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A NOTE ON ϕ-PRÜFER v-MULTIPLICATION RINGS

Xiaolei Zhang

Abstract. In this note, we show that a strongly ϕ-ring R is a ϕ-PvMR if and only if any ϕ-torsion-free R-module is ϕ-w-flat, if and only if any GV-torsion-free divisible R-module is nonnil-absolutely w-pure, if and only if any GV-torsion-free h-divisible R-module is nonnil-absolutely w-pure, if and only if any finitely generated nonnil ideal of R is w-projective.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity and all modules are unitary. We always denote by $\mathrm{Nil}(R)$ the nil radical of R, $\mathrm{Z}(R)$ the set of all zero-divisors of R and $\mathrm{T}(R)$ the total ring of fractions of R. An ideal I of R is said to be *nonnil* if there is a non-nilpotent element in I. A ring R is an NP-*ring* if $\mathrm{Nil}(R)$ is a prime ideal, and a ZN-*ring* if $\mathrm{Z}(R) = \mathrm{Nil}(R)$. A prime ideal $\mathfrak{p}$ is said to be *divided prime* if $\mathfrak{p} \subsetneq (x)$ for every $x \in R \setminus \mathfrak{p}$. Set $\mathcal{H} = \{R \,|\, R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. A ring R is a ϕ-*ring* if $R \in \mathcal{H}$. Moreover, a ZN ϕ-ring is said to be a *strongly ϕ-ring*. For a ϕ-ring R, the map $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{b}{a}) = \frac{b}{a}$ is a ring homomorphism, and the image of R, denoted by $\phi(R)$, is a strongly ϕ-ring. The notion of Prüfer domains is one of the most famous integral domains that attract many algebraists. In 2004, Anderson and Badawi [1] extended the notion of Prüfer domains to that of *ϕ-Prüfer rings* which are ϕ-rings satisfying that each finitely generated nonnil ideal is ϕ-invertible. The authors in [1] characterized ϕ-Prüfer rings from the ring-theoretic point of view. In 2018, Zhao [23] characterized ϕ-Prüfer rings using the homological properties of ϕ-flat modules. Recently, Zhang and Qi [20] gave a module-theoretic characterization of ϕ-Prüfer rings in terms of ϕ-flat modules and nonnil-FP-injective modules.

Recall that an integral domain R is called a Prüfer *v-multiplication domain* (PvMD for short) provided that any nonzero ideal of R is w-invertible (see [6] for example). In 2014, Wang et al. [13] showed that an integral domain R is a PvMD if and only if $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R.

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In 2015, Wang et al. [17] obtained that an integral domain R is a PvMD if and only if w-w.gl.dim$(R) \leq 1$, if and only if any torsion-free R-module is w-flat. In 2018, Xing et al. [19] gave a new module-theoretic characterization of PvMDs, i.e., an integral domain R is a PvMD if and only if any divisible R-module is absolutely w-pure, if and only if any h-divisible R-module is absolutely w-pure. In order to extend the notion of PvMDs to that of commutative rings in $\mathcal{H}$, the author of this paper and Zhao [22] introduced the notion of ϕ-PvMRs as the ϕ-rings in which any finitely generated nonnil ideal is ϕ-w-invertible. They also gave some ring-theoretic and homology-theoretic characterizations of ϕ-PvMRs. In this paper, we mainly study the module-theoretic characterizations of ϕ-PvMRs, which can be seen a generalization of Wang's and Xing's results in [17] and [19], respectively.

As our work involves the w-operation theory, we give a quick review as below. Let R be a commutative ring and J a finitely generated ideal of R. Then J is called a GV-*ideal* if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. The set of all GV-ideals is denoted by $\mathrm{GV}(R)$. Let M be an R-module and define

$$\mathrm{tor}_{\mathrm{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

An R-module M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is said to be a w-*module* if, for any $x \in E(M)$, there is a GV-ideal J such that $Jx \subseteq M$ where $E(M)$ is the injective envelope of M. The w-*envelope* M_w of a GV-torsion-free module M is defined by the minimal w-module that contains M. A *maximal* w-*ideal* which is maximal among the w-submodules of R is proved to be prime (see [15, Theorem 6.2.15]). The set of all maximal w-ideals is denoted by w-$\mathrm{Max}(R)$. Let M be an R-module and set $\mathrm{L}(M) = (M/\mathrm{tor}_{\mathrm{GV}}(M))_w$. Recall from [14] that M is said to be w-projective if $\mathrm{Ext}^1_R(\mathrm{L}(M), N)$ is GV-torsion for any torsion-free w-module N.

An R-homomorphism $f : M \to N$ is said to be a w-*monomorphism* (resp., w-*epimorphism*, w-*isomorphism*) if for any $\mathfrak{p} \in w$-$\mathrm{Max}(R)$, $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that f is a w-monomorphism (resp., w-epimorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$) is GV-torsion. A sequence $A \to B \to C$ is said to be w-*exact* if, for any $\mathfrak{p} \in w$-$\mathrm{Max}(R)$, $A_{\mathfrak{p}} \to B_{\mathfrak{p}} \to C_{\mathfrak{p}}$ is exact. A class $\mathcal{C}$ of R-modules is said to be *closed under* w-*isomorphisms* provided that for any w-isomorphism $f : M \to N$, if one of the modules M and N is in $\mathcal{C}$, so is the other. An R-module M is said to be of *finite type* provided that there exist a finitely generated free module F and a w-epimorphism $g : F \to M$, and it is said to be of *finitely presented type* provided that there is a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_0 and F_1 are finitely generated free modules. The classes of finite type and finitely presented type modules are all closed under w-isomorphisms (see [15, Corollary 6.4.4; Corollary 6.4.13]).

2. nonnil-absolutely w-pure modules

Recall from [18], a w-exact sequence of R-modules $0 \to N \to M \to L \to 0$ is said to be *w-pure exact* if, for any R-module K, the induced sequence $0 \to K \otimes_R N \to K \otimes_R M \to K \otimes_R L \to 0$ is w-exact. If N is a submodule of M and the exact sequence $0 \to N \to M \to M/N \to 0$ is w-pure exact, then N is said to be a *w-pure submodule* of M. Recall from [19], an R-module M is called an *absolutely w-pure module* provided that M is w-pure in every module containing M as a submodule.

