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LEONARD PAIRS GENERATED FROM $U_q(sl_2)^{\dagger}$

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ABSTRACT. Consider the quantum algebra $U_q(sl_2)$ over field $\mathcal{F}$ ($char(\mathcal{F}) = 0$) with equitable generators $x^{\pm 1}$, y and z, where q is fixed nonzero, not root of unity scalar in $\mathcal{F}$. Let V denote a finite dimensional irreducible module for this algebra. Let $\Lambda \in End(V)$, and let $\{A_1, A_2, A_3\} = \{x, y, z\}$. First we show that if Λ, A_1 is a Leonard pair, then this Leonard pair have four types, and we show that for each type there exists a Leonard pair Λ, A_1 in which Λ is a linear combination of $1, A_2, A_3, A_2A_3$. Moreover, we use Λ to construct $\Upsilon \in U_q(sl_2)$ such that Υ, A_1^{-1} is a Leonard pair, and show that $\Upsilon = I + A_1\Phi + A_1\Psi A_1$ where Φ and Ψ are linear combination of $1, A_2, A_3$.

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1. Introduction

Leonard pairs were introduced by P. Terwilliger [10] to study the sequences of orthogonal polynomials with discrete support for which the dual sequence of polynomials is also orthogonal. These polynomials are closely related to finite dimensional representations of certain quantum groups and Lie algebras. Consequently many families of Leonard pairs are constructed from these quantum groups and Lie algebras. For examples, in [3] Leonard pairs are constructed from Universal Lie algebra $U(sl_2)$, and in [1] Leonard pairs are constructed from quantum groups $U_q(sl_2)$.

A square matrix is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. A tridiagonal matrix is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero. A square matrix is said to be *upper*

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(resp. *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal, and we say the Leonard pair A, A^* has $LB-TD$ form whenever the matrix represents A^* is lower bidiagonal with subdiagonal entries all 1 and the matrix represents A is irreducible tridiagonal.

In [8] the authors constructed a Leonard pair A, A^* in which A^* is one of the equitable generators of $U_q(sl_2)$, they assumed that the Leonard pair has $LB-TD$ form, and in [9] the author showed that the Leonard pair with this property has only three types which are q-Racah, q-Hahn, or dual q-Hahn. Moreover, the A they constructed is not free from A^*.

Our work in this paper is quite different, we don't assume that the Leonard pairs A, A^* has $LB-TD$ form, and we focus our work on the Leonard pairs A, A^* in which A^* is one of the equitable generators of $U_q(sl_2)$, and A depends only on the other generators.

We can summarize our work as follows, let $\{A_1, A_2, A_3\} = \{x, y, z\}$. First we show that if Λ, A_1 is a Leonard pair, then this Leonard pair have four types which are dual q-Hahn, quantum q-Krawtchouk, affine q-krawchouk, or dual q-krawtchouk, and we show that for each type there exists a Leonard pair Λ, A_1 in which Λ is a linear combination of $1, A_2, A_3, A_2A_3$. Moreover, we use Λ to construct $\Upsilon \in U_q(sl_2)$ such that Υ, A_1^{-1} is a Leonard pair, and show that $\Upsilon = I + A_1\Phi + A_1\Psi A_1$ where Φ and Ψ are linear combination of $1, A_2, A_3$.

In [2] we described Leonard pair A, A^* similar to what we do in this paper but we use the equitable generators of Universal Lie algebra $U(sl_2)$.

Let d denote a nonnegative integer. Throughout this paper $\mathrm{Mat}_{d+1}(\mathcal{F})$ is the set of all $(d+1)\times(d+1)$ matrices where d is nonnegative integer, $\mathcal{F}$ is a field ($char(\mathcal{F}) = 0$), and $0 \neq q \in \mathcal{F}$ is not a root of unity. Let V denote a vector space over $\mathcal{F}$ with finite dimension. By *End(V)* we mean the set of all linear transformations from V into V.

We now recall some facts concerning Leonard pairs. For more details about Leonard pairs see [4, 5, 13, 14, 15].

Definition 1.1. Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair A, A^*, where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i) and (ii) below.

(1) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.
(2) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

Let $\{v_i\}_{i=0}^d$ be the eigenvectors of A in (ii) of Definition 1.1 and let $\{\theta_i\}_{i=0}^d$ be the corresponding eigenvalues, then the ordering $\{\theta_i\}_{i=0}^d$ and $\{\theta_{d-i}\}_{i=0}^d$ are said to be standard and no further ordering is standard. same result hold $\{\theta_i^*\}_{i=0}^d$.

Definition 1.2 ([12])**.** Let d denote a non negative integer. By a *parameter array* over $\mathcal{F}$ of diameter d, we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$;

$\{\varphi_j\}_{j=1}^d$, $\{\phi_j\}_{j=1}^d$) taken from $\mathcal{F}$ that satisfy the following conditions.

$$\theta_i \neq \theta_j \qquad (0 \le i < j \le d), \tag{1}$$

$$\theta_i^* \neq \theta_j^* \qquad (0 \le i < j \le d), \tag{2}$$

$$\varphi_i \neq 0 \qquad (1 \le i \le d), \tag{3}$$

$$\phi_i \neq 0 \qquad (1 \le i \le d), \tag{4}$$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \qquad (1 \le i \le d), \tag{5}$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \le i \le d), \tag{6}$$

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{j-2}^* - \theta_{j+1}^*}{\theta_{j-1}^* - \theta_j^*} \qquad (2 \le i, j \le d-1). \tag{7}$$

In [17], the author described 13 families of parameters arrays over $\mathcal{F}$, each named according to the sequences of orthogonal polynomials associated with it. Moreover, he showed that every parameter array over $\mathcal{F}$ is one of these families.

