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# ON TRANSLATION SURFACES WITH ZERO GAUSSIAN CURVATURE IN LORENTZIAN SOL$_3$ SPACE

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Abstract. In this work we classified translation invariant surfaces with zero Gaussian curvature in the 3−dimensional Sol Lie group endowed with Lorentzian metric.

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## 1. Introduction

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in the homogeneous space $Sol_3$ focusing on minimal and constant mean curvature and totally umbilic surfaces. This was initiated by R.Souam and E.Toubiana [24, 25], and by R.Lopez and M.I.Munteanu [14, 15] . More general many works are devoted to studying the geometry of surfaces in 3-homogeneous space $Sol_3$. See for example [16],[13],[18],[10],[19].
The $Sol_3$ geometry is eight models geometry of Thurston, see [27] .It is a Lie group endowed with a left-invariant metric, it is a homogeneous simply connected 3−manifold with a 3−dimensional isometry group, see [8].It is isometric to $\mathbb{R}^3$ equipped with the Lorentzian metric

$$ds^2 = e^{2z}dx^2 - e^{-2z}dy^2 + dz^2.$$

where $(x, y, z)$ the usual coordinates of $\mathbb{R}^3$.
The group structure of $Sol_3$ is given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z').$$

The isometries are

$$(x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c)$$

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where $a, b$ end $c$ are any real numbers.

A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in the Lorentzian $Sol_3$ Lie group is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}. \tag{1}$$

The Levi-Civita connection $\nabla$ of the Lorentzian $Sol_3$ Lie group with respect to this frame is

$$\begin{cases} \nabla_{E_1}E_1 = -E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = E_1 \\ \nabla_{E_2}E_1 = 0, \nabla_{E_2}E_2 = -E_3, \nabla_{E_2}E_3 = -E_2 \\ \nabla_{E_3}E_1 = 0, \nabla_{E_3}E_2 = 0, \nabla_{E_3}E_3 = 0. \end{cases} \tag{2}$$

The non-vanishing curvature tensor $R$ components are computed as

$$\begin{cases} R(E_1, E_2)E_1 = -E_2, R(E_1, E_2)E_2 = -E_1 \\ R(E_1, E_3)E_1 = E_3, R(E_1, E_3)E_3 = -E_1 \\ R(E_2, E_3)E_2 = -E_3, R(E_2, E_3)E_3 = -E_2. \end{cases} \tag{3}$$

The Ricci curvature components $\{Ric_{ij}\}$ are computed as

$$Ric_{11} = Ric_{12} = Ric_{13} = Ric_{23} = Ric_{22} = 0,\ \ Ric_{33} = -2. \tag{4}$$

The scalar curvature $\tau$ of the Lorentzian $Sol_3$ Lie group is constant and we have

$$\tau = trRic = \sum_{i=1}^{3} g(E_i, E_i)Ric(E_i, E_i) = -2. \tag{5}$$

## 2. Flat Translation Surfaces in Lorentzian Sol$_3$ space

**2.1.** In this section we classified complete flat translation surfaces $(\Sigma)$ in Lorentzian $Sol_3$ space which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x, y + c, z)$. Clearly, such a surface is generated by a curve $\gamma$ in the totally geodesic plane $\{y = 0\}$. Discarding the trivial case of a vertical plane $\{x = x_0\}$, we can assume that $\gamma$ is locally is a graph over the $x-$axis. Thus $\gamma$ is given by $\gamma(x) = (x, 0, z(x))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, y, z(x)),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on $(\Sigma)$, namely,

$$e_1 := X_x = (1, 0, z') = e^{z}E_1 + z'E_3.$$

and

$$e_2 := X_y = (0, 1, 0) = e^{-z}E_2.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= z'^2 + e^{2z},\ F =< e_1, e_2 >= 0,\ G =< e_2, e >= -e^{-2z}.$$

As a unit normal field we can take

$$N = \frac{-z'e^{-z}}{\sqrt{1+z'^2e^{-2z}}}E_1 + \frac{1}{\sqrt{1+z'^2e^{-2z}}}E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1}e_1 = 2z'e^zE_1 + (z'' - e^{2z})E_3$$
$$\widetilde{\nabla}_{e_1}e_2 = -z'e^{-z}E_2$$
$$\widetilde{\nabla}_{e_2}e_2 = -e^{-2z}E_3.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1}e_1, N >= \frac{-2z'^2 + z'' - e^{2z}}{\sqrt{1+z'^2e^{-2z}}}$$
$$m =< \widetilde{\nabla}_{e_1}e_2, N >= 0$$
$$n =< \widetilde{\nabla}_{e_2}e_2, N >= \frac{-e^{-2z}}{\sqrt{1+z'^2e^{-2z}}}.$$