Let R be an NP-ring and M an R-module. Define

$$\phi\text{-}tor(M) = \{x \in M \mid Ix = 0 \text{ for some nonnil ideal } I \text{ of } R\}.$$

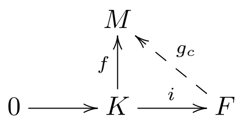
An R-module M is said to be *ϕ-torsion* (resp., *ϕ-torsion-free*) provided that ϕ-$tor(M) = M$ (resp., ϕ-$tor(M) = 0$). Now we generalize the notions in [18] and [19] to NP-rings. A w-exact sequence $0 \to M \to N \to N/M \to 0$ of R-modules is said to be *nonnil w-pure exact* provided that $0 \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(T, N/M) \to 0$ is w-exact for any finitely presented ϕ-torsion module T. In addition, if M is a submodule of N, then we say M is a *nonnil w-pure submodule* in N.

Definition 2.1. Let R be an NP-ring. An R-module M is called a *nonnil-absolutely w-pure module* provided that M is a nonnil w-pure submodule in every R-module containing M as a submodule.

Following Xing [19, Theorem 2.6], an R-module M is absolutely w-pure if and only if $\mathrm{Ext}^1_R(F, M)$ is GV-torsion for any finitely presented module F, if and only if M is a w-pure submodule in its injective envelope. Now, we give a ϕ-version of Xing's result.

Proposition 2.2. *Let R be an* NP*-ring and M an R-module. The following statements are equivalent*:

(1) *M is a nonnil-absolutely w-pure module*;
(2) $\mathrm{Ext}^1_R(T, M)$ *is* GV*-torsion for any finitely presented ϕ-torsion module T*;
(3) *M is a nonnil w-pure submodule in any injective module containing M*;
(4) *M is a nonnil w-pure submodule in its injective envelope*;
(5) *For any diagram*



with F finitely generated projective, K finitely generated and F/K ϕ-torsion, there is some $J \in \mathrm{GV}(R)$ such that any given $c \in J$, there exists $g_c : F \to M$ such that $cf = g_c i$.

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) : They hold trivially.

(2) $\Rightarrow$ (1) : Let N be an R-module containing M and T a finitely presented ϕ-torsion module. Then we have the following exact sequence

$$0 \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(T, N/M) \to \mathrm{Ext}^1_R(T, M).$$

Since $\mathrm{Ext}^1_R(T, M)$ is GV-torsion, we have

$$0 \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(T, N/M) \to 0$$

is w-exact. Hence M is a nonnil w-pure submodule in N.

(4) $\Rightarrow$ (2) : Let E be the injective envelope of M. Then, for any finitely presented ϕ-torsion module T, we have the following exact sequence:

$$0 \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, E) \to \mathrm{Hom}_R(T, E/M) \to \mathrm{Ext}^1_R(T, M) \to 0.$$

Thus we have $\mathrm{Ext}^1_R(T, M)$ is GV-torsion by (4).

(2) $\Rightarrow$ (5) : Consider the exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{\pi} F/K \to 0$ with F/K finitely presented ϕ-torsion. Thus we have the following exact sequence:

$$\mathrm{Hom}_R(F, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \to \mathrm{Ext}^1_R(F/K, M) \to 0.$$

Since F/K is finitely presented ϕ-torsion, $\mathrm{Ext}^1_R(F/K, M)$ is GV-torsion by (2). Thus i^* is a w-epimorphism. Since $f \in \mathrm{Hom}_R(K, M)$, there exists a GV-ideal J of R such that $Jf \in \mathrm{Im}(i^*)$. So, for any given $c \in J$, there exists $g_c : F \to M$ such that $g_c i = cf$.

(5) $\Rightarrow$ (2) : Let T be a finitely presented ϕ-torsion module. Then there exists a short sequence $0 \to K \xrightarrow{i} F \to T \to 0$ with F finitely generated projective and K finitely generated. Thus we have the exact sequence:

$$\mathrm{Hom}_R(F, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \to \mathrm{Ext}^1_R(T, M) \to 0.$$

For any $f \in \mathrm{Hom}_R(K, M)$, there is some $J \in \mathrm{GV}(R)$ such that any given $c \in J$, there exists $g_c : F \to M$ such that $cf = g_c i$ by (5). So $Jf \subseteq \mathrm{Im}(i^*)$. Thus i^* is a w-epimorphism, and so $\mathrm{Ext}^1_R(T, M)$ is GV-torsion. $\square$

Recall from [20, Definition 1.2] that an R-module M is called *nonnil-FP-injective* provided that $\mathrm{Ext}^1_R(T, M) = 0$ for any finitely presented ϕ-torsion module T. Thus we have the following result by Proposition 2.2.

Lemma 2.3. *Let R be an* NP*-ring. Then any nonnil-FP-injective module is nonnil-absolutely w-pure.*

Lemma 2.4. *Let T be a* GV*-torsion module. Then T is an absolutely w-pure module.*

Proof. Let T be a GV-torsion module and F a finitely presented R-module. Considering the exact sequence $0 \to K \to P \to F \to 0$ with P finitely generated projective and K finitely generated, we have the following exact sequence:

$$\mathrm{Hom}_R(K, T) \to \mathrm{Ext}^1_R(F, T) \to 0.$$

Since K is finitely generated and T is GV-torsion, $\mathrm{Hom}_R(K,T)$ is GV-torsion (see [16, Lemma 2.1(1)]). So $\mathrm{Ext}_R^1(F,T)$ is GV-torsion. Consequently, T is a absolutely w-pure module. □

Obviously, we have the following result by Proposition 2.2, [19, Theorem 2.6] and Lemma 2.4.

Corollary 2.5. *Let R be an* NP*-ring. Then any absolutely w-pure module is nonnil-absolutely w-pure. Consequently, any* GV*-torsion module is a nonnil-absolutely w-pure module.*

In order to characterize rings over which any nonnil-absolutely w-pure module is absolutely w-pure, we recall some basic facts.

Lemma 2.6 ([22, Lemma 1.6])**.** *Let R be a ϕ-ring and I a nonnil ideal of R. Then* $\mathrm{Nil}(R) = I\mathrm{Nil}(R)$.