The type of the Leonard pair is named according to the parameter array associated with it.

Theorem 1.3 ([16])**.** *Let A, A^* be a Leonard pair on V, let $(\{\theta_i\}_{i=0}^d$ (resp. $(\{\theta_i^*\}_{i=0}^d$) be standard ordering of the eigenvalues of A (resp. A^*). Via a standard basis the matrix representing A and A^* are*

$$\begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ 0 & c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & 0 & c_d & a_d \end{pmatrix}$$

and $diag(\theta_0^*, \theta_1^*, ..., \theta_d^*)$, where

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \qquad (0 \le i \le d), \tag{8}$$

$$b_i = \varphi_{i+1} \frac{\Pi_{h=0}^{i-1}(\theta_i^* - \theta_h^*)}{\Pi_{h=0}^{i}(\theta_{i+1}^* - \theta_h^*)} \qquad (0 \le i \le d-1), \tag{9}$$

$$c_i = \phi_i \frac{\Pi_{h=0}^{d-i-1}(\theta_i^* - \theta_{d-h}^*)}{\Pi_{h=0}^{d-i}(\theta_{i-1}^* - \theta_{d-h}^*)} \qquad (1 \le i \le d). \tag{10}$$

where $\varphi_0 = 0$ and $\varphi_{d+1} = 0$.

2. The Quantum algebra $U_q(sl_2)$

In this section we recall some facts concerning the quantum algebra $U_q(sl_2)$ and we state the main results in this paper.

Lemma 2.1 ([6]). *The algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathcal{F}$-algebra with generators $x^{\pm 1}, y, z$ and the following relations:*

$$xx^{-1} = x^{-1}x = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We call $x^{\pm 1}$, y, z the *equitable generators* for the quantum algebra $U_q(sl_2)$.

Lemma 2.2 ([6]). *For a nonnegative integer d, there is an irreducible finite-dimensional $U_q(sl_2)$-module V_d with basis $\mathbf{v} = \{v_0, v_1, ..., v_d\}$ such that*

$$xv_i = q^{d-2i}v_i \quad (0 \leq i \leq d),$$

$$(y - q^{2i-d}I)v_i = (q^{-d} - q^{2i+2-d})v_{i+1} \quad (0 \leq i \leq d-1),$$

$$(y - q^d I)u_d = 0,$$

$$(z - q^{2i-d}I)v_i = (q^d - q^{2i-2-d})v_{i-1} \quad (1 \leq i \leq d),$$

$$(z - q^{-d}I)v_0 = 0.$$

The main results of this paper are the following theorems

Theorem 2.3. *Let V denote an irreducible $U_q(sl_2)$-module with finite dimension. Let $\Lambda \in End(V)$, let $\{A_1, A_2, A_3\} = \{x, y, z\}$. Assume Λ, A_1 is a Leonard pair, then the type of this Leonard pair is dual q-Hahn, quantum q-Krawtchouk, affine q-krawchouk, or dual q-krawtchouk. Moreover, for each type there exists a Leonard pair Λ, A_1 in which Λ is a linear combination of* $1, A_2, A_3, A_2A_3$

Theorem 2.4. *With reference to Theorem 2.3, there exists $\Upsilon \in U_q(sl_2)$ such that Υ, A_1^{-1} is a Leonard pair, and $\Upsilon = aI + A_1\Phi + A_1\Psi A_1$ where Φ and Ψ are linear combination of* $1, A_2, A_3$.

Our work in this paper will be as follows, in the third section, we will show that if Λ, x is a Leonard pair, then the type of this pair is dual q-Hahn, quantum q-Krawtchouk, affine q-krawchouk, or dual q-krawtchouk, and for each type we construct Λ in terms of generators y and z. In section four we use the symmetry to find the linear transformations B and C such that B, y and C, z are Leonard pairs. In the last section we prove Theorem 2.4, we use Λ that appears in section three to construct Υ.

3. The Leonard pair Λ, x

Definition 3.1. Let Λ, Λ^* be a Leonard pair, let β be the common value of (7) minus one. We call β the fundamental parameter of the pair Λ, Λ^* .

