SOME POLYNOMIALS WITH UNIMODULAR ROOTS

ARTŪRAS DUBICKAS

Abstract. In this paper we consider a sequence of polynomials defined by some recurrence relation. They include, for instance, Poupard polynomials and Kreweras polynomials whose coefficients have some combinatorial interpretation and have been investigated before. Extending a recent result of Chapoton and Han we show that each polynomial of this sequence is a self-reciprocal polynomial with positive coefficients whose all roots are unimodular. Moreover, we prove that their arguments are uniformly distributed in the interval \([0, 2\pi)\).

1. Introduction

A polynomial

\[ F(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0, \]

of degree \(n\) is called self-reciprocal or palindromic if \(a_i = a_{n-i}\) for every \(i = 0, 1, \ldots, \lfloor n/2 \rfloor\). Equivalently, \(F(x) = x^n F(1/x)\). For a self-reciprocal polynomial \(F\) of degree \(n\) the polynomial \((x^{n+4} + 1) F(1) - 2x^2 F(x)\) has multiplicity at least 2 at \(x = 1\), since \(F'(1) = n F(1)/2\). Consequently, if \(F(1) \neq 0\), then

\[
\frac{(x^n + 4 + 1) F(1) - 2x^2 F(x)}{(x - 1)^2}
\]

is a polynomial of degree \(n + 2\) with leading coefficient \(F(1)\). Inserting \(x \mapsto 1/x\) into (1) and multiplying it by \(x^{n+2}\), we get the same polynomial in view of \(F(x) = x^n F(1/x)\). Thus, the polynomial (1) is self-reciprocal.

Consider a sequence of polynomials defined by \(F_0(x) = 1\), and

\[
F_n(x) = \frac{(x^{2n+2} + 1) F_{n-1}(1) - 2x^2 F_{n-1}(x)}{(x - 1)^2}
\]

for \(n = 1, 2, 3, \ldots\). Recently, in [3] Chapoton and Han showed that for each \(n \geq 1\) the polynomial \(F_n\) is a self-reciprocal polynomial with positive integer coefficients such that \(\deg F_n = 2n\), and all \(2n\) roots of \(F_n\) lie on the unit circle. The coefficients of those polynomials appear in a paper of Poupard.
[17] and have some combinatorial interpretation: see the table [17, p. 370] which corresponds to the coefficients of \( F_0(x) = 1, F_1(x) = x^2 + 2x + 1, \) \( F_2(x) = 4x^4 + 8x^3 + 10x^2 + 8x + 4, \) etc. The consecutive coefficients of these polynomials form the sequence A008301 in OEIS [20]. See also [6–8] for some calculations with the numbers in the Poupard triangle and their generalizations.

In [3], the polynomials (2) are called *Poupard polynomials*.

The proof of unimodularity of the roots of Poupard polynomials in [3] is based on a criterion of Lakatos and Losonczi [12]. See also [13] for a more general result, [14] for a historical context, [15, 18] for some other criteria for unimodularity of roots of self-inversive polynomials, and, for example, [4, 5, 9, 10, 16, 19] for some other results concerning polynomials with unimodular roots.

By a similar argument based on [12], in [3] the unimodularity of roots of the polynomials \( G_n, n = 0, 1, 2, \ldots \) of degree \( 2n + 1 \) defined by

\[
G_0(x) = x + 1 \quad \text{and} \quad \frac{(x^{2n+3} + 1)G_{n-1}(1) - 2x^2G_{n-1}(x)}{(x-1)^2}
\]

for \( n = 1, 2, 3, \ldots \) was established. Since \( G_n \) has a root at \( x = -1 \) for each \( n \geq 1 \), by setting \( H_n(x) = G_n(x)/(x + 1) \) we get the sequence of polynomials \( (H_n)_{n=0}^{\infty} \), where \( H_0(x) = 1 \) and

\[
H_n(x) = \frac{2H_{n-1}(1)(x^{2n+3} + 1)/(x + 1) - 2x^2H_{n-1}(x)}{(x-1)^2}
\]

for \( n = 1, 2, 3, \ldots \). The coefficients of the polynomials \( 2^{1-n}H_n(x) \) appear in a paper of Kreweras [11] and have some combinatorial interpretation too. The Kreweras triangle have been recently investigated in [1, 2].