Let $K_{ext}$ be the extrinsic Gauss curvature of $(\Sigma)$,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = -\frac{-2z'^2e^{-2z} + z''e^{-2z} - 1}{(1+z'^2e^{-2z})^2}. \tag{6}$$

In order to obtain the intrinsic Gauss curvature $K_{int}$, recall that $K_{int} = K_{ext} + K(e_1 \wedge e_2)$, where $K(e_1 \wedge e_2)$ is the sectional curvature of each tangent plane spanned by $e_1$ and $e_2$, and

$$K(e_1 \wedge e_2) = \frac{\langle R(e_1, e_2)e_2, e_1\rangle}{< e_1, e_1 >< e_2, e_2 > - < e_1, e_2 >^2}$$

where

$$R(e_1, e_2)e_2 = \widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_2}e_2 - \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_1}e_2 - \widetilde{\nabla}_{[e_1,e_2]}e_2$$

Now we easily compute

$$\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_2}e_2 = -e^{-z}E_1 + 2z'e^{-2z}E_3$$
$$\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_1}e_2 = z'e^{-2z}E_3$$
$$\widetilde{\nabla}_{[e_1,e_2]}e_2 = 0.$$

Thus we have

$$K(e_1 \wedge e_2) = -\frac{1 - z'^2e^{-2z}}{1 + z'^2e^{-2z}}.$$

Consequently, the intrinsic Gauss curvature is

$$K_{int} = -\frac{e^{-2z}[z'' - 2z'^2 - z'^4e^{-2z}]}{(1+z'^2e^{-2z})^2}. \tag{7}$$

So that $(\Sigma)$ is a flat surface in Lorentzian $Sol_3$ if and only if

$$K_{int} = 0,$$

that is, if and only if

$$z'' - 2z'^2 - z'^4e^{-2z} = 0 \tag{8}$$

to classify flat surfaces must solve the equation (8)
We note that for $z$ equal to a constant ($z = z_0$) is a solution of the equation (8).
If $z$ is not constant ($z' \neq 0$),suppose that $z' = p$, and

$$z'' = \frac{dp}{dx} = \frac{dp}{dz}\frac{dz}{dx} = p.p'(z)$$

equation (8) becomes

$$p.p' = 2p^2 + p^4 e^{-2z}.$$

or

$$p^{-3}.p' = 2p^{-2} + e^{-2z}. \tag{9}$$

and suppose that $p^{-2} = h$, equation (9) becomes

$$\frac{-1}{2}h' = 2h + e^{-2z}. \tag{10}$$

homogeneous solutions of equation (10) is

$$h(z) = K.e^{-4z}.$$

and general solutions of the equation (10) is

$$h(z) = e^{-4z}(a - e^{2z}),$$

where $a \in \mathbb{R}^{*,+}$ and $z \in ]-\infty, \ln(\sqrt{a})[$.Therefore

$$p(z) = \pm\frac{1}{\sqrt{h(z)}} = \pm\frac{e^{2z}}{\sqrt{a - e^{2z}}}.$$

and we have

$$z' = \pm\frac{e^{2z}}{\sqrt{a - e^{2z}}}.$$

or

$$\frac{dz}{dx} = \pm\frac{e^{2z}}{\sqrt{a - e^{2z}}}$$

so separating variables, we obtain

$$\int dx = \int \pm\frac{\sqrt{a - e^{2z}}}{e^{2z}}dz$$

i.e

$$x = \pm\int \frac{\sqrt{a - e^{2z}}}{e^{2z}}dz + \alpha,$$

where $\alpha \in \mathbb{R}$.
we substitute $\tanh(t) = \frac{\sqrt{a - e^{2z}}}{\sqrt{a}}$, $dz = -\tanh(t)dt$, and $e^{2z} = \frac{a}{\cosh^2(t)}$, therefore

$$\int \frac{\sqrt{a - e^{2z}}}{e^{2z}}dz = -\frac{-1}{\sqrt{a}}\int \sinh^2(t)dt = -\frac{1}{8\sqrt{a}}[e^{2t} - e^{-2t}] + \frac{t}{2\sqrt{a}},$$

and as $t = arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}}{1-\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}}\right)$, thus

$$\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz = -\frac{1}{8\sqrt{a}}\left[\left(\frac{\sqrt{a}+\sqrt{a-e^{2z}}}{\sqrt{a}-\sqrt{a-e^{2z}}}\right) - \left(\frac{\sqrt{a}-\sqrt{a-e^{2z}}}{\sqrt{a}+\sqrt{a-e^{2z}}}\right)\right]$$
$$+\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right)$$

and is calculated by the following

$$\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz = \frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{\sqrt{a-e^{2z}}}{2e^{2z}}.$$