Lemma 3.2. *With reference to 2.2, let $\Lambda \in End(V_d)$ such that Λ, x is a Leonard pair, then the fundamental parameter of the pair Λ, x is $\beta = q^2 + q^{-2}$.*

Proof. Note that from Lemma 2.2 the standard ordering of the eigenvalues of x is $\theta_i^* = q^{d-2i}$, $(0 \le i \le d)$, and for $(2 \le i \le d-1)$

$$\frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} = q^2 + q^{-2} + 1.$$

Hence the result hold from Definition 3.1. □

By [17], if Λ, Λ^* is a Leonard pair with fundamental parameter $\beta = q^2 + q^{-2}$, then there exist scalars a, b, c, a^*, b^*, c^*, and ζ such that the parameter array associated with this Leonard pair is

$$\begin{aligned}
\theta_i &= a + bq^{2i} + cq^{-2i} \qquad (0 \le i \le d),\\
\theta_i^* &= a^* + b^*q^{2i} + c^*q^{-2i} \qquad (0 \le i \le d),\\
\varphi_i &= (1-q^{2i})(1-q^{2d-2i+2})(\zeta - bb^*q^{2i-2} - cc^*q^{-2i-2d}) \qquad (1 \le i \le d),\\
\phi_i &= (1-q^{2i})(1-q^{2d-2i+2})(\zeta - cb^*q^{2i-2d-2} - bc^*q^{-2i}) \qquad (1 \le i \le d).
\end{aligned}$$

Moreover, four parameter arrays have the property $b^* = 0$ which are

(1) Dual q-Hahn in which $b \neq 0$, $b^* = 0$, $c \neq 0$, $c^* \neq 0$, $\zeta \neq 0$,

(2) Affine q-Krawtchouk in which $b = 0$, $b^* = 0$, $c \neq 0$, $c^* \neq 0$, $\zeta \neq 0$,

(3) Quantum q-Krawtchouk in which $b \neq 0$, $b^* = 0$, $c = 0$, $c^* \neq 0$, $\zeta \neq 0$,

(4) Dual q-Krawtchouk in which $b \neq 0$, $b^* = 0$, $c \neq 0$, $c^* \neq 0$, $\zeta = 0$.

Lemma 3.3. *Let ζ, b, $c \in \mathcal{F}$, let*

$$\begin{aligned}
\theta_i &= a + bq^{2i} + cq^{-2i} \qquad (0 \le i \le d),\\
\theta_i^* &= a^* + q^{d-2i} \qquad (0 \le i \le d),\\
\varphi_i &= (1-q^{2i})(1-q^{2d-2i+2})(\zeta - cq^{-2i-d}) \qquad (1 \le i \le d),\\
\phi_i &= (1-q^{2i})(1-q^{2d-2i+2})(\zeta - bq^{d-2i}) \qquad (1 \le i \le d).
\end{aligned}$$

Then the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ is a parameter array if and only if $q^{2i} \neq 1$, $\zeta q^{d+2i} \neq c$, $\zeta q^{2i-d} \neq b$ for $1 \le i \le d$, and that $bq^{2i-2} \neq c$ for $2 \le i \le 2d$.

Proof. Clear from the paragraph after the proof of Lemma 3.2 by taking $b^* = 0$ and $c^* = q^d$.

Note that $bq^{2i-2} \neq c$ for $2 \leq i \leq 2d$ implies that $\{\theta_i\}_{i=0}^d$ are distinct, and $q^{2i} \neq 1$, $\zeta q^{d+2i} \neq c$, $\zeta q^{2i-d} \neq b$ for $1 \leq i \leq d$ implies $\{\varphi_i\}_{i=1}^d$ and $\{\phi_i\}_{i=1}^d$ are nonzero. Moreover, if $b = 0$, then the condition $\zeta q^{2i-d} \neq b$ become $\zeta \neq 0$, and if $c = 0$, then the condition $\zeta q^{d+2i} \neq c$ become $\zeta \neq 0$. Also note that if one of b or c is zero but not both, then $\{\theta_i\}_{i=0}^d$ are distinct.

□

Now it is clear that the type of the Leonard pair Λ, x in Lemma 3.2 is dual q-Hahn, quantum q-Krawtchouk, affine q-krawchouk, or dual q-krawtchouk.

In the following work, we will find Λ in terms of the equitable generators of $U_q(sl_2)$ in each case. For the rest of the paper, let d be an integer ($d \geq 2$), $\Lambda \in U_q(sl_2)$, and V is an irreducible finite-dimensional $U_q(sl_2)$-module.

Definition 3.4. With reference to Lemma 2.2, let

$$\Omega = e_I I + e_y y + e_z z + e_{yz} yz,$$

where e_I, e_y, e_z, $e_{yz} \in \mathcal{F}$.

Lemma 3.5. *With reference to Lemma 3.2, if the Leonard pair Λ, x has type dual q-Hahn , then V have a basis $\mathbf{s} = \{s_i\}_{i=0}^d$ such that $[\Lambda]_\mathbf{s}$ is tridiagonal where*

$$[A]_\mathbf{s}(i,i) = q^{2i-d-2}(\zeta(q^{2d+2}-q^{2i}-q^{2i-2}+1)+cq^{-d})+bq^{2i-2}+a \qquad (1 \leq i \leq d+1),$$
$$[A]_\mathbf{s}(i+1,i) = (\zeta q^{2i-d} - b)(q^{2i}-1) \qquad (1 \leq i \leq d),$$
$$[A]_\mathbf{s}(i,i+1) = (q^{2i-2d-2}-1)(\zeta q^{d+2i} - c) \qquad (1 \leq i \leq d),$$

and $[x]_\mathbf{s} = diag\{q^d, q^{d-2}..., q^{2-d}, q^{-d}\}$ for some nonzero b, c, and $\zeta \in \mathcal{F}$ such that none of q^{2i}, $\zeta c^{-1} q^{d+2i}$, $\zeta b^{-1} q^{2i-d}$ is equal to 1 for $1 \leq i \leq d$, and that $bc^{-1} q^{2i-2} \neq 1$ for $2 \leq i \leq 2d$.

