\}}\,. \tag{4}$$

Sharma and Tripathi [28] proposed measures of sine intuitionistic fuzzy distance for P and Q as following:

$$\begin{aligned} &Dis^2(P,Q) \\ &= \frac{1}{2n}\sum_{i=1}^{n}\left[\left(\sin\left(\frac{|\delta_P(x_i) - \delta_Q(x_i)|}{2}\right)\pi\right) + \left(\sin\left(\frac{|\zeta_P(x_i) - \zeta_Q(x_i)|}{2}\right)\pi\right)\right] \end{aligned} \tag{5}$$

$$\begin{aligned} &Dis^3(P,Q) \\ &= \frac{1}{2n}\sum_{i=1}^{n}\left[\left(\sin\left(\frac{|\sqrt{\delta_P(x_i)}| - \sqrt{\delta_Q(x_i)}}{2}\right)\pi\right) + \left(\sin\left(\frac{|\sqrt{\zeta_P(x_i)} - \sqrt{\zeta_Q(x_i)}|}{2}\right)\pi\right)\right] \end{aligned} \tag{6}$$

Definition 2.3 (Yager [40])**.** An IFS $\widetilde{A}$ in X is given by

$$\widetilde{A} = \{\langle x, \delta_{\widetilde{A}}(x), \zeta_{\widetilde{A}}(x)\rangle \mid x \in X\}$$

where $\delta_{\widetilde{A}}(x) : X \to [0,1]$ and $\zeta_{\widetilde{A}}(x) : X \to [0,1]$, with the condition that

$$0 \le \delta^2_{\widetilde{A}}(x) + \zeta^2_{\widetilde{A}}(x) \le 1,\ \forall\ x \in X \tag{7}$$

and the degree of indeterminacy for any PFS $\widetilde{A}$ and $x \in X$ is given by

$$\eta_{\widetilde{A}}(x) = \sqrt{1 - \delta^2_{\widetilde{A}}(x) - \zeta^2_{\widetilde{A}}(x)} \tag{8}$$

This distinction in imperative conditions gives a more extensive inclusion for evidence which can be shown through graph as

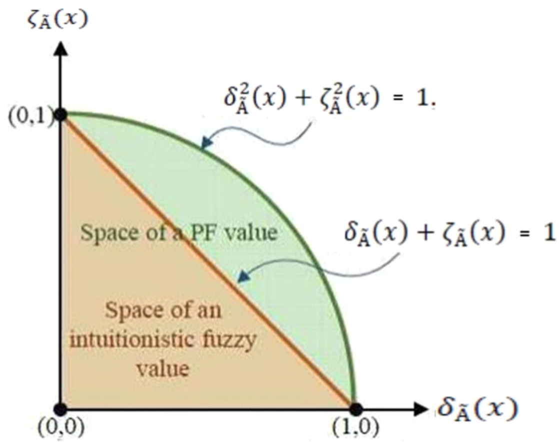

Figure 1. Comparison between IFS and PFS

3. Similarity Measures

In this stage, the current PFSs similarity measures are presented first, followed by the proposal of new PFSs.

3.1. Existing Similarity Measures.

Definition 3.1 (Ejegwa [7]). Let $P, Q \in$PFS(X) such that $X = \{x_1, x_2, \dots, x_n\}$ then new similarity measures for PFSs can be calculated as

$$Sim^1(P,Q) = 1 - \frac{1}{2n}\sum_{i=1}^{n}[|\delta_P(x_i) - \delta_Q(x_i)| + |\zeta_P(x_i) - \zeta_Q(x_i)| + |\eta_P(x_i) - \eta_Q(x_i)|] \quad (9)$$

$$Sim^2(P,Q) = 1 - \left(\frac{1}{2n}\sum_{i=1}^{n}[(\delta_P(x_i) - \delta_Q(x_i))^2 + (\zeta_P(x_i) - \zeta_Q(x_i))^2 + (\eta_P(x_i) - \eta_Q(x_i))^2]\right)^{\frac{1}{2}} \quad (10)$$

$$Sim^3(P,Q) = 1 - \frac{1}{2n}\sum_{i=1}^{n}[|\delta^2_P(x_i) - \delta^2_Q(x_i)| + |\zeta^2_P(x_i) - \zeta^2_Q(x_i)| + |\eta^2_P(x_i) - \eta^2_Q(x_i)|] \quad (11)$$