Therefore

$$x(z) = \pm\left(\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{\sqrt{a-e^{2z}}}{2e^{2z}}\right) + \alpha.$$

As conclusion, we have

**Theorem 2.1.** •*The only non extendable flat translation surfaces in Lorentzian $Sol_3$ space which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x, y + c, z)$, are the surfaces whose parametrization is $X(x, y) = (x, y, z(x))$ where $x$ and $z$ satisfy*

$$x = \pm\left(\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{\sqrt{a-e^{2z}}}{2e^{2z}}\right) + \alpha.$$

*where $a \in \mathbb{R}^{*,+}$, $\alpha \in \mathbb{R}$ and $z \in ]-\infty, \ln(\sqrt{a})[$.*
•*In particular the only complete flat translation surfaces in Lorentzian $Sol_3$ space which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x, y + c, z)$, are the planes $z = z_0$.*

**Theorem 2.2.** •*The only complete extrinsically flat translation surfaces in Lorentzian $Sol_3$ space which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x, y + c, z)$, are parameterized by*

$$X(x, y) = \left(x, y, \ln\left(\frac{1}{\sqrt{-x^2 + 2\lambda x + \mu}}\right)\right),$$

*where $\lambda, \mu \in \mathbb{R}$, and $\lambda^2 + 2\mu > 0$ $\alpha \in \mathbb{R}$ and $x \in ]\lambda - \sqrt{\lambda^2 + 2\mu}, \lambda + \sqrt{\lambda^2 + 2\mu}[$.*

*Proof.* We know that $\Sigma$ is extrinsically surface if and only if $K_{ext} = 0$, and we have $K_{ext} = 0$ equivalent to

$$2z'^2 e^{-2z} - z'' e^{-2z} = -1.$$

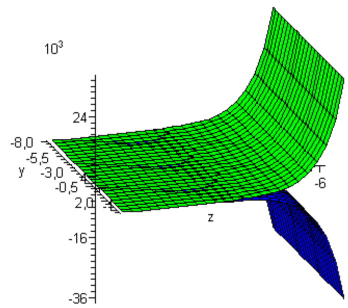


FIGURE 1. Non extendable flat surface in Lorentzian $Sol_3$ :$x(z) = \pm\left(\frac{1}{2\sqrt{0.2}}arc\tanh\left(\frac{\sqrt{0.2-e^{2z}}}{\sqrt{0.2}}\right) - \frac{\sqrt{0.2-e^{2z}}}{2e^{2z}}\right) + 2, a = 0.2,\ z = -6.. -1,,\ y = -4..8$ .

we remark that $2z'^2e^{-2z} - z''e{-}2z = (-z'e^{-2z})'$, thus

$$-z'e^{-2z} = -x + \lambda, \tag{11}$$

where $\lambda \in \mathbb{R}$,and we integrate the equation 11

$$z(x) = \ln\left(\frac{1}{\sqrt{-x^2 + 2\lambda + 2\mu}}\right),$$

where $\mu \in \mathbb{R}$, and $\lambda^2 + 2\mu > 0$ $\alpha \in \mathbb{R}$ and $x \in ]\lambda - \sqrt{\lambda^2 + 2\mu}, \lambda + \sqrt{\lambda^2 + 2\mu}[$. $\square$

**2.2.** In this section we classified complete flat translation surfaces $(\Sigma)$ in Lorentzian $Sol_3$ space which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$. Clearly, such a surface is generated by a curve $\beta$ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$, we can assume that $\beta$ is locally is a graph over the $y-$axis. Thus $\beta$ is given by $\beta(y) = (0, y, z(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, y, z(y)),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on $(\Sigma)$, namely,

$$e_1 := X_x = (1, 0, 0) = e^z E_1.$$

and

$$e_2 := X_y = (0, 1, z') = e^{-z} E_2 + z' E_3.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= e^{2z}, \; F =< e_1, e_2 >= 0, \; G =< e_2, e_2 >= z'^2 - e^{-2z}.$$