Proof. The parameter array associated with the pair Λ, x is given in Lemma 3.3. Let $a_0^* = 0$, let V be the module in Theorem 1.3 and let $\mathbf{s}$ be it is standard basis. Now the eigenvalues $\theta_i^* = q^{d-2i}$ $(0 \leq i \leq d)$, hence $[x]_\mathbf{s} = diag\{q^d, q^{d-2}..., q^{2-d}, q^{-d}\}$, and the entries of the matrix $[\Lambda]_\mathbf{s}$ hold by formulas in this theorem. □

Theorem 3.6 ([16])**.** *Let d denote a nonnegative integer, let B and B^* denote matrices in* $\mathrm{Mat}_{d+1}(\mathcal{F})$. *Assume B is lower bidiagonal and B^* is upper bidiagonal. Then the following are equivalent.*

(1) *The pair B, B^* is a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathcal{F})$.
(2) *There exists a parameter array* $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ *over $\mathcal{F}$ such that*

$$B(i,i) = \theta_i, \qquad B^*(i,i) = \theta_i^* \qquad (0 \leq i \leq d),$$
$$B(j,j-1)B^*(j-1,j) = \varphi_j \qquad (1 \leq j \leq d).$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *is uniquely determined by B, B^*.*

Lemma 3.7. *With reference to Lemma 3.5, the pair* $[\Lambda]_{\mathbf{s}}$, $[x]_{\mathbf{s}}$ *is a Leonard pair in* $Mat_{d+1}(\mathcal{F})$ *if and only if* b, c, *and* $\zeta \in \mathcal{F}$ *are nonzero, and* $q^{2i} \neq 1$, $\zeta q^{d+2i} \neq c$, $\zeta q^{2i-d} \neq b$ *for* $1 \leq i \leq d$, *and that* $bc^{-1}q^{2i-2} \neq 1$ *for* $2 \leq i \leq 2d$.

Proof. Let T be $(d+1) \times (d+1)$ matrix indexed $1, 2, ..., d+1$ such that the (i, j)-entry is

$$T_{ij} = \begin{cases} (-1)^{i+1} q^{i(i-1)-2d(j-1)} \prod_{t=i}^{j-1} (\zeta q^{2t+d} - c)(q^{2d-2t+2} - 1) & \\ \prod_{m=j-i}^{j-2} (q^{2m+2} - 1) \prod_{n=1}^{i-1} (bq^{2d-2n} - q^d \zeta) & j \geq i \\ & \\ 0 & j < i \end{cases}$$

then the matrix $T^{-1}[\Lambda]_{\mathbf{s}}T$ is lower bidiagonal with entries $T^{-1}[\Lambda]_{\mathbf{s}}T(i,i) = \theta_{i-1}$, $(1 \leq i \leq d+1)$, $T^{-1}[\Lambda]_{\mathbf{s}}T(i+1, i) = 1$, $(1 \leq i \leq d)$, and the matrix $T^{-1}[x]_{\mathbf{s}}T$ is upper bidiagonal with entries $T^{-1}[x]_{\mathbf{s}}T(i,i) = \theta^*_{i-1}$, $(1 \leq i \leq d+1)$, $T^{-1}[x]_{\mathbf{s}}T(i, i+1) = \varphi_i$, $(1 \leq i \leq d)$, where $\{\theta_i\}_{i=0}^d$, $\{\theta^*_i\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ are as in Lemma 3.3 and these entries satisfy (ii) in Theorem 3.6. Hence, $T^{-1}[\Lambda]_{\mathbf{s}}T$, $T^{-1}[x]_{\mathbf{s}}T$ is a Leonard pair if and only if b, c, and $\zeta \in \mathcal{F}$ are nonzero, and none of q^{2i}, $\zeta c^{-1}q^{d+2i}$, $\zeta b^{-1}q^{2i-d}$ is equal to 1 for $1 \leq i \leq d$, and that $bc^{-1}q^{2i-2} \neq 1$ for $2 \leq i \leq 2d$ hold, which implies that the pair $[\Lambda]_{\mathbf{s}}$, $[x]_{\mathbf{s}}$ is a Leonard pair if and only if the same conditions hold. $\square$

Lemma 3.8. *With reference to Definition 3.4,* Ω, x *is a Leonard pair of dual q-Hahn type if and only if* $e_y \neq 0$, $e_z \neq 0$, $e_{yz} \neq 0$, $e_{yz} + q^{d-2i+2}e_z \neq 0$, $e_{yz} + q^{2i-d}e_y \neq 0$, $q^{2i} \neq 1$ *for* $1 \leq i \leq d$, *and* $e_z - q^{d-2i+2}e_y \neq 0$ *for* $2 \leq i \leq 2d$.