Definition 3.2 (Nguyen et al. [22]). For two PFSs $P = \{\langle x_i, \delta_P.\zeta_P\rangle \mid x_i \in X\}$ and $Q = \{\langle x_i, \delta_Q.\zeta_Q\rangle \mid x_i \in X\}$, the two exponential similarity measures are defined as

$$Sim^4(P,Q) = e^{-|\delta^2_P(x_i) - \delta^2_Q(x_i)|} \quad (12)$$

and

$$Sim^5(P,Q) = e^{-|\zeta_P^2(x_i)-\zeta_Q^2(x_i)|} \tag{13}$$

Definition 3.3 (Zhang et al. [53])**.** Let $P = \{\langle x_i, \delta_P.\zeta_P\rangle \mid x_i \in X\}$ and $Q = \{\langle x_i, \delta_Q.\zeta_Q\rangle \mid x_i \in X\}$ be two PFSs on X, and $\eta_P(x_i) = \sqrt{1-\delta_P^2(x_i)-\zeta_P^2(x_i)}$ and $\eta_Q(x_i) = \sqrt{1-\delta_Q^2(x_i)-\zeta_Q^2(x_i)}$, then the similarity measures can be determined as

$$Sim^6(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[2^{1-(|\delta_P^2(x_i)-\delta_Q^2(x_i)| \bigvee |\zeta_P^2(x_i)-\zeta_Q^2(x_i)|)} - 1] \tag{14}$$

$$Sim^7(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[2^{1-\frac{1}{2}(|\delta_P^2(x_i)-\delta_Q^2(x_i)| + |\zeta_P^2(x_i)-\zeta_Q^2(x_i)|)} - 1] \tag{15}$$

$$Sim^8(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[2^{1-(|\delta_P^2(x_i)-\delta_Q^2(x_i)| \bigvee |\zeta_P^2(x_i)-\zeta_Q^2(x_i)| \bigvee |\eta_P^2(x_i)-\eta_Q^2(x_i)|)} - 1] \tag{16}$$

$$Sim^9(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[2^{1-\frac{1}{2}(|\delta_P^2(x_i)-\delta_Q^2(x_i)| + |\zeta_P^2(x_i)-\zeta_Q^2(x_i)| + |\eta_P^2(x_i)-\eta_Q^2(x_i)|)} - 1] \tag{17}$$

where the symbol "$\vee$" represents the maximum operation.

Definition 3.4 (Ejegwa [8])**.** Let $P, Q \in$PFS(X) such that $X = \{x_1, x_2, \ldots, x_n\}$ then new similarity measures for PFSs can be calculated as

$$\begin{aligned} &Sim^{10}(P,Q) \\ &= 1 - \frac{1}{4n}\sum_{i=1}^{n}[|\delta_P(x_i)-\delta_Q(x_i)| + ||\delta_P(x_i)-\zeta_P(x_i)| - |\delta_Q(x_i)-\zeta_Q(x_i)|| \\ &\quad + ||\delta_P(x_i)-\eta_P(x_i)| - |\delta_Q(x_i)-\eta_Q(x_i)||] \qquad (18) \\ &Sim^{11}(P,Q) \\ &= 1 - \frac{1}{4n}\sum_{i=1}^{n}[|\delta_P(x_i)-\delta_Q(x_i)| + |\zeta_P(x_i)-\zeta_Q(x_i)| + |\eta_P(x_i)-\eta_Q(x_i)| \\ &\quad + 2\max\{|\delta_P(x_i)-\delta_Q(x_i)|, |\zeta_P(x_i)-\zeta_Q(x_i)|, |\eta_P(x_i)-\eta_Q(x_i)|\}] \qquad (19) \\ &Sim^{12}(P,Q) \\ &= 1 - \Bigg(\frac{1}{4n}\sum_{i=1}^{n}[(\delta_P(x_i)-\delta_Q(x_i))^2 + (\zeta_P(x_i)-\zeta_Q(x_i))^2 + (\eta_P(x_i)-\eta_Q(x_i))^2 \end{aligned}$$

$$+ 2\max\{(\delta_P(x_i) - \delta_Q(x_i))^2, (\zeta_P(x_i) - \zeta_Q(x_i))^2, (\eta_P(x_i) - \eta_Q(x_i))^2\}]\Big)^{\frac{1}{2}} \tag{20}$$

where $\eta_P(x_i) = \sqrt{1 - \delta_P^2(x_i) - \zeta_P^2(x_i)}$ and $\eta_Q(x_i) = \sqrt{1 - \delta_Q^2(x_i) - \zeta_Q^2(x_i)}$.

