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COMPLETE HAMACHER FUZZY GRAPHS

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ABSTRACT. Our goal in this paper is to propose the advancement proposed by Alsina, Klement and other researchers in the area of fuzzy logic applied to the area of fuzzy graph theory. We introduce the notion of complete Hamacher fuzzy graphs and the notion of balanced Hamacher fuzzy graphs and discuss their properties. Moreover, several operations on these fuzzy graphs are explored via the complete notion.

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1. Introduction

Triangular conorms were presented by Menger [16] which were later on discussed by Schweizer and Sklar [19]. Alsina et al. [10] proved that conorms are one of the models for intersecting fuzzy sets. Since that time, many others have provided several operators for the same purpose [14, 15]. The min and max operators of Zadeh's have been used in many applications of fuzzy logic such as decision-making processes and fuzzy graph theory. It is a fact that from theoretical and experimental aspects other operators may work better in some situations, especially in the context of decision-making processes. To find a good operator for an application, we consider the properties they possess, their suitability to the model, their simplicity, their software and hardware implementation, etc.

Graph theory has several interesting applications in system analysis, operations research, economics and many other fields. Since most of the time the aspects of graph and graph problems are uncertain, it is a good idea to deal with these aspects via the methods of fuzzy logic. As the notions of degree, complement, completeness, regularity and many others play very important role in the crisp graph case, it is a nice idea to try to see what corresponds to these notions in the case of fuzzy graphs. Sunitha and Kumar [20] defined several new operations on fuzzy graphs and they also modified the definition of complement of a

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fuzzy graph so that to agree with the crisp case in graph. In 2011, AL-Hawary [4] introduced the new concept of balanced fuzzy graphs. He defined three new operations on fuzzy graphs and explored what classes of fuzzy graphs are balanced. Since then, many authors have studied the idea of balanced on distinct types of fuzzy graphs, see for example [7, 20]. Moreover, Al-Hawary and others explored the idea of balanced fuzzy graphs in [1, 2, 3, 4, 6, 7, 8, 9]. Fuzzy graphs have several applications in cluster analysis, database theory, decision-making and optimization of networks. Rosenfeld [18] considered fuzzy relations on fuzzy sets and modified the structure of fuzzy graphs, using max and min operations, to obtain analogs of graph-theoretical concepts. Some remarks on fuzzy graphs were provided by Bhattacharya [12]. Mordeson and Peng [17] defined some operations on fuzzy graphs and introduced the concept of strong fuzzy graphs. Later on, Bhutani and Battou [13] considered operations on fuzzy graphs preserving M-strong property. The complement of a fuzzy graph was proposed by Mordeson and Peng [17] and then modified by Sunitha and Vijayakumar [20]

In [5], Hamacher fuzzy graphs were defined and explored. Several operations on these fuzzy graphs were provided and the idea of strong Hamacher fuzzy graphs was also discussed. The main purpose of this paper is to propose the advancement proposed by Alsina, Klement and other researchers in the area of fuzzy logic in the area of fuzzy graph theory. We introduce the notion of complete Hamacher fuzzy graphs and the notion of balanced Hamacher fuzzy graphs and discuss their properties.

2. Preliminaries

A graph $G = (V, E)$ is a mathematical structure consisting of a set of vertices $V = V(G)$ and a set of edges $E = E(G)$, where each edge is an unordered pair of distinct vertices. Two vertices x and y of a graph G are adjacent if $xy \in E(G)$.