Proof. The standard ordering of the eigenvalues of x with respect to the basis $\{s_i\}_{i=0}^d$ in Lemma 3.5 is $\{q^d, q^{d-2}..., q^{2-d}, q^{-d}\}$ which is the same standard ordering with respect to the basis $\mathbf{v}$ in Lemma 2.2, hence $\{s_i = k_i v_i\}_{i=0}^d$ for some nonzero scalars $k_i \in \mathcal{F}$. Let $k_i = 1$ $(0 \leq i \leq d)$, then routine calculations shows that $[\Lambda]_{\mathbf{s}} = [\Omega]_{\mathbf{v}}$ if and only if

$$e_I = a + q^{d+2}\zeta, \qquad e_z = cq^{-d},$$
$$e_y = q^d b, \qquad e_{yz} = -q^{d+2}\zeta.$$

Note that $\zeta c^{-1}q^{d+2i} \neq 1$ implies $e_{yz} + q^{d-2i+2}e_z \neq 0$, $\zeta b^{-1}q^{2i-d} \neq 1$ implies $e_{yz} + q^{d-2i+2}e_y \neq 0$, and $bc^{-1}q^{2i-2} \neq 1$ implies $e_y - q^{2i-2d-2}e_z \neq 0$. $\square$

For the three other cases note that the parameter arrays associated with them are similar to one associated with dual q-Hahn but some scalars become zeros.

We start with Leonard pair of affine q-krawchouk type, let $b = 0$ in the parameter array associated with dual q-Hahn, then the result parameter array will be of affine q-krawchouk type, hence we have the following Lemmas

Lemma 3.9. *With reference to Lemma 3.2, if the Leonard pair* Λ, x *has type affine q-krawchouk , then* V *have a basis* $\mathbf{s} = \{s_i\}_{i=0}^{d}$ *such that* $[\Lambda]_{\mathbf{s}}$ *is tridiagonal where*

$$[\Lambda]_{\mathbf{s}}(i,i) = q^{2i-d-2}(\zeta(q^{2d+2} - q^{2i-2} - q^{2i} + 1) + q^{-d}c) + a \qquad (1 \leq i \leq d+1),$$
$$[\Lambda]_{\mathbf{s}}(i+1,i) = \zeta q^{2i-d}(q^{2i} - 1) \qquad (1 \leq i \leq d),$$
$$[\Lambda]_{\mathbf{s}}(i,i+1) = (q^{2i-2d-2} - 1)(\zeta q^{2i+d} - c) \qquad (1 \leq i \leq d),$$

and $[x]_{\mathbf{s}} = diag\{q^d, q^{d-2} ..., q^{2-d}, q^{-d}\}$ *for some nonzero* c, *and* $\zeta \in \mathcal{F}$, *and none of* q^{2i}, $\zeta c^{-1} q^{2i+d}$ *is equal to* 1 *for* $1 \leq i \leq d$.

Proof. The parameter array associated with the pair Λ, x is given in Lemma 3.3 which is the same of the parameter array of dual q-Hahn but $b = 0$. Hence, let $b = 0$ in Lemma 3.5 to get the result.

□

Lemma 3.10. *With reference to Lemma 3.9, the pair* $[\Lambda]_{\mathbf{s}}$, $[x]_{\mathbf{s}}$ *is a Leonard pair in* $Mat_{d+1}(\mathcal{F})$ *if and only if* c, *and* $\zeta \in \mathcal{F}$ *are nonzero, and none of* q^{2i}, $\zeta c^{-1} q^{2i+d}$ *is equal to* 1 *for* $1 \leq i \leq d$.

Proof. Clear from Lemmas 3.7 and 3.9 □

Lemma 3.11. *With reference to Definition 3.4,* Ω, x *is a Leonard pair of affine q-krawchouk type if and only if* $e_y = 0$, $e_z \neq 0$, $e_{yz} \neq 0$, $e_{yz} + q^{d-2i+2} e_z \neq 0$, *and* $q^{2i} \neq 1$ *for* $1 \leq i \leq d$.

Proof. Let $b = 0$ in 3.8 and note that $[\Lambda]_{\mathbf{s}} = [\Omega]_{\mathbf{v}}$ if and only if

$$e_I = a + q^{d+2}\zeta, \qquad e_z = cq^{-d},$$
$$e_y = 0, \qquad e_{yz} = -q^{d+2}\zeta.$$

Now $\zeta c^{-1} q^{d+2i} \neq 1$ implies $\frac{-q^{2i} e_{yz}}{q^{d+2} e_z} \neq 1$ implies $e_{yz} + q^{d-2i+2} e_z \neq 0$ for $1 \leq i \leq d$.

So the result hold from Lemma 3.10. □

For th Leonard pair of quantum q-Krawtchouk type, let $c = 0$ in the in parameter array associated with dual q-Hahn, then the result parameter array will be of quantum q-Krawtchouk type, hence we have the following Lemmas

Lemma 3.12. *With reference to Lemma 3.2, assume that the type of the pair* Λ, x *is a quantum q-Krawtchouk , then the module* V *have a basis* $\mathbf{s} = \{s_i\}_{i=0}^{d}$ *such that* $[\Lambda]_{\mathbf{s}}$ *is tridiagonal where*

$$[\Lambda]_{\mathbf{s}}(i,i) = q^{2i-d-2}(\zeta(q^{2d+2} - q^{2i} - q^{2i-2} + 1)) + bq^{2i-2} + a \qquad (1 \leq i \leq d+1),$$

$$[\Lambda]_{\mathbf{s}}(i+1,i) = (\zeta q^{2i-d} - b)(q^{2i}-1) \qquad (1 \le i \le d),$$
$$[\Lambda]_{\mathbf{s}}(i,i+1) = \zeta q^{d+2i}(q^{2i-2d-2}-1) \qquad (1 \le i \le d),$$

and $[x]_{\mathbf{s}} = diag\{q^d, q^{d-2} ..., q^{2-d}, q^{-d}\}$ *for some nonzero* b*, and* $\zeta \in \mathcal{F}$ *such that none of* q^{2i}*,* $\zeta b^{-1} q^{2i-d}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. The parameter array associated with the Leonard pair Λ, x is given in Lemma 3.3, let $c = 0$ in Lemma 3.5 to get the result.