Lemma 2.7 ([20, Proposition 1.5])**.** *Let R be a ϕ-ring and M an FP-injective $R/\mathrm{Nil}(R)$-module. Then M is nonnil-FP-injective over R.*

Now, we recall the special injective w-module constructed in [21]. Let $R\{x\}$ be the w-Nagata ring of R, that is, the localization of $R[X]$ at the multiplicative closed set $S_w = \{f \in R[x] \mid c(f) \in \mathrm{GV}(R)\}$, where $c(f)$ is the content of f. Let M be an R-module. Set $M\{x\} = M \otimes_R R\{x\}$. Then $\{\mathfrak{m}\{x\} \mid \mathfrak{m} \in w\text{-Max}(R)\}$ is the set of all maximal ideals of $R\{x\}$ by [14, Proposition 3.3(4)]. Set

$$E' = \prod_{m \in w\text{-Max}(R)} E_R(R\{x\}/\mathfrak{m}\{x\}),$$

where $E_R(R\{x\}/\mathfrak{m}\{x\})$ is the injective envelope of the R-module $R\{x\}/\mathfrak{m}\{x\}$. Since $R\{x\}/\mathfrak{m}\{x\}$ is a w-module over R by [15, Theorem 6.6.19(2)], it follows that E' is an injective w-module over R. Set

$$\tilde{E} := \mathrm{Hom}_R(R\{x\}, E').$$

Then $\tilde{E}$ is trivially an $R\{x\}$-module. Since $R\{x\}$ is a flat R-module, $\tilde{E}$ is an injective w-module over R by [15, Theorem 6.1.18] and [5, Theorem 3.2.9].

Lemma 2.8 ([21, Corollary 3.11])**.** *Let M be an R-module. The following statements are equivalent*:

(1) *M is* GV*-torsion*;
(2) $\mathrm{Hom}_R(M,E) = 0$ *for any injective w-module E*;
(3) $\mathrm{Hom}_R(M,\tilde{E}) = 0$.

Theorem 2.9. *Let R be a ϕ-ring. Then R is an integral domain if and only if any nonnil-absolutely w-pure module is absolutely w-pure.*

Proof. If R is an integral domain, then any nonnil-absolutely w-pure module is absolutely w-pure obviously.

Assume that any nonnil-absolutely w-pure module is absolutely w-pure. Note that we have $\mathrm{Hom}_R(R/\mathrm{Nil}(R), \widetilde{E})$ is an injective $R/\mathrm{Nil}(R)$-module by

[5, Proposition 3.1.6]. Thus by Lemma 2.7, $\mathrm{Hom}_R(R/\mathrm{Nil}(R), \widetilde{E})$ is a nonnil-FP-injective R-module, and so is a nonnil-absolutely w-pure R-module. Thus we have $\mathrm{Hom}_R(R/\mathrm{Nil}(R), \widetilde{E})$ is an absolutely w-pure R-module by assumption. That is,

$$\mathrm{Ext}_R^1(F, \mathrm{Hom}_R(R/\mathrm{Nil}(R), \widetilde{E})) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(F, R/\mathrm{Nil}(R)), \widetilde{E})$$

is a GV-torsion module for any finitely presented R-module F as $\widetilde{E}$ is an injective R-module. Since $\widetilde{E}$ is a w-module, $\mathrm{Hom}_R(\mathrm{Tor}_1^R(F, R/\mathrm{Nil}(R)), \widetilde{E})$ is also a w-module by [15, Theorem 6.1.18]. Thus we have

$$\mathrm{Hom}_R(\mathrm{Tor}_1^R(F, R/\mathrm{Nil}(R)), \widetilde{E}) = 0.$$

Hence $\mathrm{Tor}_1^R(F, R/\mathrm{Nil}(R))$ is GV-torsion by Lemma 2.8. Let s be a nilpotent element in R and set $F = R/\langle s\rangle$. Then

$$\begin{aligned}\mathrm{Tor}_1^R(F, R/\mathrm{Nil}(R)) &= \mathrm{Tor}_1^R(R/\langle s\rangle, R/\mathrm{Nil}(R)) \\ &\cong \langle s\rangle \cap \mathrm{Nil}(R)/s\mathrm{Nil}(R) = \langle s\rangle/s\mathrm{Nil}(R)\end{aligned}$$

is GV-torsion (see [15, Exercise 3.20]). Thus there is a GV-ideal J such that $sJ \subseteq s\mathrm{Nil}(R)$. Since J is a GV-ideal, it is a nonnil ideal, thus $\mathrm{Nil}(R) = J\mathrm{Nil}(R)$ by Lemma 2.6. So $sJ \subseteq s\mathrm{Nil}(R) = sJ\mathrm{Nil}(R) \subseteq sJ$. That is, $sJ = sJ\mathrm{Nil}(R)$. Since sJ is finitely generated, $sJ = 0$ by Nakayama's lemma. Since $J \in \mathrm{GV}(R)$, $sR \subseteq R$ is GV-torsion-free, then $s = 0$. Consequently, $\mathrm{Nil}(R) = 0$. Since R is a ϕ-ring, $\mathrm{Nil}(R) = 0$ is the unique minimal prime ideal. So R is an integral domain. □

Lemma 2.10. *Let R be a ring. If R is a (strongly) ϕ-ring, then $R_{\mathfrak{p}}$ is a (strongly) ϕ-ring for any prime ideal $\mathfrak{p}$ of R.*

Proof. Let R be a ϕ-ring and $\mathfrak{p}$ a prime ideal of R. Then $R_{\mathfrak{p}}/\mathrm{Nil}(R_{\mathfrak{p}}) \cong (R/\mathrm{Nil}(R))_{\overline{\mathfrak{p}}}$ which is certainly an integral domain, where $\overline{\mathfrak{p}} = \mathfrak{p}/\mathrm{Nil}(R)$. So $\mathrm{Nil}(R_{\mathfrak{p}})$ is a prime ideal of $R_{\mathfrak{p}}$. Let $\frac{r}{s} \in R_{\mathfrak{p}} \setminus \mathrm{Nil}(R_{\mathfrak{p}})$ and $\frac{r_1}{s_1} \in \mathrm{Nil}(R_{\mathfrak{p}})$. Note $r \in R \setminus \mathrm{Nil}(R)$ and $r_1 \in \mathrm{Nil}(R)$. Then $r_1 = rt$ for some $t \in \mathrm{Nil}(R)$. Thus $\frac{r_1}{s_1} = \frac{rt}{s_1} = \frac{rts}{ss_1} = \frac{r}{s}\frac{ts}{s_1} \in \langle \frac{r}{s}\rangle$. So $\mathrm{Nil}(R_{\mathfrak{p}})$ is a divided prime ideal of $R_{\mathfrak{p}}$. Hence $R_{\mathfrak{p}}$ is a ϕ-ring. Now suppose R is a strongly ϕ-ring. Let $\frac{r}{s} \in R_{\mathfrak{p}} \setminus \mathrm{Nil}(R_{\mathfrak{p}})$. Then r is non-nilpotent, and thus r is regular. Assume $\frac{r}{s}\frac{r_1}{s_1} = 0$ in $R_{\mathfrak{p}}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $rr_1t = 0$. Thus $r_1t = 0$. Hence r_1 and thus $\frac{r_1}{s_1}$ is equal to 0 since t is also regular. Consequently, $R_{\mathfrak{p}}$ is a strongly ϕ-ring. □