The standard products of graphs: the direct product (tensor product), the cartesian product, the semi-strong product and the strong product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ will be denoted by $G_1 \times G_2, G_1 \Box G_2, G_1 \cdot G_2$ and $G_1 \blacksquare G_2$, respectively. Let

$(x_1, x_2), (y_1, y_2) \in V_1 \times\ V_2$, where $V_1 \times\ V_2$ is the vertex set of each product graph, then

$E(G_1 \times G_2) = \{(x_1, x_2), (y_1, y_2) | x_1y_1 \in E_1 \textit{ and } x_2y_2 \in E_2\}$,

$E(G_1 \Box G_2) = \{(x_1, x_2), (y_1, y_2) | x_1 = y_1 \textit{ and } x_2y_2 \in E_2 \textit{ or } x_1y_1 \in E_1 \textit{ and } x_2 = y_2\}$

$E(G_1 \bullet G_2) = \{(x_1, x_2), (y_1, y_2) | x_1 = y_1 \textit{ and } x_2y_2 \in E_2 \textit{ or } x_1y_1 \in E_1 \textit{ and } x_2y_2 \in E_2\}$

$E(G_1 \blacksquare G_2) = E(G_1 \times G_2) \cup E(G_1 \Box G_2)$.

Throughout this paper, V represents a crisp universe of generic elements, G^* stands for the crisp graph and G is the Hamacher fuzzy graph.

Definition 2.1. [18] A *fuzzy graph* with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0, 1]$ is a fuzzy subset and $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$.

Definition 2.2. [11] A binary operation $S : [0,1]^2 \to [0,1]$ defined by $S(x,y) = \frac{x+y-2xy}{1-xy}$ is called a Hamshar conorm.

Next, we recall some definitions and results from [5] that will be needed throughout this paper.

Definition 2.3. A Hamacher fuzzy graph with a finite set V as its underlying set is a pair $G = (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset of V and $\mu : V \times V \to [0,1]$ is a symmetric fuzzy relation on σ such that

$$\mu(x,y) \leq \frac{\mu(x) + \mu(y) - 2\mu(x)\mu(y)}{1 - \mu(x)\mu(y)}.$$

σ is called the Hamacher fuzzy vertex set of G and μ is called the Hamacher fuzzy edge set of G. The underlying crisp graph of G is denoted by $G^* = (V, E)$.

Definition 2.4. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of E_i for i=1,2. The direct product of G_1 and G_2 is the Hamacher fuzzy graph $G_1 \times G_2 = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ where

$$(\sigma_1 \times \sigma_2)(x_1, x_2) = \frac{\sigma_1(x_1) + \sigma_2(x_2) - 2\sigma_1(x_1)\sigma_2(x_2)}{1 - \sigma_1(x_1)\sigma_2(x_2)}$$

for all $(x_1, x_2) \in V_1 \times V_2$ and

$$(\mu_1 \times \mu_2)(x_1, x_2) = \frac{\mu_1(x_1) + \mu_2(x_2) - 2\mu_1(x_1)\mu_2(x_2)}{1 - \mu_1(x_1)\mu_2(x_2)}$$

for all $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$.

Lemma 2.5. *If G_1 and G_2 are the Hamacher fuzzy graphs of the graphs G_1^* and G_2^*, respectively, the direct product $G_1 \times G_2$ is the Hamacher fuzzy graph of $G_1^* \times G_2^*$.*

Definition 2.6. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of E_i for i=1,2. The Cartesian product of G_1 and G_2 is the Hamacher fuzzy graph $G_1 \Box G_2 = (\sigma_1 \Box \sigma_2, \mu_1 \Box \mu_2)$ where

$$(\sigma_1 \Box \sigma_2)(x_1, x_2) = \frac{\sigma_1(x_1) + \sigma_2(x_2) - 2\sigma_1(x_1)\sigma_2(x_2)}{1 - \sigma_1(x_1)\sigma_2(x_2)}$$

for all $(x_1, x_2) \in V_1 \times V_2$,

$$(\mu_1 \Box \mu_2)((x, x_2)(x, y_2)) = \frac{\sigma_1(x) + \mu_2(x_2y_2) - 2\sigma_1(x)\mu_2(x_2y_2)}{1 - \sigma_1(x)\mu_2(x_2y_2)}$$

for all $x \in V_1$ and $x_2y_2 \in E_2$ and

$$(\mu_1 \Box \mu_2)((x_1, z)(y_1, z)) = \frac{\sigma_2(z) + \mu_1(x_1y_1) - 2\sigma_2(z)\mu_1(x_1y_1)}{1 - \sigma_2(z)\mu_1(x_1y_1)}$$

for all $z \in V_2$ and $x_1y_1 \in E_1$.