As a unit normal field we can take

$$N = \frac{z' e^z}{\sqrt{|-1 + z'^2 e^{2z}|}} E_2 + \frac{1}{\sqrt{|-1 + z'^2 e^{2z}|}} E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1} e_1 = -e^{2z} E_3,$$
$$\widetilde{\nabla}_{e_1} e_2 = z' e^z E_1,$$
$$\widetilde{\nabla}_{e_2} e_2 = -2z' e^{-z} E_2 + (z'' - e^{-2z}) E_3.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1} e_1, N >= \frac{-e^{2z}}{\sqrt{|-1 + z'^2 e^{2z}|}}$$

$$m =< \widetilde{\nabla}_{e_1} e_2, N >= 0$$

$$n =< \widetilde{\nabla}_{e_2} e_2, N >= \frac{2z'^2 + z'' - e^{-2z}}{\sqrt{|-1 + z'^2 e^{2z}|}}.$$

Let $K_{ext}$ be the extrinsic Gauss curvature of $(\Sigma)$,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{2z'^2 e^{2z} + z'' e^{2z} - 1}{(-1 + z'^2 e^{2z})^2}. \tag{12}$$

In order to obtain the intrinsic Gauss curvature $K_{int}$, recall that $K_{int} = K_{ext} + K(e_1 \wedge e_2)$, where $K(e_1 \wedge e_2)$ is the sectional curvature of each tangent plane spanned by $e_1$ and $e_2$, and

$$\begin{aligned} K(e_1 \wedge e_2) &= \tfrac{\langle R(e_1, e_2) e_2, e_1 \rangle}{< e_1, e_1 >< e_2, e_2 > - < e_1, e_2 >^2} \\ &= \tfrac{R_{1212} + z'^2 R_{1313}}{-1 + z'^2 e^{2z}} \\ &= \tfrac{-1 - z'^2 e^{2z}}{-1 + z'^2 e^{2z}}. \end{aligned}$$

Consequently, the intrinsic Gauss curvature is

$$K_{int} = \frac{e^{2z}[z'' + 2z'^2 - z'^4 e^{2z}]}{(-1 + z'^2 e^{2z})^2}. \tag{13}$$

So that $(\Sigma)$ is a flat surface in Lorentzian $Sol_3$ if and only if

$$K_{int} = 0,$$

that is, if and only if

$$z'' + 2z'^2 - z'^4 e^{2z} = 0 \tag{14}$$

to classify flat surfaces must solve the equation (14)
We note that for $z$ equal to a constant ($z = z_0 \in \mathbb{R}$) is a solution of the equation (14).
If $z$ is not constant ($z' \neq 0$),suppose that $z' = q$, and

$$z'' = \frac{dq}{dx} = \frac{dq}{dz}\frac{dz}{dx} = q.q'(z)$$

equation (14) becomes

$$q.q' = -2q^2 + q^4 e^{2z}.$$

or

$$q^{-3}.q' = -2q^{-2} + e^{2z}. \tag{15}$$

and suppose that $q^{-2} = g$, equation (15) becomes

$$\frac{-1}{2}g' = -2g + e^{2z}. \tag{16}$$

homogeneous solutions of equation (16) is

$$g(z) = K.e^{4z}.$$

and general solutions of the equation (16) is

$$g(z) = e^{4z}(a + e^{-2z}),$$

where $a \in \mathbb{R}^{*,-}$ and $z \in ]-\infty, \ln(\frac{1}{\sqrt{-a}})[$. Therefore

$$q(z) = \pm\frac{1}{\sqrt{g(z)}} = \pm\frac{e^{-2z}}{\sqrt{a + e^{-2z}}}.$$

and we have

$$z' = \pm\frac{e^{-2z}}{\sqrt{a + e^{-2z}}}.$$

or

$$\frac{dz}{dy} = \pm\frac{e^{-2z}}{\sqrt{a + e^{-2z}}}$$

so separating variables, we obtain

$$\int dy = \int \pm\frac{\sqrt{a + e^{-2z}}}{e^{-2z}}dz$$

i.e

$$y = \int \pm\frac{\sqrt{a + e^{-2z}}}{e^{-2z}}dz + \delta,$$

where $\delta \in \mathbb{R}$.
we substitute $\tanh(t) = \frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}$, $dz = \tanh(t)dt$, and $e^{-2z} = \frac{a}{\cosh^2(t)} = a(1 - \tanh^2(t))$, therefore