3.2. Proposed Similarity Measures. To begin, the axiomatic proposition of similarity for PFSs is recalled.

Proposition 1 (Ejegwa [7])**.** *Let X be non-empty set and $P, Q, R \in PFS(X)$. The similarity measure Sim between P and Q is a function $Sim : PFS \times PFS \to [0, 1]$ satisfies*

(P1) *Boundedness:* $0 \leq Sim(P, Q) \leq 1$.
(P2) *Separability:* $Sim(P, Q) = 1 \Leftrightarrow P = Q$.
(P3) *Symmetric:* $Sim(P, Q) = Sim(Q, P)$.
(P4) *Inequality: If R is a PFS in X and $P \subseteq Q \subseteq R$, then $Sim(P, R) \leq Sim(P, Q)$ and $Sim(P, R) \leq Sim(Q, R)$.*

Wei and Wei [37] proposed two cosine similarity measures for two PFSs P and Q, as follows:

$$Sim^{13}(P, Q) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi}{2}\left(\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right| \bigvee \left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\right)\right] \tag{21}$$

$$Sim^{14}(P, Q) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi}{4}\left(\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right| + \left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\right)\right] \tag{22}$$

In several circumstances, the weight of the elements $x_i \in X$ must be considered. For instance, in decision making, the attributes usually have distinct significance, and thus ought to be designated unique weights. As a result, two weighted cosine similarity measure were proposed by Wei and Wei [37] as

$$Sim^{15}(P, Q) = \frac{1}{n}\sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{2}\left(\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right| \bigvee \left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\right)\right] \tag{23}$$

$$Sim^{16}(P, Q) = \frac{1}{n}\sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{4}\left(\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right| + \left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\right)\right] \tag{24}$$

Based on the above measures, we propose the following trigonometric similarity measures between P and Q, as follows:
Let $P, Q \in$PFS(X) such that $X = \{x_1, x_2, \ldots, x_n\}$ then

$$S_{PFS1}(P, Q) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left[\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\right\} + \sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\right\}\right] \tag{25}$$

$$S_{PFS2}(P,Q) = 1 - \frac{1}{3n}\sum_{i=1}^{n}\Big[\sin\Big\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\Big\} + \sin\Big\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\Big\} + \sin\Big\{\frac{\pi}{2}|\eta_P^2(x_i) - \eta_Q^2(x_i)|\Big\}\Big] \tag{26}$$

$$S_{PFS3}(P,Q) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\omega_i\Big[\sin\Big\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\Big\} + \sin\Big\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\Big\}\Big] \tag{27}$$

$$S_{PFS4}(P,Q) = 1 - \frac{1}{3n}\sum_{i=1}^{n}\omega_i\Big[sin\Big\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\Big\} + \sin\Big\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\Big\} + \sin\Big\{\frac{\pi}{2}|\eta_P^2(x_i) - \eta_Q^2(x_i)|\Big\}\Big] \tag{28}$$

$\eta_P(x_i) = \sqrt{1 - \delta_P^2(x_i) - \zeta_P^2(x_i)}$ and $\eta_Q(x_i) = \sqrt{1 - \delta_Q^2(x_i) - \zeta_Q^2(x_i)}$,
where $\omega = (\omega_1,\ \omega_2, \ldots, \omega_n)^T$ is the weight vector of x_i $(i = 1, 2, \ldots, n)$, with $\omega_k \in [0, 1]$, $k = 1, 2, \ldots, n$, $\sum_{k=1}^{n}\omega_k = 1$. If $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)^T$, then the weighted sine similarity measure reduces to proposed sine similarity measures. If we take $\omega_k = 1$, $k = 1, 2, \ldots, n$, then $S_{PFS3}(P,Q) = S_{PFS1}(P,Q)$. Similarly, it can be verified that $S_{PFS4}(P,Q) = S_{PFS2}(P,Q)$.