Definition 2.7. If a fuzzy membership degree is attached from $[0,1]$ to each edge of the Hamacher fuzzy graph G of a graph G^* and each vertex is crisply in G, then G is called the Hamacher fuzzy edge graph.

Lemma 2.8. *Let G_1 and G_2 be the Hamacher fuzzy edge graphs of the graphs G_1^* and G_2^*, respectively. The cartesian product $G_1 \square G_2$ of G_1 and G_2 is the Hamacher fuzzy edge graph of $G_1 \square G_2$.*

Definition 2.9. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of E_i for i=1,2. The semi-strong product of G_1 and G_2 is the Hamacher fuzzy graph $G_1 \cdot G_2 = (\sigma_1 \cdot \sigma_2, \mu_1 \cdot \mu_2)$ where

$$(\sigma_1 \cdot \sigma_2)(x_1, x_2) = \frac{\sigma_1(x_1) + \sigma_2(x_2) - 2\sigma_1(x_1)\sigma_2(x_2)}{1 - \sigma_1(x_1)\sigma_2(x_2)}$$

for all $(x_1, x_2) \in V_1 \times V_2$,

$$(\mu_1 \cdot \mu_2)((x, x_2)(x, y_2)) = \frac{\sigma_1(x) + \mu_2(x_2y_2) - 2\sigma_1(x)\mu_2(x_2y_2)}{1 - \sigma_1(x)\mu_2(x_2y_2)}$$

for all $x \in V_1$ and $x_2y_2 \in E_2$ and

$$(\mu_1 \cdot \mu_2)((x_1, x_2)(y_1, y_2)) = \frac{\mu_1(x_1y_1) + \mu_2(x_2y_2) - 2\mu_1(x_1y_1)\mu_2(x_2y_2)}{1 - \mu_1(x_1y_1)\mu_2(x_2y_2)}$$

for all $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$.

Lemma 2.10. *Let G_1 and G_2 be the Hamacher fuzzy edge graphs of the graphs G_1^* and G_2^*, respectively. The semi-strong product $G_1 \bullet G_2$ of G_1 and G_2 is the Hamacher fuzzy edge graph of $G_1^* \bullet G_2^*$.*

Definition 2.11. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of E_i for i=1,2. The strong product of G_1 and G_2 is the Hamacher fuzzy graph $G_1 \blacksquare G_2 = (\sigma_1 \blacksquare \sigma_2, \mu_1 \blacksquare \mu_2)$ where

$$(\sigma_1 \blacksquare \sigma_2)(x_1, x_2) = \frac{\sigma_1(x_1) + \sigma_2(x_2) - 2\sigma_1(x_1)\sigma_2(x_2)}{1 - \sigma_1(x_1)\sigma_2(x_2)}$$

for all $(x_1, x_2) \in V_1 \times V_2$,

$$(\mu_1 \blacksquare \mu_2)((x, x_2)(x, y_2)) = \frac{\sigma_1(x) + \mu_2(x_2y_2) - 2\sigma_1(x)\mu_2(x_2y_2)}{1 - \sigma_1(x)\mu_2(x_2y_2)}$$

for all $x \in V_1$ and $x_2y_2 \in E_2$,

$$(\mu_1 \blacksquare \mu_2)((x_1, z)(y_1, z)) = \frac{\sigma_2(z) + \mu_1(x_1y_1) - 2\sigma_2(z)\mu_1(x_1y_1)}{1 - \sigma_2(z)\mu_1(x_1y_1)}$$

for all $z \in V_2$ and $x_1y_1 \in E_1$ and

$$(\mu_1 \blacksquare \mu_2)((x_1, x_2)(y_1, y_2)) = \frac{\mu_1(x_1y_1) + \mu_2(x_2y_2) - 2\mu_1(x_1y_1)\mu_2(x_2y_2)}{1 - \mu_1(x_1y_1)\mu_2(x_2y_2)}$$

for all $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$.