It is worth mentioning that (as observed in [3]) the constant terms of Poupard polynomials and Kreweras polynomials are related to the reduced tangent numbers and so-called Genocchi numbers respectively: see the sequences A002105 and A001469 in [20].

In this paper we consider a sequence of polynomials \( (F_n)_{n=0}^{\infty} \) defined by

\[
F_n(x) = u_n \frac{(x^{2n+2} + 1)F_{n-1}(1) - 2x^2F_{n-1}(x)}{(x-1)^2} + v_n \frac{x^{2n+2} - 1}{x^2 - 1}
\]

for \( n = 1, 2, 3, \ldots \), where \((u_n)_{n=1}^{\infty}\) is a sequence of positive numbers and \((v_n)_{n=1}^{\infty}\) is a sequence of nonnegative numbers.

Note that the polynomials (4) include all those defined by (2) and (3). Indeed, selecting in (4) \( c_0 = 1, u_n = 1 \) and \( v_n = 0 \) for each \( n \geq 1 \), we get the sequence of Poupard polynomials (1), while the choice \( c_0 = 1, u_n = 1 \) and \( v_n = H_{n-1}(1) \) for \( n \geq 1 \) leads to the polynomials (3).

With the above assumptions on \( u_n, v_n \) (\( n \geq 1 \)) we will not only show that \( F_n \) defined in (4) is a self-reciprocal polynomial with positive coefficients whose roots are unimodular, but also that the roots of \( F_n \) are uniformly distributed along the unit circle as \( n \to \infty \).
Theorem 1.1. For each $n \in \mathbb{N}$ the polynomial defined by (4) is a self-reciprocal polynomial of degree $2n$ with positive coefficients if $c_0 > 0$, $u_i > 0$ and $v_i \geq 0$ for $i = 1, \ldots, n$. If $c_0, u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{Z}$, then $F_n \in \mathbb{Z}[x]$.

Furthermore, for each $n \in \mathbb{N}$ all $2n$ roots of $F_n$ are unimodular. More precisely, for $n \geq 2$ they are of the form $e^{\pm i \psi_1}, \ldots, e^{\pm i \psi_n}$, with arguments $0 < \psi_1 < \cdots < \psi_n < \pi$ satisfying
\begin{align*}
\psi_{2k-1} &\in \left( \frac{\pi (2k-1)}{n+1}, \frac{2\pi k}{n+1} \right) \\
\text{for } k = 1, \ldots, \lfloor (n+1)/2 \rfloor \\
\psi_{2k} &\in \left( \frac{2\pi k}{n+1}, \frac{\pi (2k+1)}{n+1} \right) \\
\text{for } k = 1, \ldots, \lfloor n/2 \rfloor.
\end{align*}

Note that for $n = 1$ the polynomial $F_1$ may have a double root, and so condition $n \geq 2$ is necessary. Indeed, by (4), we find that
$$F_1(x) = u_1 c_0 (x+1)^2 + v_1 (x^2 + 1).$$
It is a self-reciprocal quadratic polynomial in $\mathbb{R}[x]$ with nonpositive discriminant, so it has two unimodular roots. However, in the case when $v_1 = 0$ it has a double root at $x = -1$.

The proof of the first assertion of the theorem (without claiming positivity of the coefficients of $F_n$) is straightforward. Fix $n \geq 1$ and suppose $F_{n-1}$ is a self-reciprocal polynomial of degree $2n-2$. Then, the first summand on the right hand side of (4) is a degree $2n$ self-reciprocal polynomial by (1). The second summand, $v_n (1 + x^2 + \cdots + x^{2n})$, is either a self-reciprocal polynomial of degree $2n$ or zero (when $v_n = 0$), which implies the first claim of the theorem by induction on $n$. Also, if $F_{n-1} \in \mathbb{Z}[x]$ and $u_n, v_n \in \mathbb{Z}$, then $F_n \in \mathbb{Z}[x]$ by (4). This proves the second assertion of the theorem, namely, $F_n \in \mathbb{Z}[x]$.