Theorem 1. *The Pythagorean fuzzy similarity measures $S_{PFS1}(P,Q)$ and $S_{PFS2}(P,Q)$ defined in equation (25)-(28) are valid measures of Pythagorean fuzzy similarity.*

Proof. All the necessary four conditions to be a similarity measure are satisfied by the new similarity measures as follows:

(P1) *Boundedness:* $0 \le S_{PFS1}(P,Q)$, $S_{PFS2}(P,Q) \le 1$

Proof. For $S_{PFS1}(P,Q)$: Since the value of the sine function is within $[0, 1]$, the similarity measure based on the sine function is also within $[0, 1]$. Thus, we have,

$$0 \le \sin\Big\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\Big\} + \sin\Big\{\frac{\pi}{2}\left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\Big\} \le 2$$

$$\Rightarrow \quad 0 \le \frac{1}{2}\Big[\sin\Big\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\Big\} + \sin\Big\{\frac{\pi}{2}\left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\Big\}\Big] \le 1$$

$$\Rightarrow \quad 0 \le \frac{1}{2}\Big[\sin\Big\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\Big\} + \sin\Big\{\frac{\pi}{2}\left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\Big\}\Big] \le 1$$

$$\Rightarrow \quad 0 \le \frac{1}{2n}\sum_{i=1}^{n}\Big[\sin\Big\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\Big\} + \sin\Big\{\frac{\pi}{2}\left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\Big\}\Big] \le 1$$

$$\Rightarrow \quad 0 \leq 1 - \frac{1}{2n}\sum_{i=1}^{n}\left[\sin\left\{\frac{\pi}{2}\left|\delta_P^2(x_i) - \delta_Q^2(x_i)\right|\right\} + \sin\left\{\frac{\pi}{2}\left|\zeta_P^2(x_i) - \zeta_Q^2(x_i)\right|\right\}\right] \leq 1$$

$$\Rightarrow \quad 0 \leq S_{PFS1}(P,Q) \leq 1.$$

Measure $S_{PFS2}(P,Q)$ can be proved similarly.

(P2) *Separability:* $S_{PFS1}(P,Q), S_{PFS2}(P,Q) = 1 \Leftrightarrow P = Q$.

Proof. For $S_{PFS1}(P,Q)$: For two PFSs P and Q in $X = \{x_1, x_2, \ldots, x_n\}$, if $P = Q$, then $\delta_P^2(x_i) = \delta_Q^2(x_i)$ and $\zeta_P^2(x_i) = \zeta_Q^2(x_i)$. Thus, $|\delta_P^2(x_i) - \delta_Q^2(x_i)| = 0$ and $|\zeta_P^2(x_i) - \zeta_Q^2(x_i)| = 0$. Since $\sin 0 = 0$, therefore, $S_{PFS2}(P,Q) = 1$.
If $S_{PFS1}(P,Q) = 1$, this implies $|\delta_P^2(x_i) - \delta_Q^2(x_i)| = 0$ and $|\zeta_P^2(x_i) - \zeta_Q^2(x_i)| = 0$. Since $\sin 0 = 0$, therefore $\delta_P^2(x_i) = \delta_Q^2(x_i)$ and $\zeta_P^2(x_i) = \zeta_Q^2(x_i)$. Hence $P = Q$.
Measure $S_{PFS2}(P,Q)$ can be proved similarly.

(P3) *Symmetric:* $S_{PFSj}(P,Q) = S_{PFSj}(Q,P)$ for $j = 1,2$.
Proofs are self-explanatory and straight forward.

(P4) *Inequality:* If R is a PFS in X and $P \subseteq Q \subseteq R$, then $S_{PFSj}(P,R) \leq S_{PFSj}(P,Q)$ and $S_{PFSj}(P,R) \leq S_{PFSj}(Q,R)$ where $j = 1,2$.

Proof. For $S_{PFS1}(P,Q)$: If $P \subseteq Q \subseteq R$, then for $x_i \in X$, we have $0 \leq \delta_P(x_i) \leq \delta_Q(x_i) \leq \delta_R(x_i) \leq 1$ and $1 \geq \zeta_P(x_i) \geq \zeta_Q(x_i) \geq \zeta_R(x_i) \geq 0$.
This implies that $0 \leq \delta_P^2(x_i) \leq \delta_Q^2(x_i) \leq \delta_R^2(x_i) \leq 1$ and $1 \geq \zeta_P^2(x_i) \geq \zeta_Q^2(x_i) \geq \zeta_R^2(x_i) \geq 0$. This we have,

$$|\delta_P^2(x_i) - \delta_Q^2(x_i)| \leq |\delta_P^2(x_i) - \delta_R^2(x_i)|\ ; |\delta_Q^2(x_i) - \delta_R^2(x_i)| \leq |\delta_P^2(x_i) - \delta_R^2(x_i)|$$

and

$$|\zeta_P^2(x_i) - \zeta_Q^2(x_i)| \leq |\zeta_P^2(x_i) - \zeta_R^2(x_i)|\ ; |\zeta_Q^2(x_i) - \zeta_R^2(x_i)| \leq |\zeta_P^2(x_i) - \zeta_R^2(x_i)|$$