$$\int \frac{\sqrt{a+e^{-2z}}}{e^{-2z}} dz = \frac{1}{\sqrt{a}} \int \sinh^2(t)dt = -\frac{1}{8\sqrt{a}}[e^{2t} - e^{-2t}] + \frac{t}{2\sqrt{a}},$$

and as $t = arc\tanh\left(\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}}{1-\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}}\right)$, thus

$$\int \frac{\sqrt{a+e^{-2z}}}{e^{-2z}} dz = -\frac{1}{8\sqrt{a}}\left[\left(\frac{\sqrt{a}+\sqrt{a+e^{-2z}}}{\sqrt{a}-\sqrt{a+e^{-2z}}}\right) - \left(\frac{\sqrt{a}-\sqrt{a+e^{-2z}}}{\sqrt{a}+\sqrt{a+e^{-2z}}}\right)\right]$$
$$+\frac{1}{2\sqrt{a}} arc\tanh\left(\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}\right)$$

and is calculated by the following

$$\int \frac{\sqrt{a+e^{-2z}}}{e^{-2z}} dz = \frac{1}{2\sqrt{a}} arc\tanh\left(\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}\right) + \frac{\sqrt{a+e^{-2z}}}{2e^{-2z}}.$$

As conclusion, we have

**Theorem 2.3.** •*The only non extendable flat translation surfaces in Lorentzian $Sol_3$ which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x+c, y, z)$, are the surfaces whose parametrization is $X(x, y) = (x, y, z(y))$ where $y$ and $z$ satisfy*

$$y = \pm\left(\frac{1}{2\sqrt{a}} arc\tanh\left(\frac{\sqrt{a+e^{-2z}}}{\sqrt{a}}\right) + \frac{\sqrt{a+e^{-2z}}}{2e^{-2z}}\right) + \delta,$$

*where $a \in \mathbb{R}^{*,-}$, $\delta \in \mathbb{R}$ and $z \in ]-\infty, \ln(\frac{1}{\sqrt{-a}})[$.*

•*In particular the only complete flat translation surfaces in Lorentzian $Sol_3$ which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$, are the planes $z = z_0$.*

**Theorem 2.4.** •*The only complete extrinsically flat translation surfaces in Lorentzian $Sol_3$ which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$, are parameterized by*

$$X(x, y) = \left(x, y, \ln\left(\sqrt{y^2 + 2\lambda y + \mu}\right)\right),$$

*where $\lambda, \mu \in \mathbb{R}$, and $\lambda^2 - 2\mu > 0$ $\alpha \in \mathbb{R}$ and $y \in ]-\infty, \lambda - \sqrt{\lambda^2 - 2\mu}[\cup]\lambda + \sqrt{\lambda^2 - 2\mu}, +\infty[$.*

*Proof.* We know that $\Sigma$ is extrinsically flat surface if and only if $K_{ext} = 0$, and we have $K_{ext} = 0$ equivalent to

$$2z'^2 e^{2z} + z'' e^{2z} = 1.$$

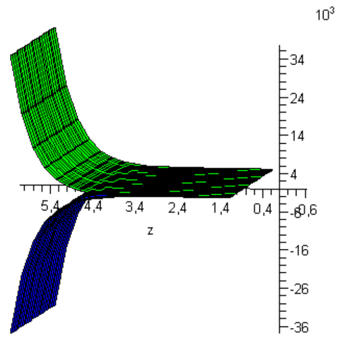


FIGURE 2. Non extendable flat surface in Lorentzian $Sol_3$ :$y(z) = \pm\left(\frac{1}{2\sqrt{0.2}} arc\tanh\left(\frac{\sqrt{0.2-e^{-2z}}}{\sqrt{0.2}}\right) - \frac{\sqrt{0.2-e^{-2z}}}{2e^{-2z}}\right) + 2, a = 0.2$, $z = -0.5..6,$, $x = -4..8$ .

we remark that $2z'^2e^{2z} + z''e^{2z} = (z'e^{2z})'$, thus

$$z'e^{2z} = y + \lambda, \tag{17}$$

where $\lambda \in \mathbb{R}$,and we integrate the equation 17

$$z(y) = \ln\left(\sqrt{y^2 + 2\lambda y + 2\mu}\right),$$

where $\mu \in \mathbb{R}$, and $\lambda^2 - 2\mu > 0$ $\alpha \in \mathbb{R}$ and $y \in ]-\infty, \lambda - \sqrt{\lambda^2 - 2\mu}[\cup]\lambda + \sqrt{\lambda^2 - 2\mu}, +\infty[$. □

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