Remark 2.11. Note that the converse of Lemma 2.10 is not true in general. Indeed, let R be a von Neumann regular ring which is not a field. Then $R_{\mathfrak{p}}$ is a field for any prime ideal $\mathfrak{p}$ of R. However, R is not a ϕ-ring since $\mathrm{Nil}(R) = 0$ is not a prime ideal in this case.

Let R be an NP-ring. Recall from [24] that an R-module M is said to be *ϕ-flat* if for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a monomorphism; a ϕ-ring R is said to be

ϕ-von Neumann if any R-module is ϕ-flat. The authors in [24, Theorem 4.1] proved that a ϕ-ring R is ϕ-von Neumann if and only if the Krull dimension of R is 0. It was shown in [20, Theorem 1.7] that a ϕ-ring R is ϕ-von Neumann if and only if $R/\mathrm{Nil}(R)$ is a field, if and only if every non-nilpotent element is invertible, if and only if any R-module is nonnil-FP-injective. Recall from [22, Definition 1.3] that an R-module M is said to be *ϕ-w-flat* if, for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a w-monomorphism. It was proved in [22, Theorem 3.1] that a ϕ-ring R is ϕ-von Neumann if and only if any R-module is ϕ-w-flat. Now we give a new characterization of ϕ-von Neumann rings.

Lemma 2.12. *Let R be a ϕ-ring. Then R is a ϕ-von Neumann regular ring if and only if $R_{\mathfrak{m}}$ is a ϕ-von Neumann regular ring for any $\mathfrak{m} \in w$-$Max(R)$.*

Proof. Assume that R is a ϕ-von Neumann regular ring. Let $\mathfrak{m}$ be a prime ideal. Let $\frac{r}{s}$ be a non-nilpotent element in $R_{\mathfrak{m}}$. Then r is non-nilpotent. So r is invertible by [20, Theorem 1.7]. Hence $\frac{r}{s}$ is also invertible in $R_{\mathfrak{m}}$, whence $R_{\mathfrak{m}}$ is a ϕ-von Neumann regular ring by [20, Theorem 1.7] and Lemma 2.10.

Now let r be a non-nilpotent element in R. Then $\frac{r}{1}$ is a non-nilpotent element in $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in w$-$Max(R)$, since R is a ϕ-ring. By [20, Theorem 1.7], $\frac{r}{1}$ is invertible in $R_{\mathfrak{m}}$. Thus $r \notin \mathfrak{m}$ for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$. So $\langle r \rangle_w = R$, and hence r is invertible by [15, Exercise 6.11(2)]. □

Theorem 2.13. *Let R be a ϕ-ring. Then R is a ϕ-von Neumann regular ring if and only if any R-module is nonnil-absolutely w-pure.*

Proof. Suppose R is a ϕ-von Neumann regular ring and let M be an R-module. Then any non-nilpotent element of R is invertible by [20, Theorem 1.7]. So the only nonnil ideal of R is R itself. Let T be a finitely presented ϕ-torsion module. Then $T = \phi\text{-}tor(T) = \{x \in T \mid Ix = 0$ for some nonnil ideal I of $R\} = 0$. It follows that $\mathrm{Ext}^1_R(T, M) = 0$, which is GV-torsion. Consequently, M is nonnil-absolutely w-pure.

Assume that any R-module is nonnil-absolutely w-pure and let I be a finitely generated nonnil ideal of R. Since for any R-module M, $\mathrm{Ext}^1_R(R/I, M)$ is GV-torsion, it follows that R/I is finitely generated w-projective. Thus $R_{\mathfrak{m}}/I_{\mathfrak{m}}$ is a finitely generated projective $R_{\mathfrak{m}}$-module for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$ by [15, Theorem 6.7.18]. Then $I_{\mathfrak{m}}$ is an idempotent ideal of $R_{\mathfrak{m}}$ by [8, Theorem 1.2.15]. By [7, Chapter I, Proposition 1.10], $I_{\mathfrak{m}}$ is generated by an idempotent $e_{\mathfrak{m}} \in R_{\mathfrak{m}}$. Thus $R_{\mathfrak{m}}$ is a ϕ-von Neumann regular ring by [24, Theorem 4.1] and Lemma 2.10. So R is ϕ-von Neumann regular by Lemma 2.12. □

3. Some new characterizations of ϕ-Prüfer v-multiplication rings

Following [15], a ring R is said to be *w-coherent* if any finite type ideal of R is of finitely presented type. Recall from [10] that a ϕ-ring R is said to be a *nonnil-coherent ring* if any finitely generated nonnil ideal of R is of finitely presented. Now, we generalize both w-coherent rings and nonnil-coherent rings.

Definition 3.1. A ϕ-ring R is said to be a *nonnil-w-coherent ring* provided that any finite type nonnil ideal of R is of finitely presented type.