From the above we can write,

$$\begin{aligned} &\Rightarrow \quad \frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)| \leq \frac{\pi}{2}|\delta_P^2(x_i) - \delta_R^2(x_i)| \\ &\Rightarrow \quad \sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\right\} \leq sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_R^2(x_i)|\right\} \end{aligned} \tag{29}$$

Also,

$$\sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\right\} \leq \sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_R^2(x_i)|\right\}. \tag{30}$$

Adding (29) and (30), we have

$$\begin{aligned} &\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\right\} + \sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\right\} \\ &\qquad \leq \sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_R^2(x_i)|\right\} + \sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_R^2(x_i)|\right\} \end{aligned}$$

$$\Rightarrow \quad \frac{1}{2n}\sum_{i=}^{n}\left[\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i) - \delta_Q^2(x_i)|\right\} + \sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i) - \zeta_Q^2(x_i)|\right\}\right]$$

$$\leq \frac{1}{2n}\sum_{i=}^{n}\left[\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i)-\delta_R^2(x_i)|\right\}+\sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i)-\zeta_R^2(x_i)|\right\}\right]$$

$$\Rightarrow \quad 1-\frac{1}{2n}\sum_{i=}^{n}\left[\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i)-\delta_Q^2(x_i)|\right\}+\sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i)-\zeta_Q^2(x_i)|\right\}\right]$$

$$\geq 1-\frac{1}{2n}\sum_{i=}^{n}\left[\sin\left\{\frac{\pi}{2}|\delta_P^2(x_i)-\delta_R^2(x_i)|\right\}+\sin\left\{\frac{\pi}{2}|\zeta_P^2(x_i)-\zeta_R^2(x_i)|\right\}\right]$$

$$\Rightarrow \quad S_{PFS1}(P,R) \leq S_{PFS1}(P,Q). \tag{31}$$

Similarly, $S_{PFS1}(P,R) \leq S_{PFS1}(Q,R)$.

Similar proofs can be made for $S_{PFS2}(P,R) \leq S_{PFS2}(P,Q)$ and $S_{PFS2}(P,R) \leq S_{PFS2}(Q,R)$.

Analogous to the proofs done above, we can also validate properties depicted in Proposition 1 for weighted similarity measures $S_{PFS3}(P,Q)$ and $S_{PFS4}(P,Q)$ accordingly.

3.3. Numerical Verification of the Similarity Measures. Based on the parameters suggested by Wei and Wei [37], we verify whether proposed similarity measures satisfy above four properties:

Example 1. Let $P,Q,R \in PFS(X)$ for $X=\{x_1,x_2,x_3\}$. Suppose
$P=\{\langle x_1,0.6,0.2\rangle,\langle x_2,0.4,0.6\rangle,\langle x_3,0.5,0.3\rangle\}$,
$Q=\{\langle x_1,0.8,0.1\rangle,\langle x_2,0.7,0.3\rangle,\langle x_3,0.6,0.1\rangle\}$ and
$R=\{\langle x_1,0.9,0.2\rangle,\langle x_2,0.8,0.2\rangle,\langle x_3,0.7,0.3\rangle\}$

The following are the steps for calculating similarity using the recommended similarity measures:

$$\begin{aligned}
&S_{PFS1}(P,Q)\\
&=1-\frac{1}{6}\left[\sin\left\{\frac{\pi}{2}|0.6^2-0.8^2|\right\}+\sin\left\{\frac{\pi}{2}|0.2^2-0.1^2|\right\}+\sin\left\{\frac{\pi}{2}|0.4^2-0.7^2|\right\}\right.\\
&\quad\left.+\sin\left\{\frac{\pi}{2}|0.6^2-0.3^2|\right\}+\sin\left\{\frac{\pi}{2}|0.5^2-0.6^2|\right\}+\sin\left\{\frac{\pi}{2}|0.3^2-0.1^2|\right\}\right]\\
&=1-\frac{1}{6}[0.42578+0.04710+0.495458+0.411514+0.17193+0.12533]\\
&=1-\frac{1}{6}(1.677112)=0.72048.
\end{aligned}$$