In all what follows we will prove that the coefficients of $F_n$ are all positive and also that the roots of $F_n$ are unimodular with arguments as indicated in (5) and (6). There are indeed $n$ arguments in $(0, \pi)$, since $\lfloor (n+1)/2 \rfloor + \lfloor n/2 \rfloor = n$.

In the next section we first prove a useful lemma and then state the remaining part of Theorem 1.1 in terms of cosine polynomials (see Theorem 2.2). In Section 3 we will complete the proof of Theorem 2.2 and so that of Theorem 1.1.

2. Reduction of the problem to cosine trigonometric polynomials

Throughout, for $n \geq 0$ let $\mathcal{T}_n$ be the set of cosine trigonometric polynomials
$$a_n \cos(nx) + a_{n-1} \cos((n-1)x) + \cdots + a_0$$
with positive coefficients $a_n, a_{n-1}, \ldots, a_0$. Set

$$
\phi_n(x) = \begin{cases} \frac{\sin((n+1)x)}{\sin(x)} & \text{if } x \neq \pi k, \ k \in \mathbb{Z}, \\ \frac{\sin((n+1)x)}{n+1} & \text{if } x = \pi k, \ k \in \mathbb{Z}. \end{cases}
$$

By L'Hôpital's rule, the function $\phi_n$ is continuous for each $n \geq 0$. Likewise, for $n \geq j \geq 0$ we define the function

$$
\phi_{n,j}(x) = \begin{cases} \cos((n+1)x) - \cos(jx) & \text{if } x \neq 2\pi k, \ k \in \mathbb{Z}, \\ \frac{(n+1)^2 - j^2}{(n+1)^2 - 1} & \text{if } x = 2\pi k, \ k \in \mathbb{Z}. \end{cases}
$$

Applying L'Hôpital's rule twice, we see that the function $\phi_{n,j}$ is continuous.

Now, we will prove the following lemma:

**Lemma 2.1.** For $n > j \geq 0$ the functions $\phi_n$ and $\phi_{n,j}$ are cosine trigonometric polynomials of degree $n$. Moreover, $\phi_n$ has nonnegative coefficients and $\phi_{n,j} \in T_n$.

**Proof.** Note that for $x \neq \pi k, \ k \in \mathbb{Z},$

$$
\frac{\sin((n+1)x)}{\sin(x)} = e^{inx} + e^{i(n-2)x} + \cdots + e^{-i(n-2)x} + e^{-inx},
$$

since both sides of this identity are equal to $\frac{e^{i(n+1)x} - e^{-i(n+1)x}}{e^{ix} - e^{-ix}}$. The right hand side of (9) can be written as

$$
2\cos(nx) + 2\cos((n-2)x) + \cdots + 2\cos(x)
$$

for $n$ odd, and

$$
2\cos(nx) + 2\cos((n-2)x) + \cdots + 2\cos(2x) + 1
$$

for $n$ even.

At $x = \pi k$ each of $(n+1)/2$ summands of (10) is equal to $2(-1)^k$, since $n$ is odd. So the sum on the right hand side of (9) is $(n+1)(-1)^k$, which equals $(n+1)(-1)^{nk}$. Similarly, at $x = \pi k$ each of $n/2$ first summands of (11) is equal to 2, since $n$ is even. Thus, the sum on the right hand side of (9) is $2 \cdot (n/2) + 1 = n + 1 = (n+1)(-1)^{nk}$. Therefore, in view of (7) and (9) we obtain

$$
\phi_n(x) = e^{inx} + e^{i(n-2)x} + \cdots + e^{-i(n-2)x} + e^{-inx}
$$

for each $n \geq 0$ and each $x \in \mathbb{R}$. By (10), (11), this implies that