Lemma 3.2. *A ϕ-ring R is a nonnil-w-coherent ring if and only if any finitely generated nonnil ideal of R is of finitely presented type.*

Proof. Let I be a finite type nonnil ideal of a nonnil-w-coherent ring R. Then there exists a finitely generated sub-ideal K of I such that I/K is GV-torsion. Since I is a nonnil ideal, there is a non-nilpotent element $s \in I$ such that $Js \subseteq K$ for some $J \in \mathrm{GV}(R)$. Since J is nonnil and R is a ϕ-ring, K is also nonnil. So K is of finitely presented type, and hence I is also of finitely presented type. □

Proposition 3.3. *Suppose R is a strongly ϕ-ring. Then R is nonnil-w-coherent if and only if $R/\mathrm{Nil}(R)$ is w-coherent.*

Proof. Suppose R is nonnil-w-coherent. If $I/\mathrm{Nil}(R)$ is a finitely generated nonzero $R/\mathrm{Nil}(R)$-ideal, then I is a finitely generated nonnil R-ideal. Since R is nonnil-w-coherent, I is of finitely presented type. So there are two exact sequences $0 \to T_1 \to L \to N \to 0$ and $0 \to N \to J \to T_2 \to 0$, where T_1 and T_2 are GV-torsion, L is finitely presented. Now, we have an $R/\mathrm{Nil}(R)$-exact sequence $0 \to N/\mathrm{Nil}(R) \to J/\mathrm{Nil}(R) \to T_2 \to 0$. By [11, Lemma 2.11(a)], T_2 is a GV-torsion $R/\mathrm{Nil}(R)$-module and $N/\mathrm{Nil}(R)$ is a finitely generated $R/\mathrm{Nil}(R)$-module. We have the exact sequence $T_1 \otimes_R R/\mathrm{Nil}(R) \to L \otimes_R R/\mathrm{Nil}(R) \to N \otimes_R R/\mathrm{Nil}(R) \to 0$. By [11, Lemma 2.9(a)], we have $N \otimes_R R/\mathrm{Nil}(R) \cong N/N\mathrm{Nil}(R) \cong N/\mathrm{Nil}(R)$. Then there is an exact sequence $0 \to T \to L \otimes_R R/\mathrm{Nil}(R) \to N/\mathrm{Nil}(R) \to 0$. By [11, Lemma 2.11(a)] again, T is a GV-torsion $R/\mathrm{Nil}(R)$-module. Since $L \otimes_R R/\mathrm{Nil}(R)$ is a finitely presented $R/\mathrm{Nil}(R)$-module, it follows that $J/\mathrm{Nil}(R)$ is a finitely presented type $R/\mathrm{Nil}(R)$-module. Hence, $R/\mathrm{Nil}(R)$ is w-coherent.

Assume that $R/Nil(R)$ is w-coherent and let I be a finitely generated nonnil ideal of R. Then $I/\mathrm{Nil}(R)$ is a finitely generated $R/\mathrm{Nil}(R)$-ideal. Since $R/\mathrm{Nil}(R)$ is a w-coherent domain, $I/\mathrm{Nil}(R)$ is a finitely presented type $R/\mathrm{Nil}(R)$-ideal. Write $I/\mathrm{Nil}(R) = (\overline{x_1}, \ldots, \overline{x_n})$, where x_i is a non-nilpotent element in R $(i = 1, \ldots, n)$. Set $I = (x_1, \ldots, x_n)$. We will show I is of finitely presented type by induction on n. If $n = 1$, $I = Rx_1 \cong R$ is of finitely presented type since R is a strongly ϕ-ring. For general case, $I = (x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}) + Rx_n$. By induction, we have $(x_1, \ldots, x_{n-1})$ and Rx_n are all of finitely presented type. Since $R/\mathrm{Nil}(R)$ is a w-coherent domain, we have $(x_1, \ldots, x_{n-1})/\mathrm{Nil}(R) \cap Rx_n/\mathrm{Nil}(R) = ((x_1, \ldots, x_{n-1}) \cap Rx_n)/\mathrm{Nil}(R)$ is a finite type nonzero $R/\mathrm{Nil}(R)$-ideal. So we have $(x_1, \ldots, x_{n-1}) \cap Rx_n$ is a finite type nonnil ideal by [11, Lemma 2.9]. Consider the exact sequence

$$0 \to (x_1, \ldots, x_{n-1}) \cap Rx_n \to (x_1, \ldots, x_{n-1}) \oplus Rx_n \to (x_1, \ldots, x_n) \to 0.$$

It follows from [15, Theorem 6.4.11] that I is of finitely presented type. □

Let R be a ring and M an R-module. We recall from [9] the idealization $R(+)M$ of R by M. Let $R(+)M$ be an R-module isomorphic to $R\oplus M$. Define

(1) $(r,m)+(s,n)=(r+s,m+n)$,

(2) $(r,m)(s,n)=(rs,sm+rn)$.

Then $R(+)M$ is a ring with identity $(1,0)$. The next example shows that nonnil-w-coherent rings can neither be w-coherent nor be nonnil-coherent.

Example 3.4. Let D be a non-coherent w-coherent domain (see [15, Example 9.1.18]) with K its quotient field. Then K is not a finitely generated D-module. Set $R = D(+)K$. Then D is a strongly ϕ-ring (see [2, Remark 1]). Since $\mathrm{Nil}(R) = 0(+)K$, it follows that $R/\mathrm{Nil}(R) \cong D$ is a non-coherent w-coherent domain. By Proposition 3.3, R is a nonnil-w-coherent ring. By [10, Remark 2.1], R is not nonnil-coherent. Next we will show R is not w-coherent. Note that $(0,1)R$ is a finitely generated ideal of R. Consider the exact sequence $0 \to L \to R \to (0,1)R \to 0$. Then $L = \mathrm{Nil}(R) = 0(+)K$. Since the w-module K is not finitely generated over D, K is also not of finite type. By [4, Lemma 2.2], the w-ideal $\mathrm{Nil}(R)$ is not of finite type. So $(0,1)R$ is not of finitely presented type. Hence R is not w-coherent.