Lemma 2.12. *Let G_1 and G_2 be the Hamacher fuzzy edge graphs of the graphs G_1^* and G_2^*, respectively. The strong product $G_1 \blacksquare G_2$ of G_1 and G_2 is the Hamacher fuzzy edge graph of $G_1^* \blacksquare G_2^*$.*

Definition 2.13. Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of E_i, i $= 1, 2$ where $V_1 \cap V_2 = \varnothing$. Define the union $G_1 \cup G_2 = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ of the Hamacher fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ as follows:$(\sigma_1 \cup \sigma_2)(x) = \sigma_1(x)$ if $x \in V_1 \backslash V_2$, $(\sigma_1 \cup \sigma_2)(x) = \sigma_2(x), if x \in V_2 \backslash V$, $(\mu_1 \cup \mu_2)(xy) = \mu_1(xy)$ if $xy \in E_1 \backslash E_2$ and $(\mu_1 \cup \mu_2)(xy) = \mu_2(xy)$ if $xy \in E_2 \backslash E_1$.

Theorem 2.14. *The union $G_1 \cup G_2$ of G_1 and G_2 is the Hamacher fuzzy graph of $G_1^* \cup G_2^*$ if and only if G_1 and G_2 are the Hamacher fuzzy graphs of G_1^* and G_2^*, respectively, where σ_1, σ_2, μ_1 and μ_2 are fuzzy subsets of V_1, V_2, E_1 and E_2, respectively, and $V_1 \cap V_2 = \emptyset$.*

Definition 2.15. Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of E_i, $i = 1, 2$ where $V_1 \cap V_2 = \varnothing$. Define the ring sum $G_1 \oplus G_2 = (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)$ of the Hamacher fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ as follows:

$(\sigma_1 \oplus \sigma_2)(x) = (\sigma_1 \cup \sigma_2)(x)$ if $x \in V_1 \cup V_2$, $(\mu_1 \oplus \mu_2)(xy) = \mu_1(xy)$ if $xy \in E_1 \backslash E_2$ and $(\mu_1 \oplus \mu_2)(xy) = \mu_2(xy)$ if $xy \in E_2 \backslash E_1$.

Lemma 2.16. *The ring sum $G_1 \oplus G_2$ of two Hamacher fuzzy graphs G_1 and G_2 of G_1 and G_2 is the Hamacher fuzzy graph of $G_1^* \oplus G_2^*$.*

Definition 2.17. A homomorphism $\psi : G_1 \to G_2$ of two Hamacher fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ is a mapping $\psi : V_1 \to V_2$ satisfying the following conditions:

(a) $\sigma_1(x) \leq \sigma_2(\psi(x))$ for all $x \in V_1$, (b) $\mu_1(xy) \leq \mu_2(\psi(x)\psi(y))$ for all $x, y \in V_1$. An isomorphism $\psi : G_1 \to G_2$ of two Hamacher fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ (denoted by $G_1 \simeq G_2$) is a bijective mapping $\psi : V_1 \to V_2$ which is also an homomorphism.

Definition 2.18. The complement of a Hamacher fuzzy graph $G = (\sigma, \mu)$ of a graph $G^* = (V, E)$ is the Hamacher fuzzy graph $\bar{G} = (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma}\,(x) = \sigma(x)$ for all $x \in E$ and $\bar{\mu}(xy) = \frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)} - \mu(xy)$ for all $x, y \in V$.

We remark that this definition of complement for Hamacher fuzzy graphs agrees with that of crisp graphs in the sense that the complement of a complement is isomorphic to the original graph.