□

Lemma 3.13. *With reference to Lemma 3.12, the pair* $[\Lambda]_{\mathbf{s}}$*,* $[x]_{\mathbf{s}}$ *is a Leonard pair in* $Mat_{d+1}(\mathcal{F})$ *if and only if* b*, and* $\zeta \in \mathcal{F}$ *are nonzero, and none of* q^{2i}*,* $\zeta b^{-1} q^{2i-d}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. Clear from Lemmas 3.7 and 3.12.

□

Lemma 3.14. *With reference to Definition 3.4,* Ω*,* x *is a Leonard pair of quantum q-Krawtchouk type if and only if* $e_y \neq 0$*,* $e_z = 0$*,* $e_{yz} \neq 0$*,* $e_{yz} + q^{2i-d} e_y \neq 0$ *for* $1 \le i \le d$*.*

Proof. Let $c = 0$ in 3.8 and note that $[\Lambda]_{\mathbf{s}} = [\Omega]_{\mathbf{v}}$ if and only if

$$e_I = a + q^{d+2}\zeta, \qquad e_z = 0,$$
$$e_y = q^d b, \qquad e_{yz} = -q^{d+2}\zeta.$$

Now $\zeta b^{-1} q^{2i-d}$ implies $e_y + q^{2i-d-2} e_{yz} \neq 0$, So the result hold from Lemma 3.13.

□

Lemma 3.15. *With reference to Lemma 3.2, assume that the pair* Λ*,* x *has dual q-Krawtchouk type, then the module* V *have a basis* $\mathbf{s} = \{s_i\}_{i=0}^d$ *such that* $[\Lambda]_{\mathbf{s}}$ *is tridiagonal where*

$$[\Lambda]_{\mathbf{s}}(i,i) = cq^{2i-2d-2} + bq^{2i-2} + a \qquad (1 \le i \le d+1),$$
$$[\Lambda]_{\mathbf{s}}(i+1,i) = -b(q^{2i}-1) \qquad (1 \le i \le d),$$
$$[\Lambda]_{\mathbf{s}}(i,i+1) = -c(q^{2i-2d-2}-1) \qquad (1 \le i \le d),$$

and $[x]_{\mathbf{s}} = diag\{q^d, q^{d-2} ..., q^{2-d}, q^{-d}\}$ *for some nonzero* b*, and* $c \in \mathcal{F}$ *such that* $q^{2i} \neq 1$*, for* $1 \le i \le d$*, and that* $bc^{-1} q^{2i-2} \neq 1$ *for* $2 \le i \le 2d$*.*

Proof. For the Leonard pair of dual q-Krawtchouk type, let $\zeta = 0$ in the in parameter array associated with dual q-Hahn, then the result parameter array will be of dual q-Krawtchouk type, hence taking $\zeta = 0$ in Lemma 3.5 gives the result.

□

Lemma 3.16. *With reference to Lemma 3.15, the pair* $[\Lambda]_{\mathbf{s}}$, $[x]_{\mathbf{s}}$ *is a Leonard pair in* $Mat_{d+1}(\mathcal{F})$ *if and only if* b, *and* $c \in \mathcal{F}$ *are nonzero, and none of* $q^{2i} \neq 1$, *for* $1 \leq i \leq d$, *and* $bc^{-1}q^{2i-2} \neq 1$ *for* $2 \leq i \leq 2d$.

Proof. Clear from Lemmas 3.7 and 3.15. □

Lemma 3.17. *With reference to Definition 3.4,* Ω, x *is a Leonard pair of dual q-Krawtchouk type if and only if* $e_y \neq 0$, $e_z \neq 0$, $e_{yz} = 0$, $e_z - q^{2i-2d-2}e_y \neq 0$ *for* $2 \leq i \leq 2d$.

Proof. Let $\zeta = 0$ in 3.8, to see that $[\Lambda]_{\mathbf{s}} = [\Omega]_{\mathbf{v}}$ if and only if

$$e_I = a, \qquad e_z = q^{-d}c,$$
$$e_y = q^d b, \qquad e_{yz} = 0.$$

Note that $bc^{-1}q^{2i-2} \neq 1$ implies $e_z - q^{2i-2d-2}e_y \neq 0$, so the result hold from Lemma 3.16. □

4. The Leonard pairs B, y and C, z

In this section we find B and C in terms of the equitable generators of $U_q(sl_2)$ such that B, y and C, z are Leonard pairs. We recall some formulas that we will use in this section.