Lemma 3.5. *Let R be a nonnil-w-coherent ring. Let T be a finitely generated ϕ-torsion module of finitely presented type. Suppose T is generated by $\{t_1,\dots,t_k,t_{k+1}\}$ with $k\geq 1$ and T_k the submodule of T generated by $\{t_1,\dots,t_k\}$. Then T_k is of finitely presented type.*

Proof. Note $T/T_k = (T_k + Rt_{k+1})/T_{k+1} \cong Rt_{k+1}/(T_k \cap Rt_{k+1}) \cong R/I$ where $I = (0 :_R t_{k+1} + T_k \cap Rt_{k+1})$ is an ideal of R. Since T is a ϕ-torsion module of finitely presented type and T_k is finitely generated, it follows by [15, Theorem 6.4.14] that I is a finite type nonnil ideal of R . Since R is nonnil-w-coherent, then I is of finitely presented type. Consider the following pull-back:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X & \to & F & \to & R/I & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_k & \to & T & \to & R/I & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where F is finitely generated free. Then K is of finite type by [15, Theorem 6.4.11]. Since I is of finitely presented type, $I\{x\}$ is finitely presented $R\{x\}$-ideal by [14, Theorem 3.9]. So we have an $R\{x\}$-exact sequence: $0 \to X\{x\} \to$

$F\{x\} \to R\{x\}/I\{x\} \to 0$. So $X\{x\}$ is a finitely presented $R\{x\}$-module by [8, Theorem 2.1.2]. Hence X is of finitely presented type by [14, Theorem 3.9] again. Thus T_k is finitely presented by [15, Theorem 6.4.12]. □

Proposition 3.6. *Let R be a nonnil-w-coherent ring and T is a finitely generated ϕ-torsion module of finitely presented type. Suppose $0 \to K \to R^n \to T \to 0$ is an exact sequence. Then K is of finitely presented type.*

Proof. We will show K is of finitely presented type by induction on n. If $n = 1$, then K is a finite type nonnil ideal of R. Thus K is of finitely presented type since R is nonnil-w-coherent. Suppose $n = k+1$. Then there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \cap R^k & \longrightarrow & R^k & \longrightarrow & R^k/K \cap R^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & R^{k+1} & \longrightarrow & R^{k+1}/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0,
\end{array}$$

where $I = K/K \cap R^k$ is an ideal of R. Since $T = R^{k+1}/K$ is finitely generated ϕ-torsion of finitely presented type, it follows by Lemma 3.5 that $R^k/K \cap R^k$ is a ϕ-torsion module of finitely presented type. Thus R/I is ϕ-torsion of finitely presented type by [15, Theorem 6.4.11]. Since $R^k/K \cap R^k$ is generated by k elements, $K \cap R^k$ and I are of finitely presented type by induction. Thus K is also of finitely presented type by [15, Theorem 6.4.12]. □

Proposition 3.7. *Let R be a nonnil-w-coherent ring and S a multiplicative subset of R. Suppose T is a finitely presented ϕ-torsion module and E is a* GV*-torsion-free module. Then there is a natural isomorphism*:

$$\mathrm{Ext}^1_R(T, E)_S \cong \mathrm{Ext}^1_{R_S}(T_S, E_S).$$

Proof. Let T be a finitely presented ϕ-torsion R-module. Then there is an exact sequence $0 \to K \to F_1 \xrightarrow{f} F_0 \to T \to 0$, where F_1 and F_0 are finitely generated free. Set $A = \mathrm{Im}(f)$. Then A is of finitely presented type by Proposition 3.6, and hence K is of finite type by [15, Theorem 6.4.14]. So there is a finitely generated submodule L of K such that K/L is GV-torsion. Consider the following pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & F_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K/L & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We have X is finitely presented. Consider the following commutative diagram with rows exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(A,E)_S & \longrightarrow & \mathrm{Hom}_R(X,E)_S & \longrightarrow & \mathrm{Hom}_R(K/L,E)_S & = 0 \\
 & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & \\
0 \rightarrow & \mathrm{Hom}_{R_S}(A_S,E_S) & \rightarrow & \mathrm{Hom}_{R_S}(X_S,E_S) & \rightarrow & \mathrm{Hom}_{R_S}((K/L)_S,E_S) & \rightarrow 0.
\end{array}$$

Note that f_2 is an isomorphism by [15, Theorem 2.6.16]. Since f_3 is a monomorphism, f_1 is also an isomorphism by Five Lemma. Now we consider the following commutative diagram with rows exact:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F_0,E)_S & \longrightarrow & \mathrm{Hom}_R(A,E)_S & \longrightarrow & \mathrm{Ext}^1_R(T,E)_S & \longrightarrow & 0 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g} & & \\
\mathrm{Hom}_{R_S}((F_0)_S,E_S) & \rightarrow & \mathrm{Hom}_{R_S}(A_S,E_S) & \rightarrow & \mathrm{Ext}^1_{R_S}(T_S,E_S) & \rightarrow & 0.
\end{array}$$

Since f_1 and g_1 are isomorphisms, we have g is also an isomorphism by Five Lemma. □

Recall from [3] that a ϕ-ring R is said to be a *ϕ-chain ring* (ϕ-CR for short) if for any non-nilpotent elements $a, b \in R$, either $a \mid b$ or $b \mid a$ in R. A ϕ-ring R is said to be a ϕ-Prüfer *ring* if any finitely generated nonnil ideal I is ϕ-invertible, i.e., $\phi(I)\phi(I^{-1}) = \phi(R)$ where $I^{-1} = \{x \in \mathrm{T}(R) \,|\, Ix \subseteq R\}$. It follows from [1, Corollary 2.10] that a ϕ-ring R is ϕ-Prüfer, if and only if $R_{\mathfrak{m}}$ is a ϕ-CR for any maximal ideal $\mathfrak{m}$ of R, if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is a Prüfer ring.

Let R be a ϕ-ring. Recall from [11] that a nonnil ideal J of R is said to be a ϕ-GV-*ideal* (resp., *ϕ-w-ideal*) of R if $\phi(J)$ is a GV-ideal (resp., w-ideal) of $\phi(R)$. An ideal I of R is *ϕ-w-invertible* if $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ where W is the w-operation of $\phi(R)$. In order to extend PvMDs to ϕ-rings, the authors in [22] gave the notion of ϕ-Prüfer v-multiplication rings: A ϕ-ring R is said to be a ϕ-Prüfer *v-multiplication ring* (ϕ-PvMR for short) provided that any finitely generated nonnil ideal is ϕ-w-invertible. They also show that a ϕ-ring R is a ϕ-PvMR if and only if $R_{\mathfrak{m}}$ is a ϕ-CR for any $\mathfrak{m} \in w\text{-}Max(R)$, if and only if $R/\mathrm{Nil}(R)$ is a PvMD, if and only if $\phi(R)$ is a PvMR.