Definition 2.19. A Hamacher fuzzy graph $G = (\sigma, \mu)$ is said to be self-complementary if $G \simeq \bar{G}$.

Lemma 2.20. *a) Let $G : (\sigma, \mu)$ be a self-complemetary Hamacher fuzzy graph. Then $\sum_{x,y \in V} \mu(x, y) = (1/2) \sum_{x,y \in V} (\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})$.*

b) Let $G : (\sigma, \mu)$ be a fuzzy graph with $\mu(x, y) = (1/2)(\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})$ for all $x, y \in V$. Then G is self-complemetary.

c) Any two isomorphic fuzzy Hamacher graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ satisfy $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{x,y \in V_1} \mu_1(x, y) = \sum_{x,y \in V_2} \mu_2(x, y)$.

Definition 2.21. A Hamacher fuzzy graph $G = (\sigma, \mu)$ is called strong if $\mu(xy) = \frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)}$ for all $xy \in E$.

3. Strong and complete Hamacher fuzzy graphs

We start this section by defining the relatively new notion of complete Hamacher fuzzy graph.

Definition 3.1. A Hamacher fuzzy graph $G = (\sigma, \mu)$ is called complete if $\mu(xy) = \frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)}$ for all $x, y \in V$.

It is clear that every complete Hamacher fuzzy graph is strong, but the converse needs not be true as the Hamacher fuzzy graph on two vertices with no edges is strong but not complete.

The proof of the following result follows at once from the proof of Lemma 2.20 and Theorem 2.14:

Lemma 3.2. *If G_1 and G_2 are strong (complete) Hamacher fuzzy graphs, then $G_1 \times G_2$ is also a strong (complete) Hamacher fuzzy graph.*

The proof of the following result follows at once from the proof of Lemma 2.8 :

Lemma 3.3. *If G_1 and G_2 are strong (complete) Hamacher fuzzy edge graphs, then $G_1 \square G_2$, $G_1 \blacksquare G_2$ and $G_1 \cdot G_2$ are also strong (complete) Hamacher fuzzy edge graphs.*

4. Balanced Hamacher fuzzy graphs

We begin this section by defining the density of a Hamacher fuzzy graph and balanced Hamacher fuzzy graphs. We then show that any complete Hamacher fuzzy graph is balanced, but the converse need not be true.

Definition 4.1. The *density* of a fuzzy Hamacher graph $G : (\sigma, \mu)$ is

$$D(G) = 2(\sum_{x,y \in V} \mu(x, y))/ \sum_{x,y \in V} (\frac{\sigma(u) + \sigma(v) - 2\sigma(u)\sigma(v)}{1 - \sigma(u)\sigma(v)}).$$

G is *balanced* if. $D(H) \leq D(G)$ for all fuzzy non-empty Hamacher subgraphs H of G.

Theorem 4.2. *Any complete fuzzy Hamacher graph is balanced.*

Proof. Let G be a complete fuzzy Hamacher graph. Then

$$\begin{aligned} D(G) &= 2(\sum_{x,y \in y} \mu(x, y))/(\sum_{x,y \in y} (\frac{\sigma(x) + \sigma(y) - 2\sigma(x)\sigma(y)}{1 - \sigma(x)\sigma(y)}) \\ &= 2(\sum_{x,y \in y} \mu(x, y))/(\sum_{x,y \in y} \mu(x, y))) \\ &= 2. \end{aligned}$$

If H is a non-empty fuzzy Hamacher subgraph of G,then

$$\begin{aligned} & D(H) \\ = \ & 2(\sum_{x,y\in y(H)} \mu(x,y))/(\sum_{x,y\in y(H)} (\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)}) \\ \leq \ & 2(\sum_{x,y\in y(H)} (\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})/(\sum_{x,y\in y(H)} (\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)}) \\ = \ & 2 \\ = \ & D(G). \end{aligned}$$

Thus G is balanced. ■

The converse of the preceding result need not be true.