For all integers k and for all nonnegative integers n, m write

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \qquad [n]! = [1][2]\cdots[n],$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[n-m]![m]!} & n \geq m \\ 0 & n < m \end{cases}$$

and let

$$P_{ij} = (-1)^j q^{(j-d)(i-1)} \begin{bmatrix} i \\ d-j \end{bmatrix}, \quad Q_{ij} = (-1)^j q^{j(d-i-1)} \begin{bmatrix} d-i \\ j \end{bmatrix} \quad (0 \leq i, j \leq d).$$

Lemma 4.1 ([11])**.** *With reference to Lemma 2.2,*

(1) *Let* $u_j = \sum_{i=d-j}^{d} P_{ij}v_i$ $(0 \leq j \leq d)$, *then* $\mathbf{u} = \{u_0, u_1, \ldots, u_d\}$ *is a standard* y*-eigenbasis of* V_d.
(2) *Let* $w_j = \sum_{i=0}^{d-j} Q_{ij}v_i$ $(0 \leq j \leq d)$, *then* $\mathbf{w} = \{w_0, w_1, \ldots, w_d\}$ *is a standard* z*-eigenbasis of* V_d.

Lemma 4.2 ([11])**.** *With reference to Lemma 4.1,*

$$[z]_{\mathbf{v}} = [x]_{\mathbf{u}} = [y]_{\mathbf{w}}, \qquad [x]_{\mathbf{v}} = [y]_{\mathbf{u}} = [z]_{\mathbf{w}}, \qquad [y]_{\mathbf{v}} = [z]_{\mathbf{u}} = [x]_{\mathbf{w}}.$$

Lemma 4.3. *Let k_0, k_1, k_2 and k_3 be scalars in $\mathcal{F}$, let*

$$A = k_0I{+}k_1y{+}k_2z{+}k_3yz, \quad B = k_0I{+}k_1z{+}k_2x{+}k_3zx, \quad C = k_0I{+}k_1x{+}k_2y{+}k_3xy.$$

If $[A]_{\mathbf{v}}$, $[x]_{\mathbf{v}}$ is a Leonard pair, then both $[B]_{\mathbf{u}}$, $[y]_{\mathbf{u}}$ and $[C]_{\mathbf{w}}$, $[z]_{\mathbf{w}}$ are Leonard pairs. Moreover the three pairs have the same type.

Proof. Clear from Lemma 4.2 and note that $[A]_{\mathbf{v}} = [B]_{\mathbf{u}} = [C]_{\mathbf{w}}$. □

proof of Theorem 2.3. Clear from Lemmas 3.8, 3.11, 3.14, and 3.17, and 4.3. □

5. The Leonard pair Υ, $\mathbf{x^{-1}}$

In this section we find Υ in terms of the equitable generators of $U_q(sl_2)$ such that Υ, x^{-1} is a Leonard pair. By [16], if $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array, then $(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d; \{\varphi_{d-j+1}\}_{j=1}^d, \{\phi_{d-j+1}\}_{j=1}^d)$ is also a parameter array.

We will find Υ for the case A, x is a Leonard pair of affine q-Krawchouk type, the other cases will be similar.

Lemma 5.1. *Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ be a parameter array of affine q-Krawchouk type, replace q by q^{-1} and let*

$$\begin{aligned}
\epsilon_i &= a + c'q^{2i} \qquad (0 \le i \le d),\\
\epsilon_i^* &= a^* + (c^*)'q^{2i} \qquad (0 \le i \le d),\\
\varphi_i' &= (1-q^{-2i})(1-q^{-2d+2i-2})(\zeta - c'(c^*)'q^{2i+2d}) \qquad (1 \le i \le d),\\
\phi_i' &= (1-q^{-2i})(1-q^{-2d+2i-2})\zeta' \qquad (1 \le i \le d),
\end{aligned}$$

for some nonzero c', $(c^*)'$, and $\zeta' \in \mathcal{F}$, and none of q^{-2i}, $\zeta'(c'(c^*)')^{-1}q^{-2d-2i}$ is equal to 1 for $1 \le i \le d$. Then $\Phi = (\{\epsilon_i\}_{i=0}^d, \{\epsilon_i^*\}_{i=0}^d; \{\varphi'\}_{j=1}^d, \{\phi'\}_{j=1}^d)$ is a parameter array.

Proof. Replace c' by cq^{-2d}, $(c^*)'$ by c^*q^{-2d}, ζ' by ζq^{-2d-2} to show that $\epsilon_i = \theta_{d-i}$, $\epsilon_i^* = \theta_{d-i}^*$, $\varphi_i' = \varphi_{d-i+1}$ and $\phi_i' = \phi_{d-i+1}$. where $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array of affine q-Krawchouk type given in the paragraph after Lemma 3.2. Hence the result hold.

□

In the next work we will use the notation c, c^* and ζ instead of c', $(c^*)'$ and ζ'.