Recall that an R-module E is said to be *divisible* if $sM = M$ for any regular element $s \in R$, and an R-module M is said to be *h-divisible* provided that M is a quotient of an injective R-module. Evidently, any injective R-module is h-divisible, and any h-divisible module is divisible. The authors in [20] introduced the notion of *nonnil-divisible* modules E in which for any $m \in E$ and any non-nilpotent element $a \in R$, there exists $x \in E$ such that $ax = m$.

Lemma 3.8 ([20, Lemma 2.2])**.** *Let R be an* NP*-ring and E an R-module. Consider the following statements*:

(1) *E is nonnil-divisible*;
(2) *E is divisible*;
(3) $\mathrm{Ext}^1_R(R/\langle a\rangle, E) = 0$ *for any* $a \notin \mathrm{Nil}(R)$.

Then we have (1) $\Rightarrow$ (2) *and* (1) $\Rightarrow$ (3)*. Moreover, if R is a* ZN*-ring, all statements are equivalent.*

Lemma 3.9 ([20, Lemma 2.4])**.** *Let R be an* NP*-ring and E a nonnil-divisible R-module. Then $E_{\mathfrak{p}}$ is a nonnil-divisible $R_{\mathfrak{p}}$-module for any prime ideal $\mathfrak{p}$ of R.*

Let M be an R-module. Recall from [14] that M is said to have *w-rank n* if, for any maximal w-ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module of rank n. Let τ denote the trace map of M, that is, $\tau : M \otimes_R \mathrm{Hom}(M, R) \to R$ defined by $\tau(x \otimes f) = f(x)$ for $x \in M$ and $f \in \mathrm{Hom}_R(M, R)$. M is said to be *w-invertible*, if the trace map τ is a w-isomorphism. It was proved in [14, Theorem 4.13] that an R-module M is w-invertible if and only if M is of finite type and has w-rank 1, if and only if M is w-projective of finite type and has w-rank 1.

Proposition 3.10. *Let R be a strongly ϕ-ring and I a finitely generated nonnil ideal of R. If I is w-projective, then I is ϕ-w-invertible.*

Proof. Let I be a finitely generated nonnil ideal of the strongly ϕ-ring R. Then I is a regular ideal of R. Let $\mathfrak{m}$ be a maximal w-ideal of R. Since I is w-projective R-ideal, $I_{\mathfrak{m}}$ is a free ideal of $R_{\mathfrak{m}}$ by [15, Theorem 6.7.11]. Then $I_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ or $I_{\mathfrak{m}} = 0$. We claim that $I_{\mathfrak{m}} \cong R_{\mathfrak{m}}$. Indeed, let r be a regular element in I. If $I_{\mathfrak{m}} = 0$, then there is an element $s \in R - \mathfrak{m}$ such that $rs = 0$. So $s = 0$, which is a contradiction. Hence $I_{\mathfrak{m}}$ is of rank 1 for any maximal w-ideal $\mathfrak{m}$ of R. By [14, Theorem 4.13], $\phi(I) = I$ is w-invertible since R is a strongly ϕ-ring. Hence I is ϕ-w-invertible. □

Lemma 3.11 ([20, Proposition 2.12])**.** *Let R be an* NP*-ring, $\mathfrak{p}$ a prime ideal of R and M an R-module. Then M is ϕ-torsion over R if and only $M_{\mathfrak{p}}$ is ϕ-torsion over $R_{\mathfrak{p}}$.*

Lemma 3.12. *Let R be an* NP*-ring, M an R-module. If M is ϕ-torsion-free over R, then $M_{\mathfrak{m}}$ is ϕ-torsion-free over $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R. Moreover, if M is* GV*-torsion-free, then the converse also holds.*

Proof. Suppose M is a ϕ-torsion-free R-module. Let $\mathfrak{m}$ be a maximal w-ideal of R and $\frac{m}{s} \in M_{\mathfrak{m}}$. Suppose $I_{\mathfrak{m}}$ is a nonnil ideal of $R_{\mathfrak{m}}$ and $I_{\mathfrak{m}}\frac{m}{s} = 0$ in $M_{\mathfrak{m}}$. Then there exists $t \notin \mathfrak{m}$ such that $tIm = 0$ in R. Since I is nonnil in R by [22, Lemma 1.1], we have It is also nonnil as t is non-nilpotent. Since M is a ϕ-torsion-free, m and thus $\frac{m}{s}$ is equal to 0.

Suppose M is a GV-torsion-free R-module such that $M_{\mathfrak{m}}$ is ϕ-torsion-free over $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R. Let $m \in M$ such that $Im = 0$ for some nonnil ideal I of R. Then $I_{\mathfrak{m}}\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$. Since $I_{\mathfrak{m}}$ is nonnil in $R_{\mathfrak{m}}$ by [22, Lemma 1.1], $\langle m\rangle_{\mathfrak{m}} = 0$ for any maximal w-ideal $\mathfrak{m}$ of R. Thus $\langle m\rangle$ is GV-torsion in M by [15, Theorem 6.2.15]. Since M is GV-torsion-free by assumption, we have $m = 0$. $\square$

It is well-known that an integral domain R is a PvMD if and only if any torsion-free R-module is w-flat, if and only if any (h-)divisible R-module is absolutely w-pure (see [17, 19]). Recently, the authors in [20] characterized ϕ-Prüfer rings in terms of nonnil-FP-injective modules, that is, a strongly ϕ-ring R is a ϕ-Prüfer ring if and only if any ϕ-torsion-free R-module is ϕ-flat, if and only if any (h-)divisible module is nonnil-FP-injective. Now, we characterize ϕ-PvMRs in terms of ϕ-w-flat modules, nonnil-absolutely w-pure modules and w-projective modules, which can be seen as a generalization of the results in [17, 19, 20].

Theorem 3.13. *Let R be a strongly ϕ-ring. The following statements are equivalent for R*:

(1) *R is a ϕ*-PvMR;
(2) *any ϕ-torsion-free R-module is ϕ-w-flat*;
(3) *any nonnil ideal of R is w-flat*;
(4) *any ideal of R is ϕ-w-flat*;
(5) *any* GV*-torsion-free divisible R-module is nonnil-absolutely w-pure*;
(6) *any* GV*-torsion-free h-divisible R-module is nonnil-absolutely w-pure*;
(7) *any finitely generated nonnil ideal of R is w-projective*;
(8) *any finite type nonnil ideal of R is w-projective.*

Proof. (1) $\Rightarrow$ (2): Let $\mathfrak{m}$ be a maximal w-ideal of R and M a ϕ-torsion-free R-module. By Lemma 3.12, $M_{\mathfrak{m}}$ is ϕ-torsion-free over $R_{\mathfrak{m}}$. Since R is a ϕ-PvMR, $R_{\mathfrak{m}}$ is a ϕ-CR by [22, Theorem 3.3]. Then $M_{\mathfrak{m}}$ is ϕ-flat by [23, Theorem 4.3], and thus M is ϕ-w-flat by [22, Theorem 1.4].