Example 4.3. *The fuzzy Hamacher graph* $G : (\sigma, \mu)$ *on* $V = \{a, b, c)$ *where* $\sigma(a) = .5$, $\sigma(b) = \sigma(c) = 1$ *and* $\mu(a, b) = \mu(a, c) = \mu(b, c) = .25$ *is a balanced graph that is not complete as* $\mu(b, c) = .25 \neq 0 = \frac{\sigma(b)+\sigma(c)-2\sigma(b)\sigma(c)}{1-\sigma(b)\sigma(c)}$.

We next provide two types of fuzzy graphs with density equals 1.

Theorem 4.4. *Every self-complementary fuzzy Hamacher graph has density equals 1.*

Proof. Let G be a self-complementary fuzzy Hamacher graph. Then by Lemma 2.20, $D(G) = 2(\sum_{x,y\in V} \mu(x, y))/(\sum_{x,y\in V} (\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})$
$= 2(\sum_{x,y\in V} \mu(x, y))/(2\sum_{x,y\in V} \mu(x, y))) = 1$. ■

Theorem 4.5. *Let* $G : (\sigma, \mu)$ *be a fuzzy Hamacher graph such that* $\mu(x, y) = (1/2)(\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})$, *for all* $x, y \in V$. *Then* $D(G) = 1$.

Proof. Let $G : (\sigma, \mu)$ be a fuzzy Hamacher graph such that $\mu(u, v) = (1/2)(\frac{\sigma(x)+\sigma(y)-2\sigma(x)\sigma(y)}{1-\sigma(x)\sigma(y)})$, for all $x, y \in V$. Then by Lemma 2.8, G is self-complementary and thus by the preceding Theorem $D(G) = 1$. ■

Next, we prove the following Lemma that we use to give necessary and sufficient conditions for the direct product, semi-strong product and strong product of two fuzzy Hamacher balanced graphs to be balanced:

Lemma 4.6. *Let* G_1 *and* G_2 *be fuzzy Hamacher graphs. Then* $D(G_i) \leq D(G_1 \times G_2)$ *for* $i = 1, 2$ *if and only if* $D(G_1) = D(G_2) = D(G_1 \times G_2)$.

If $D(G_i) \leq D(G_1 \times G_2)$ for $i = 1, 2$, then if we let $a = \frac{\sigma_2(v_1)+\sigma_2(v_2)-2\sigma_2(v_1)\sigma_2(v_2)}{1-\sigma_2(v_1)\sigma_2(v_2)}$ and $b = \frac{\sigma_1(u_1)+\sigma_1(u_2)-2\sigma_1(u_1)\sigma_1(u_2)}{1-\sigma_1(u_1)\sigma_1(u_2)}$, $D(G_1) = 2(\sum_{u_1,u_2\in V_1} \mu_1(u_1, u_2)/\sum_{u_1,u_2\in V_1} b)$ and if the sum is taking over all $u_1, u_2 \in V_1, v_1, v_2 \in V_2$, then

$$\begin{aligned} & D(G_1) \\ \geq \ & 2(\sum \frac{\mu_1(u_1,u_2)+a-2\mu_1(u_1,u_2)a}{1-\mu_1(u_1,u_2)a} / \sum \frac{b+a-2ab}{1-ab}) \end{aligned}$$

$$\begin{aligned} &= 2\Big(\sum \frac{\mu_1(u_1,u_2)+\mu_2(v_1,v_2)-2\mu_1(u_1,u_2)\mu_2(v_1,v_2))}{1-\mu_1(u_1,u_2)\mu_2(v_1,v_2))} \Big/ \sum \frac{b+a-2ab}{1-ab}\Big) \\ &= 2\Big(\frac{\sum \mu_1\times\mu_2((u_1,u_2)(v_1,v_2))}{\sum \frac{(\sigma_1\times\sigma_2)(u_1,v_1)+(\sigma_1\times\sigma_2)(u_2,v_2)-2(\sigma_1\times\sigma_2)(u_1,v_1)(\sigma_1\times\sigma_2)(u_2,v_2)}{1-(\sigma_1\times\sigma_2)(u_1,v_1)(\sigma_1\times\sigma_2)(u_2,v_2)}}\Big) \\ &= D(G_1\times G_2). \end{aligned}$$