Lemma 5.2. *Let B, B^* be the Leonard pair associated with the parameter array in Lemma 5.1, then there is an irreducible finite-dimensional $U_q(sl_2)$-module V_d with basis $\mathbf{t} = \{t_i\}_{i=0}^d$ such that $[B]_{\mathbf{t}}$ is tridiagonal where*

$$[B]_{\mathbf{t}}(i,i) = q^{d-2i+2}\zeta(q^{-2d-2}{-}q^{-2i+2}{-}q^{-2i}{+}1){+}cq^{2d-2i+2}{+}a \qquad (1 \le i \le d{+}1),$$

$$[B]_{\mathbf{t}}(i+1,i) = \zeta q^{d-2i}(q^{-2i}-1) \qquad (1 \le i \le d),$$

$$[B]_{\mathbf{t}}(i,i+1) = (q^{2d-2i+2}-1)(\zeta q^{-d-2i}-c) \qquad (1 \le i \le d),$$

and $[B^*]_{\mathbf{t}} = diag\{q^{-d}, q^{2-d}, ..., q^{d-2}, q^d\}$ *for some nonzero* c*, and* $\zeta \in \mathcal{F}$*, and none of* q^{-2i}*,* $\zeta c^{-1} q^{-d-2i}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. The parameter array associated with the pair B, B^* in Lemma 5.1 is obtained by replacing q by q^{-1} in the parameter array associated with Leonard pair of affine q-Krawchouk type, so by replacing q by q^{-1} in Lemma 3.9 and taking $c^* = q^{-d}$. we can see that $[B^*]_{\mathbf{t}} = diag\{q^{-d}, q^{2-d}, ..., q^{d-2}, q^d\}$, and the entries of the matrix $[B]_{\mathbf{t}}$ as they appear in the lemma. □

Lemma 5.3. *With reference to Lemma 5.2, the pair* $[B]_{\mathbf{t}}$*,* $[B^*]_{\mathbf{t}}$ *is a Leonard pair iff* c*, and* $\zeta \in \mathcal{F}$ *are nonzero, and none of* q^{-2i}*,* $\zeta c^{-1} q^{-d-2i}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. Similar to proof of Lemma 3.7 but take $b = 0$ and replace q by q^{-1}. □

Lemma 5.4. *Let* $\Upsilon \in U_q(sl_2)$ *such that* Υ*,* x^{-1} *is a Leonard pair with parameter array as in Lemma 5.1, then* $\Upsilon = a_I I + x(b_I I + b_z z) + x(c_I I + c_y y + c_z z)x$ *where*

$$a_I = a + c, \qquad b_I = \zeta(1 - q^{-2})(1 - q^{-2d}) + cq^d, \qquad b_z = -c,$$
$$c_I = -\zeta q^{-d}(1 + q^{-2}), \qquad c_y = \zeta q^{-2}, \qquad c_z = \zeta q^{-2d}.$$

for nonzero c*, and* $\zeta \in \mathcal{F}$*, and none of* q^{-2i}*,* $\zeta c^{-1} q^{-d-2i}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. The action of the generators x, y and z on the basis $\mathbf{v} = \{v_i\}_{i=0}^d$ is described in 2.2, now routine calculation shows that $[\Upsilon]_{\mathbf{v}} = [B]_{\mathbf{t}}$ and $[x^{-1}]_{\mathbf{v}} = [B^*]_{\mathbf{t}}$ as in Lemma 5.2. □

Lemma 5.5. *Let* $\Upsilon \in U_q(sl_2)$ *such that* $\Upsilon = a_I I + x(b_I I + b_z z) + x(c_I I + c_y y + c_z z)x$*, then* Υ*,* x^{-1} *is a Leonard pair iff there exist nonzero* r*, and* $s \in \mathcal{F}$ *such that*

$$b_I = s(1 - q^{-2})(1 - q^{-2d}) + rq^d, \qquad b_z = -r,$$
$$c_I = -sq^{-d}(1 + q^{-2}), \qquad c_y = sq^{-2}, \qquad c_z = sq^{-2d}.$$

and none of q^{-2i}*,* $sr^{-1}q^{-d-2i}$ *is equal to* 1 *for* $1 \le i \le d$*.*

Proof. Clear from Lemmas 5.3, and 5.4, let $s = \zeta$, $r = c$. □

For the other types of Leonard pairs Λ, x that appear in previous sections, we can use same work to find $\Upsilon \in U_q(sl_2)$ such that Υ, x^{-1} is a Leonard pair, we summarize this in the following lemma

Lemma 5.6. *Assume that* A*,* x *is a Leonard pair, then there exists* $\Upsilon \in U_q(sl_2)$*, such that* Υ*,* x^{-1} *is a Leonard pair. Moreover*

(1) *If the type of* A*,* x *is dual q-Hahn, then* $\Upsilon = aI + x\Phi + x\Psi x$ *where* Φ *and* Ψ *are linear combination of* I, y, z*.*
(2) *If the type of* A*,* x *is quantum q-Krawtchouk, then* $\Upsilon = aI + x\Phi + x\Psi x$ *where* Φ *is a linear combination of* I, y*, and* Ψ *is a linear combination of* I, y, z*.*

(3) *If the type of A, x is dual q-Krawtchouk, then $\Upsilon = aI + x\Phi$ where Φ is a linear combination of I, y, z.*

Proof. Similar to proof of Lemma 5.4, use Lemmas 3.8, 3.14 and 3.17. $\square$

proof of Theorem 2.4. Clear from Lemmas 5.4, 5.6 and the fact in the paragraph after Lemma 4.1. $\square$

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