(2) $\Rightarrow$ (4): It follows from the fact that R is ϕ-torsion-free since R is a strongly ϕ-ring (see [23, Proposition 2.2]).

(4) $\Leftrightarrow$ (3): Let J be a nonnil ideal of R and I an ideal of R. We have

$$\mathrm{Tor}_1^R(R/J, I) \cong \mathrm{Tor}_2^R(R/J, R/I) \cong \mathrm{Tor}_1^R(R/I, J).$$

Now the assertion follows.

(4) ⇒ (1): See [22, Theorem 3.8].

(3) ⇒ (7): Let I be a finitely generated nonnil ideal of R. Then $I\{x\}$ is a flat $R\{x\}$-ideal. Since R is a strongly ϕ-ring, there exists a non-zero-divisor in I. So $I[x]$ is a regular ideal of $R[x]$, and hence $I\{x\}$ is also a regular ideal of $R\{x\}$. By [12, Corollary 3.1], $I\{x\}$ is a projective $R\{x\}$-ideal. Hence, I is w-projective by [15, Theorem 6.7.18].

(7) ⇒ (1): It follows from Proposition 3.10.

(1) + (7) ⇒ (5): First we claim that R is a nonnil-w-coherent ring. Indeed, let I be a finitely generated nonnil ideal of R. Then I is w-projective by (7). Hence $I\{x\}$ is a finitely generated projective $R\{x\}$-ideal (which implies $I\{x\}$ is a finitely presented $R\{x\}$-ideal) by [15, Theorem 6.7.18]. So I is of finitely presented type by [14, Theorem 3.9]. The claim holds by Lemma 3.2. By (1) and [22, Theorem 3.3], $R_{\mathfrak{m}}$ is a ϕ-chained ring. Note that by Lemma 3.11, $T_{\mathfrak{m}}$ is a finitely presented ϕ-torsion $R_{\mathfrak{m}}$-module. It follows from [23, Theorem 4.1] that $T_{\mathfrak{m}} \cong \bigoplus_{i=1}^{n} R_{\mathfrak{m}}/R_{\mathfrak{m}}x_i$ for some regular element $x_i \in R_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$ is a strongly ϕ-ring by Lemma 2.10. It follows by Lemma 3.8 and Lemma 3.9 that $E_{\mathfrak{m}}$ is a divisible module over $R_{\mathfrak{m}}$. Thus, by Proposition 3.7, we have $\mathrm{Ext}_R^1(T,E)_{\mathfrak{m}} \cong \mathrm{Ext}_{R_{\mathfrak{m}}}^1(T_{\mathfrak{m}},E_{\mathfrak{m}}) = \bigoplus_{i=1}^{n} \mathrm{Ext}_{R_{\mathfrak{m}}}^1(R_{\mathfrak{m}}/R_{\mathfrak{m}}x_i, E_{\mathfrak{m}}) = 0$ by Lemma 3.8. It follows that $\mathrm{Ext}_R^1(T,E)$ is a GV-torsion module. Therefore E is a nonnil-absolutely w-pure module.

(5) ⇒ (6) and (8) ⇒ (7): They hold trivially.

(6) ⇒ (7): Let N be a w-module and I a finitely generated nonnil ideal of R. The short exact sequence $0 \to I \to R \to R/I \to 0$ induces a long exact sequence as follows:

$$0 = \mathrm{Ext}_R^1(R,N) \to \mathrm{Ext}_R^1(I,N) \to \mathrm{Ext}_R^2(R/I,N) \to \mathrm{Ext}_R^2(R,N) = 0.$$

Let $0 \to N \to E \to K \to 0$ be an exact sequence where E is the injective envelope of N. Then E also is a w-module, and hence K is a GV-torsion-free R-module by [15, Theorem 6.1.17]. There exists a long exact sequence as follows:

$$0 = \mathrm{Ext}_R^1(R/I,E) \to \mathrm{Ext}_R^1(R/I,K) \to \mathrm{Ext}_R^2(R/I,N) \to \mathrm{Ext}_R^2(R/I,E) = 0.$$

Thus $\mathrm{Ext}_R^1(I,N) \cong \mathrm{Ext}_R^2(R/I,N) \cong \mathrm{Ext}_R^1(R/I,K)$ is a GV-torsion module as K is nonnil-absolutely w-pure by (6). It follows that I is a w-projective ideal of R.

(7) ⇒ (8): Let I be a finite type nonnil ideal of R. Then there is a finitely generated sub-ideal K of I such that K/I is GV-torsion (see [15, Proposition 6.4.2(3)]). Then I is w-isomorphic to K. We claim that K is a nonnil ideal. Indeed, since I is nonnil, there is an non-nilpotent element $s \in I$. Thus there is a GV-ideal J of R such that $Js \subseteq K$. Since J is nonnil and R is a ϕ-ring, we have K is a nonnil ideal of R. By (7), K is w-projective. And thus I is w-projective by [15, Proposition 6.7.8(1)]. □

Obviously, any nonnil-FP-injective module is nonnil-absolutely w-pure. The following example shows that the converse does not hold in general.

Example 3.14. Let D be a PvMD but not a Prüfer domain, K the quotient field of D. Then the idealization $R = D(+)K$ is a ϕ-PvMR but not a ϕ-Prüfer ring. Note that R is a strongly ϕ-ring by [2, Remark 1]. Thus there is a nonnil-absolutely w-pure divisible module M which is not nonnil-FP-injective by Theorem 3.13 and [20, Theorem 2.13].

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Xiaolei Zhang
School of Mathematical Sciences
Qufu Normal University
Jining 273165, P. R. China
and
School of Mathematics and Statistics
Shandong University of Technology
Zibo 255049, P. R. China
Email address: zxlrghj@163.com