Hence $D(G_1) \geq D(G_1 \times G_2)$ and thus $D(G_1) = D(G_1 \times G_2)$. Similarly, $D(G_2) = D(G_1 \times G_2)$. Therefore, $D(G_1) = D(G_2) = D(G_1 \times G_2)$.**Proof.**

Theorem 4.7. *Let G_1 and G_2 be fuzzy Hamacher balanced graphs. Then $G_1 \times G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \times G_2)$.*

Proof. If $G_1 \times G_2$ is balanced, then $D(G_i) \leq D(G_1 \times G_2)$ for $i = 1, 2$ and by Lemma 4.6, $D(G_1) = D(G_2) = D(G_1 \times G_2)$.

Conversely, if $D(G_1) = D(G_2) = D(G_1 \times G_2)$ and H is a fuzzy Hamacher subgraph of $G_1 \times G_2$, then there exist fuzzy Hamacher subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = n_1/r_1$, then $D(H_1) = a_1/b_1 \leq n_1/r_1$ and $D(H_2) = a_2/b_2 \leq n_1/r_1$. Thus $a_1r_1 + a_2r_1 \leq b_1n_1 + b_2n_1$ and hence $D(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D(G_1 \times G_2)$. Therefore, $G_1 \times G_2$ is balanced. ■

By similar arguments to that in Lemma 4.6 and Theorem 4.7, we can prove the following result:

Theorem 4.8. *Let G_1 and G_2 be fuzzy balanced graphs. Then*
(1) $G_1 \bullet G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \bullet G_2)$.
(2) $G_1 \square G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \square G_2)$.
(3) $G_1 \blacksquare G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \blacksquare G_2)$.

We end this section by showing that isomorphism between fuzzy Hamacher graphs preserve balanced.

Theorem 4.9. *Let G_1 and G_2 be isomorphic fuzzy Hamacher graphs. If G_2 is balanced, then G_1 is balanced.*

Proof. Let $h : V_1 \to V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$. By Lemma 2.20, $\sum_{x\in V_1} \sigma_1(x) = \sum_{x\in V_2} \sigma_2(x)$ and $\sum_{x,y\in V_1} \mu_1(x, y) = \sum_{x,y\in V_2} \mu_2(x, y)$. If $H_1 = (\sigma_1', \mu_1')$ is a fuzzy Hamacher subgraph of G_1 with underlying set W, then $H_2 = (\sigma_2', \mu_2')$ is a fuzzy Hamacher subgraph of G_2 with underlying set $h(W)$ where $\sigma_2'(h(x)) = \sigma_1'(x)$ and $\mu_2'(h(x), h(y)) = \mu_1'(x, y)$ for all $x, y \in W$. Since G_2 is balanced, $D(H_2) \leq D(G_2)$ and so

$$\begin{aligned} & 2\Big(\sum_{x,y\in W} \mu_2'(h(x),h(y))\Big)\Big/\Big(\sum_{x,y\in W} \frac{\sigma_2'(x)+\sigma_2'(y)-2\sigma_2'(x)\sigma_2'(y)}{1-\sigma_2'(x)\sigma_2'(y)} \\ \leq\ & 2\Big(\sum_{x,y\in V_2} \mu_2(x,y)\Big)\Big/\Big(\sum_{x,y\in V_2} \frac{\sigma_2(x)+\sigma_2(y)-2\sigma_2(x)\sigma_2(y)}{1-\sigma_2(x)\sigma_2(y)} \end{aligned}$$

and so

$$\begin{aligned}
& 2(\sum_{x,y\in W}\mu_1(x,y))/(\sum_{x,y\in W}\frac{\sigma_2'(x)+\sigma_2'(y)-2\sigma_2'(x)\sigma_2'(y)}{1-\sigma_2'(x)\sigma_2'(y)} \\
\leq \quad & 2(\sum_{x,y\in V_1}\mu_1(x,y))/(\sum_{x,y\in V_2}\frac{\sigma_2(x)+\sigma_2(y)-2\sigma_2(x)\sigma_2(y)}{1-\sigma_2(x)\sigma_2(y)} \\
\leq \quad & D(G_1).
\end{aligned}$$