$$\begin{aligned}
&S_{PFS1}(P,R)\\
&=1-\frac{1}{6}\left[\sin\left\{\frac{\pi}{2}|0.6^2-0.9^2|\right\}+\sin\left\{\frac{\pi}{2}\left|0.2^2-0.2^2\right|\right\}+\sin\left\{\frac{\pi}{2}|0.4^2-0.8^2|\right\}\right]\\
&\quad+\sin\left\{\frac{\pi}{2}|0.6^2-0.2^2|\right\}+\sin\left\{\frac{\pi}{2}|0.5^2-0.7^2|\right\}+\sin\left\{\frac{\pi}{2}|0.3^2-0.3^2|\right\}\\
&=1-\frac{1}{6}[0.649448+0.0+0.684547+0.4817536+0.368124+0.0]
\end{aligned}$$

$$= 1 - \frac{1}{6}(2.1838726) = 0.636021.$$

$$S_{PFS1}(Q, R)$$

$$= 1 - \frac{1}{6}\left[\sin\left\{\frac{\pi}{2}|0.8^2 - 0.9^2|\right\} + \sin\left\{\frac{\pi}{2}|0.1^2 - 0.2^2|\right\} + \sin\left\{\frac{\pi}{2}|0.7^2 - 0.8^2|\right\}\right]$$

$$+ \sin\left\{\frac{\pi}{2}|0.3^2 - 0.2^2|\right\} + \sin\left\{\frac{\pi}{2}|0.6^2 - 0.7^2|\right\} + \sin\left\{\frac{\pi}{2}|0.1^2 - 0.3^2|\right\}$$

$$= 1 - \frac{1}{6}[0.26387 + 0.04710 + 0.233445 + 0.078459 + 0.202787 + 0.12533]$$

$$= 1 - \frac{1}{6}(0.950991) = 0.8415015.$$

Similarly, we can find values for $S_{PFS2}(P, Q)$, $S_{PFS2}(P, R)$, and $S_{PFS2}(Q, R)$ as 0.755442, 0.616652 and 0.806625, respectively.

Similarly, we can find values for $S_{PFS3}(P, Q)$, $S_{PFS3}(P, R)$, and $S_{PFS3}(Q, R)$ as 0.905335, 0.875293 and 0.947552 respectively and we can find values for $S_{PFS4}(P, Q)$, $S_{PFS4}(P, R)$, and $S_{PFS4}(Q, R)$ as 0.911446, 0.864312 and 0.935455, respectively.

Numerical Justification: From the above computations, it supports that

P1: $0 \leq S_{PFSj}(P, Q) \leq 1$; $j = 1, 2, 3, 4$
P2: $S_{PFSj}(P, Q) = 1 \Leftrightarrow P = Q$; $j = 1, 2, 3, 4$
P3: It follows that $S_{PFSj}(P, Q) = S_{PFSj}(Q, P)$; $j = 1, 2, 3, 4$.
($\because$ use of square and absolute value)
P4: $S_{PFSj}(P, R) \leq S_{PFSj}(P, Q)$ and $S_{PFSj}(P, R) \leq S_{PFSj}(Q, R)$; $j = 1, 2, 3, 4$.

4. Applications of Pythagorean Fuzzy Sets

To determine the legitimacy of PFSs, pattern recognition and medical diagnosis approaches to the decision making are presented in this section.

(a) Pattern Recognition. Assume that there are three known patterns A_1, A_2, and A_3. Each pattern can be conveyed by PFSs in $X = \{x_1, x_2, x_3\}$ as follows:

$$A_1 = \{\langle x_1, 1, 0\rangle, \langle x_2, 0.8, 0\rangle, \langle x_3, 0.7, 0.1\rangle\},$$
$$A_2 = \{\langle x_1, 0.8, 0.1\rangle, \langle x_2, 1, 0\rangle, \langle x_3, 0.9, 0.1\rangle\},$$
$$A_3 = \{\langle x_1, 0.6, 0.2\rangle, \langle x_2, 0.8, 0\rangle, \langle x_3, 1, 0\rangle\}.$$

The model B which needs to be identified is as follows: $B = \{\langle x_1, 0.5, 0.3\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.8, 0.1\rangle\}$. The goal of this task is to sort the pattern B into one of three categories: A_1, A_2, and A_3. For the recommended weighted similarity measures, weight of x_1, x_2, x_3 are assumed to be 0.5, 0.3, and 0.2, respectively. To achieve this, the proposed similarities and the measures suggested by Wei and Wei [37] have been calculated from B to A_1, A_2, and A_3 as exhibited (Table 1).