$$
\phi_n(x) = 2\cos(nx) + 2\cos((n-2)x) + \cdots + 2\cos(x)
$$

for $n$ odd, and

$$
\phi_n(x) = 2\cos(nx) + 2\cos((n-2)x) + \cdots + 2\cos(2x) + 1
$$

for $n$ even. In both cases, $n$ even or odd, $\phi_n$ is a cosine trigonometric polynomial of degree $n$ with nonnegative coefficients.
In order to prove that $\phi_{n,j}$ is in $T_n$, we fix two positive integers $m$ and $s$ satisfying $m + s = 2n$. Consider the product of two sums

$$e^{imx/2} + e^{i(m-2)x/2} + \cdots + e^{-i(m-2)x/2} + e^{-imx/2} = \phi_m(x/2)$$

and

$$e^{isx/2} + e^{i(s-2)x/2} + \cdots + e^{-i(s-2)x/2} + e^{-isx/2} = \phi_s(x/2)$$

(see (12)). Note that the coefficient for each $e^{\pm ix}$, where $0 \leq \ell \leq n$, will be positive, and the coefficient for $e^{i\ell x}$, $1 \leq \ell \leq n$, is the same as that for $e^{-i\ell x}$.

Consequently, for any positive integers $m$, $s$ satisfying $m + s = 2n$.

$$(13) \quad \phi_m(x/2)\phi_s(x/2) \in T_n$$

for any $x \neq 2\pi k$, $k \in \mathbb{Z}$, we have

$$\phi_m(x/2)\phi_s(x/2) = \frac{\sin((m + 1)x/2)\sin((s + 1)x/2)}{\sin^2(x/2)} = \frac{\cos((m + s + 2)x/2) - \cos((m - s)x/2)}{\cos(x) - 1}.$$

In particular, selecting $m = n + j$ and $s = n - j$, we find that

$$\frac{\cos((n + 1)x) - \cos(jx)}{\cos(x) - 1} = \phi_{n+j}(x/2)\phi_{n-j}(x/2).$$

At $x = 2\pi k$, $k \in \mathbb{Z}$, by (7), we get

$$\phi_{n+j}(\pi k)\phi_{n-j}(\pi k) = (n + j + 1)(-1)^{(n+j)}k(n - j + 1)(-1)^{(n-j)}k$$

$$= (n + j + 1)(n - j + 1) = (n + 1)^2 - j^2.$$

Now, taking into account (8) and (13) we conclude that

$$\phi_{n,j}(x) = \phi_{n+j}(x/2)\phi_{n-j}(x/2) \in T_n.$$

This completes the proof of the lemma. \( \square \)

Suppose $F_n \in \mathbb{R}[x]$ is a self-reciprocal polynomial of degree $2n$. Then,

$$(14) \quad U_n(x) = e^{-inx}F_n(e^{ix})$$

is a cosine trigonometric polynomial of degree $n$. If $U_n$ has $n$ roots in $(0, \pi)$, say $0 < \psi_1 < \cdots < \psi_n < \pi$, then $e^{\pm i\psi_1}, \ldots, e^{\pm i\psi_n}$ are the roots of $F_n$ and vice versa. In particular, all $2n$ roots of $F_n$ are unimodular if and only if $U_n$ has $2n$ roots in $[0, 2\pi)$. The coefficients of $F_n$ are positive if and only if $U_n \in T_n$.

Inserting $e^{ix}$ instead of $x$ into (4) and using (14), by the identities

$$\frac{e^{2(n+1)ix} + 1}{(e^{ix} - 1)^2} F_{n-1}(1) = e^{inx} F_{n-1}(1) \frac{2 \cos((n + 1)x)}{2 \cos(x) - 2}$$