Therefore, G_1 is balanced. ■

5. Conclusion

In this paper, the new concepts of complete and balanced Hamacher fuzzy graph were introduced. It has been proved that the direct product, join and ring sum of two complete Hamacher fuzzy graphs are complete Hamacher fuzzy graphs. Moreover, these results on Hamacher fuzzy graphs are considered preserving complete property. The Dombi fuzzy graph can well describe the uncertainty of all kinds of networks.

References

1. M. Akram, D. Saleem and T. Al-Hawary, *Spherical fuzzy graphs with application to decision-making*, Math. and Comp. Appl. **25** (2020), 1-32.
2. T. Al-Hawary, *Fuzzy flats*, Indian J. Math. **55** (2013), 223-236.
3. T. Al-Hawary, S. Al-Shalaldeh and M. Akram, *Certain Matrices and Energies of Fuzzy Graphs*, TWMS J. App. Eng. Math. **11** (2022), 1-17.
4. T. Al-Hawary, *Complete fuzzy graphs*, Inter. J. Math. Cobin. **4** (2011), 26-34.
5. T. Al-Hawary, *Hamacher fuzzy graphs*, to appear in Graphs and Combinatorics.
6. T. Al-Hawary, *Density Results for Perfectly Regular and Perfectly Edge-regular Fuzzy Graphs*, to appear in J. Disc. Math. Sci. & Cryptography.
7. T. Al-Hawary, *Operations on Greedoids*, Mathematica **64(87)** (2022), 3-8.
8. T. Al-Hawary and B. Horani, *On Intuitionistic Product Fuzzy Graphs*, Ital. J. Pure. Appl. Math. **38** (2017), 113-126.
9. T. Al-Hawary and B. Horani, *On product fuzzy graphs*, Annals of fuzzy math. and Inform. **12** (2016), 279-294.
10. C. Alsina, E. Trillas and L. Valverde, *On some logical connectives for fuzzy sets theory*, J. Math. Anal. Appl. **93** (1983), 15-26.
11. S. Ashraf, S. Naz and H. Rashmanlou et al, *Regularity of graphs in single Valued neutrosophic environment*, J. Intell. Fuzzy Syst. **33** (2017), 529-542.
12. P. Bhattacharya, *Some remarks on fuzzy graphs*, Pattern Recognition Lett. **6** (1987), 297-302.
13. K.R. Bhutani and A. Battou, *On M-strong fuzzy graphs*, Inf. Sci. **155** (2003), 103-109.
14. J. Hamacher, *A General class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators*, Fuzzy Sets Syst. **8** (1982), 149-163.
15. H. Hamacher, *Uber logische Verknupfungen unscharfer Aussagen und deren Augehörige Bewertungsfunctionen*, Progress in cybernetics and systems research, **3** Hemisphere, Washington, D.C. 1978.
16. K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. **28** (1942), 535-537.
17. J.N. Mordeson and C.S. Peng, *Operations on fuzzy graphs*, Inf. Sci. **79** (1994), 159-170.

18. A. Rosenfeld, *Fuzzy graphs*, Fuzzy sets and their applications, Academic Press, New York, 1975.
19. B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North-Holland, Amsterdam, 1983.
20. M.S. Sunitha, *Complement of a fuzzy graph*, Indian J. Pure. Appl. Math. **33** (2002), 1451-1464.

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