TABLE 1. Similarity measure between classes A_1, A_2, A_3 and B

	(A_1, B)	(A_2, B)	(A_3, B)
$Sim^{13}(P,Q)$	0.753293	0.772845	**0.911421**
$Sim^{14}(P,Q)$	0.917269	0.92816	**0.972924**
$Sim^{15}(P,Q)$	0.219088	0.254245	**0.310956**
$Sim^{16}(P,Q)$	0.294755	0.307587	**0.326082**
$S_{PFS1}(P,Q)$	0.702201	0.688112	**0.784918**
$S_{PFS2}(P,Q)$	0.638988	0.620872	**0.747197**
$S_{PFS3}(P,Q)$	0.879058	0.887487	**0.936321**
$S_{PFS4}(P,Q)$	0.854095	0.866164	**0.928437**

We may deduce from the numerical findings in table 1 that the degree of similarity between A_3 and B is the highest, based on eight similarity metrics. According to the recognition principle of maximum degree of similarity between PFSs, all eight similarity measures assign the unknown class B to the known class A_3.

(b) Medical Diagnosis. In a conventional problem of medicinal diagnosis, we assume that if a doctor desires to diagnose some of patients "$P = \{$Alex, Chris, James, Mike and Shawn$\}$" under some demarcated diagnosis "$D = \{$Viral fever, Malaria, Typhoid, Stomach problem and Chest problem$\}$" and a set of symptoms "$S = \{$Temperature, Headche, Stomach pain, Chough and Chest pain$\}$". The subsequent Table 2 and Table 3 provide the intent of the recommended computational presentation. Suggested trigonometric similarity measures have been given in Table 4 and Table 5. However, if we assume the weights of these symptoms to be 0.15, 0.25, 0.20, 0.15, and 0.25, we can use the suggested weighted similarity measures in Tables 6 and 7.

TABLE 2. Symptom - Disease Pythagorean relation

	Viral Fever	Malaria	Typhoid	Stomach Pain	Chest Pain
Temperature	$\langle 0.8, 0.1\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.1, 0.6\rangle$
Headache	$\langle 0.9, 0.1\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.2, 0.7\rangle$
Stomach Pain	$\langle 0.0, 0.7\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.1, 0.2\rangle$
Cough	$\langle 0.7, 0.1\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.0, 0.5\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.0, 0.6\rangle$
Chest pain	$\langle 0.5, 0.1\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.3, 0.4\rangle$

TABLE 3. Patient - Symptom Pythagorean relation

	Temperature	Headache	Stomach Pain	Cough	Chest pain
Alex	$\langle 0.4, 0.0\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.1, 0.7\rangle$
Chris	$\langle 0.7, 0.0\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.0, 0.9\rangle$	$\langle 0.7, 0.0\rangle$	$\langle 0.1, 0.8\rangle$
James	$\langle 0.3, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.1, 0.9\rangle$
Mike	$\langle 0.1, 0.7\rangle$	$\langle 0.2, 0.4\rangle$	$\langle 0.8, 0.0\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.2, 0.7\rangle$
Shawn	$\langle 0.1, 0.8\rangle$	$\langle 0.0, 0.8\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.8, 0.1\rangle$

TABLE 4. Patient - Disease Pythagorean relation for $S_{PFS1}(P, Q)$

	Alex	Chris	James	Mike	Shawn
Viral Fever	0.633762309	0.659151182	0.602614371	0.342805574	0.418028836
Malaria	0.671998058	**0.662154189**	0.649195485	0.477973271	0.407924197
Typhoid	0.604393011	0.472212706	0.566920195	0.755991033	0.690886493
Stomach Pain	0.640998054	0.508047521	**0.731854056**	0.494562368	0.657566065
Chest Pain	**0.68537**	0.558406752	0.672762627	**0.756366058**	**0.693944024**

TABLE 5. Patient - Disease Pythagorean relation for $S_{PFS2}(P, Q)$

	Alex	Chris	James	Mike	Shawn
Viral Fever	0.612709859	0.643829909	0.593963527	0.445030908	0.516884038
Malaria	**0.679716983**	**0.669339697**	0.667841574	0.550341744	0.480216191
Typhoid	0.633905625	0.511042541	0.579320543	**0.707672464**	0.691988097
Stomach Pain	0.672216121	0.61344757	**0.771484629**	0.586326454	**0.694560623**
Chest Pain	0.668778778	0.564903325	0.656651545	0.691738507	0.629092371