$$= e^{inx} U_{n-1}(0) \frac{\cos((n + 1)x)}{\cos(x) - 1},$$
\[
\frac{2e^{2ix} F_{n-1}(e^{ix})}{(e^{ix} - 1)^2} = e^{inx} \frac{U_{n-1}(x)}{\cos(x) - 1},
\]
and
\[
\frac{e^{2(n+1)ix} - 1}{e^{2ix} - 1} = e^{inx} \frac{\sin((n+1)x)}{\sin(x)},
\]
we obtain
\[
U_n(x) = u_n \frac{\cos((n+1)x)U_{n-1}(0) - U_{n-1}(x)}{\cos(x) - 1} + v_n \frac{\sin((n+1)x)}{\sin(x)}
\]
when \(x \neq \pi k, k \in \mathbb{Z}\). Here, for \(x = \pi k, k \in \mathbb{Z}\), the second summand is defined by (9), (12), while for \(x = 2\pi k, k \in \mathbb{Z}\), the function
\[
U_{n-1}(0) \cos((n+1)x) - U_{n-1}(x) \cos(x) - 1 + v_n \phi_n(x)
\]
is defined by continuity (see also (18) for an explicit expression in terms of (7), (8) and the coefficients of \(U_{n-1}\)).

The unimodularity of the roots of \(F_n\) for \(n = 1\) has been explained below Theorem 1.1. The remaining parts of Theorem 1.1 follow from the next theorem.

**Theorem 2.2.** Let \((u_n)_{n=1}^{\infty}\) be a sequence of positive real numbers, and let \((v_n)_{n=1}^{\infty}\) be a sequence of nonnegative real numbers. Consider the sequence defined by \(U_0(x) = c_0 > 0\) and
\[
U_n(x) = u_n \frac{\cos((n+1)x)U_{n-1}(0) - U_{n-1}(x)}{\cos(x) - 1} + v_n \phi_n(x)
\]
for \(n = 1, 2, 3, \ldots\). Then,
\[
U_n \in \mathcal{T}_n
\]
for each \(n \geq 0\). Furthermore, for every \(n \geq 2\) this cosine trigonometric polynomial \(U_n\) in the interval \([-\pi, \pi]\) has \(2n\) roots \(\pm \psi_1, \ldots, \pm \psi_n\), where \(0 < \psi_1 < \cdots < \psi_n < \pi\) belong to the intervals as described in (5) and (6).

### 3. Proof of Theorem 2.2

The claim (16) is trivial for \(n = 0\). Fix \(n \in \mathbb{N}\) and assume that \(U_{n-1} \in \mathcal{T}_{n-1}\), that is,
\[
U_{n-1}(x) = b_{n-1} \cos((n-1)x) + \cdots + b_1 \cos(x) + b_0,
\]
where \(b_{n-1}, \ldots, b_1, b_0 > 0\).

By Lemma 2.1, \(v_n \phi_n(x)\) is a degree \(n\) cosine trigonometric polynomial with nonnegative coefficients (2, 0 and possibly 1 if \(n\) is even) or zero identically (if \(v_n = 0\)). So, in order to show that \(U_n \in \mathcal{T}_n\) it suffices to prove that
\[
\frac{U_{n-1}(0) \cos((n+1)x) - U_{n-1}(x)}{\cos(x) - 1} \in \mathcal{T}_n.
\]
Indeed, from $U_{n-1}(0) = \sum_{k=0}^{n-1} b_k$ it follows that
\[
\frac{U_{n-1}(0) \cos((n+1)x) - U_{n-1}(x)}{\cos(x) - 1} = \sum_{j=0}^{n-1} b_j \frac{\cos((n+1)x) - \cos(jx)}{\cos(x) - 1} = \sum_{j=0}^{n-1} b_j \phi_{n,j}(x) \in T_n
\]
by (8) and Lemma 2.1. This finishes the proof of (17) and so that of (16). Note that, by (15), we have
\[
(18) \quad U_n(x) = u_n \sum_{j=0}^{n-1} b_j \phi_{n,j}(x) + v_n \phi_n(x).
\]
Next, we will investigate the cosine polynomial $U_n$ in the interval $[0, \pi]$ for $n \geq 2$. Using (7), (8) and (18) at $x = 0$ we derive that
\[
U_n(0) = u_n \sum_{j=0}^{n-1} b_j ((n+1)^2 - j^2) + v_n (n+1) > 0.
\]
Set
\[
y_k = \frac{2\pi k}{n+1} \quad \text{and} \quad z_k = \frac{\pi(2k+1)}{n+1} \quad \text{for} \quad k = 0, 1, \ldots, n.
\]
Then,
\[
0 = y_0 < z_0 < y_1 < z_1 < \cdots < y_n < z_n < 2\pi.
\]
We have just shown that $U_n(y_k) = U_n(0) > 0$. We next claim that $U_n(y_k) < 0$ for $k = 1, 2, \ldots, n$ and $U_n(z_k) > 0$ for $k = 0, 1, \ldots, n$.

For $k = 1, \ldots, n$ it is clear that
\[
\phi_n(y_k) = \frac{\sin((n+1)y_k)}{\sin(y_k)} = 0,
\]
unless $y_k = \pi$. This is only possible if $n + 1$ is even and $k = (n + 1)/2$. Then, by (7) (with $k = 1$), $\phi_n(\pi) = (n + 1)(-1)^n = -n - 1$. So, for each $k = 1, \ldots, n$ we have $v_n \phi_n(y_k) \leq 0$.

Therefore, in order to show that $U_n(y_k) < 0$ for $k = 1, \ldots, n$, by (15), $\cos((n+1)y_k) = 1$ and $\cos(y_k) - 1 < 0$, it suffices to verify the inequality $U_{n-1}(0) > U_{n-1}(y_k)$. This is indeed the case in view of $n \geq 2$ and $U_{n-1} \in T_{n-1}$, since then
\[
U_{n-1}(0) = \sum_{j=0}^{n-1} b_j > \sum_{j=0}^{n-1} b_j \cos(jy_k) = U_{n-1}(y_k),
\]
which is true by $b_{n-1}, \ldots, b_1, b_0 > 0$ and $\cos(y_k) < 1$. Hence, $U_n(y_k) < 0$ for $k = 1, \ldots, n$.

The proof of the inequality $U_n(z_k) > 0$ for $k = 0, 1, \ldots, n$ is similar. It is clear that $\phi_n(z_k) = 0$, unless $z_k = \pi$. In that case, $n$ is even and $k = n/2$. Then, by (7), $\phi_n(\pi) = n + 1$. So, $v_n \phi_n(z_k) \geq 0$ for each $k = 0, 1, \ldots, n$. 
Now, as \( \cos((n + 1)z_k) = -1 \), in order to show that \( U_n(z_k) > 0 \) for \( k = 0, 1, \ldots, n \), by (15) and \( \cos(z_k) - 1 < 0 \), it suffices to verify the inequality
\[
U_{n-1}(0) + U_{n-1}(z_k) = \sum_{j=0}^{n-1} b_j(1 + \cos(jz_k)) > 0.
\]
This is true in view of \( b_{n-1}, \ldots, b_1, b_0 > 0 \).

The inequalities \( U_n(y_k) < 0 \), \( k = 1, 2, \ldots, n \), and \( U_n(z_k) > 0 \), \( k = 0, 1, \ldots, n \), imply that \( U_n(x) \) has a root in each of the open intervals
\[
(y_0, z_1), (z_1, y_2), \ldots, (z_{n-1}, y_n), (y_n, z_n).
\]
Consequently, the intervals lying in \([0, \pi]\) that contain a root of \( U_n \) can be described as in (5) and (6). In particular, the interval \((0, \pi)\) contains precisely \( \lfloor (n + 1)/2 \rfloor + \lfloor n/2 \rfloor = n \) roots of \( U_n \). Finally, since \( U_n \in T_n \), any \( \psi \in (0, \pi) \) is a root of \( U_n \) whenever \( -\psi \) is a root of \( U_n \).

References


Artūras Dubickas
Institute of Mathematics
Faculty of Mathematics and Informatics
Vilnius University
Naugarduko 24
LT-03225 Vilnius, Lithuania
Email address: arturas.dubickas@mif.vu.lt