TABLE 6. Patient - Disease Pythagorean relation for $S_{PFS3}(P, Q)$

	Alex	Chris	James	Mike	Shawn
Viral Fever	0.921729577	0.919845164	0.921544403	0.871946093	0.88755709
Malaria	0.930462526	**0.922976582**	0.932164343	0.900239879	0.884228595
Typhoid	0.922248876	0.901133253	0.908121904	**0.948555276**	0.936024499
Stomach Pain	0.921944595	0.895724146	**0.940830927**	0.893706762	0.931947321
Chest Pain	**0.938884504**	0.91887858	0.92766981	0.948268155	**0.936910114**

Table 7. Patient - Disease Pythagorean relation for $S_{PFS4}(P,Q)$

	Alex	Chris	James	Mike	Shawn
Viral Fever	0.919279797	0.91769755	0.918157037	0.888069357	0.904094338
Malaria	0.933418263	**0.925606823**	0.933831811	0.910927351	0.897622596
Typhoid	0.927958061	0.906229853	0.912704345	**0.939215346**	0.9377672
Stomach Pain	0.930718445	0.918609925	**0.951119265**	0.912198115	**0.940820646**
Chest Pain	**0.934359834**	0.916444138	0.925143018	0.934829072	0.925058742

Observations.

(a) Taking into an account of numerical computations of above tables, it is being determined that for the similarity measure S_{PFS1} (P,Q), Alex, Shawn and Mike are suffering from chest pain; Chris - Malaria; James - stomach pain (Table 4).

(b) For the measure S_{PFS2} (P,Q), it is being observed that Shawn and James are suffering from stomach pain, Alex and Chris are suffering from malaria, whereas Mike is suffering from typhoid stomach (Table 5).

(c) For the measure S_{PFS3} (P,Q), it is being observed that James is suffering from stomach pain, Chris is suffering from malaria, whereas Mike is suffering from typhoid and Alex and Shawn are suffering from chest pain (Table 6).

(d) For the measure S_{PFS4} (P,Q), it is being observed that Shawn and James are suffering from stomach pain, Chris is suffering from malaria, whereas Mike is suffering from typhoid and Alex is suffering from chest pain (Table 7).

This analysis is done on the grounds that higher value of the patient against every similarity measure demonstrates the greater likelihood of having the disease.

5. Comparative Study

A comparison between the proposed similarity measure and current similarity measures is undertaken based on the numerical scenarios presented to establish the proposed similarity measure's dominance. Table 8 shows a complete analysis of the PFS similarity measures.

From the numerical results presented in the Tables 8, analysis has been done between the similarity measures proposed by Wei and Wei [37] and the results attained using our proposed similarity measures for PFSs. It has been noticed that the results produced by applying our proposed similarity measures based on the idea of greatest degree of similarity between PFSs are comparable to those shown in equations (21)-(24).

TABLE 8. Comparison of existing measures with the proposed similarity measures

Comparison	Alex	Chris	James	Mike	Shawn
$Sim^{13}(P,Q)$	Malaria	Malaria	Stomach Pain	Typhoid	Typhoid
$Sim^{14}(P,Q)$	Chest Pain	Malaria	Chest Pain	Typhoid	Chest Pain
$Sim^{15}(P,Q)$	Chest Pain	Malaria	Stomach Pain	Typhoid	Typhoid
$Sim^{16}(P,Q)$	Chest Pain	Malaria	Malaria	Typhoid	Chest Pain
$S_{PFS1}(P,Q)$	Chest Pain	Malaria	Stomach Pain	Chest Pain	Chest Pain
$S_{PFS2}(P,\ Q)$	Malaria	Malaria	Stomach Pain	Typhoid	Stomach Pain
$S_{PFS3}(P,Q)$	Chest Pain	Malaria	Stomach Pain	Typhoid	Chest Pain
$S_{PFS4}(P,Q)$	Chest Pain	Malaria	Stomach Pain	Typhoid	Stomach Pain

6. Conclusion

In this paper, novel trigonometric and weighted similarity measures that are compatible with the traditional PFS criteria are offered. Numerical computation to confirm the validity of the proposed similarity measures has been done.Further,applications of these similarity measures to decision making problems such as,pattern recognition and medical diagnosis has been provided. It is verified that the proposed similarity measures can effectively overcome the limitations of the existing similarity measures. Also, comparative analysis of the investigated similarity measures was performed to determine the effectiveness of the proposed measures. These intended measures can be applied to complex decision making, linguistic sets and risk analysis in the future.

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