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SOME POLYNOMIALS WITH UNIMODULAR ROOTS

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Abstract. In this paper we consider a sequence of polynomials defined by some recurrence relation. They include, for instance, Poupard polynomials and Kreweras polynomials whose coefficients have some combinatorial interpretation and have been investigated before. Extending a recent result of Chapoton and Han we show that each polynomial of this sequence is a self-reciprocal polynomial with positive coefficients whose all roots are unimodular. Moreover, we prove that their arguments are uniformly distributed in the interval $[0, 2\pi)$.

1. Introduction

A polynomial

$$F(x) = a_n x^n + \cdots + a_1 x + a_0,\ a_n \neq 0,$$

of degree n is called *self-reciprocal* or *palindromic* if $a_i = a_{n-i}$ for every $i = 0, 1, \ldots, \lfloor n/2 \rfloor$. Equivalently, $F(x) = x^n F(1/x)$. For a self-reciprocal polynomial F of degree n the polynomial $(x^{n+4}+1)F(1) - 2x^2 F(x)$ has multiplicity at least 2 at $x = 1$, since $F'(1) = nF(1)/2$. Consequently, if $F(1) \neq 0$, then

$$\frac{(x^{n+4}+1)F(1) - 2x^2 F(x)}{(x-1)^2} \tag{1}$$

is a polynomial of degree $n+2$ with leading coefficient $F(1)$. Inserting $x \mapsto 1/x$ into (1) and multiplying it by x^{n+2}, we get the same polynomial in view of $F(x) = x^n F(1/x)$. Thus, the polynomial (1) is self-reciprocal.

Consider a sequence of polynomials defined by $F_0(x) = 1$, and

$$F_n(x) = \frac{(x^{2n+2}+1)F_{n-1}(1) - 2x^2 F_{n-1}(x)}{(x-1)^2} \tag{2}$$

for $n = 1, 2, 3, \ldots$. Recently, in [3] Chapoton and Han showed that for each $n \geq 1$ the polynomial F_n is a self-reciprocal polynomial with positive integer coefficients such that $\deg F_n = 2n$, and all $2n$ roots of F_n lie on the unit circle. The coefficients of those polynomials appear in a paper of Poupard

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[17] and have some combinatorial interpretation: see the table [17, p. 370] which corresponds to the coefficients of $F_0(x) = 1$, $F_1(x) = x^2 + 2x + 1$, $F_2(x) = 4x^4 + 8x^3 + 10x^2 + 8x + 4$, etc. The consecutive coefficients of these polynomials form the sequence A008301 in OEIS [20]. See also [6–8] for some calculations with the numbers in the Poupard triangle and their generalizations. In [3], the polynomials (2) are called *Poupard polynomials.*

The proof of unimodularity of the roots of Poupard polynomials in [3] is based on a criterion of Lakatos and Losonczi [12]. See also [13] for a more general result, [14] for a historical context, [15, 18] for some other criteria for unimodularity of roots of self-inversive polynomials, and, for example, [4, 5, 9, 10,16,19] for some other results concerning polynomials with unimodular roots.

By a similar argument based on [12], in [3] the unimodularity of roots of the polynomials G_n, $n = 0, 1, 2, \dots$, of degree $2n+1$ defined by $G_0(x) = x+1$ and

$$G_n(x) = \frac{(x^{2n+3}+1)G_{n-1}(1) - 2x^2 G_{n-1}(x)}{(x-1)^2}$$

for $n = 1, 2, 3, \dots$ was established. Since G_n has a root at $x = -1$ for each $n \geq 0$, by setting $H_n(x) = G_n(x)/(x+1)$ we get the sequence of polynomials $(H_n)_{n=0}^{\infty}$, where $H_0(x) = 1$ and

$$H_n(x) = \frac{2H_{n-1}(1)(x^{2n+3}+1)/(x+1) - 2x^2 H_{n-1}(x)}{(x-1)^2} \tag{3}$$

for $n = 1, 2, 3, \dots$. The coefficients of the polynomials $2^{1-n}H_n(x)$ appear in a paper of Kreweras [11] and have some combinatorial interpretation too. The Kreweras triangle have been recently investigated in [1, 2].

It is worth mentioning that (as observed in [3]) the constant terms of Poupard polynomials and Kreweras polynomials are related to the reduced tangent numbers and so-called Genocchi numbers respectively: see the sequences A002105 and A001469 in [20].

In this paper we consider a sequence of polynomials $(F_n)_{n=0}^{\infty}$ defined by $F_0(x) = c_0 > 0$ and

$$F_n(x) = u_n \frac{(x^{2n+2}+1)F_{n-1}(1) - 2x^2 F_{n-1}(x)}{(x-1)^2} + v_n \frac{x^{2n+2}-1}{x^2-1} \tag{4}$$

for $n = 1, 2, 3, \dots$, where $(u_n)_{n=1}^{\infty}$ is a sequence of positive numbers and $(v_n)_{n=1}^{\infty}$ is a sequence of nonnegative numbers.

Note that the polynomials (4) include all those defined by (2) and (3). Indeed, selecting in (4) $c_0 = 1$, $u_n = 1$ and $v_n = 0$ for each $n \geq 1$, we get the sequence of Poupard polynomials (1), while the choice $c_0 = 1$, $u_n = 1$ and $v_n = H_{n-1}(1)$ for $n \geq 1$ leads to the polynomials (3).

With the above assumptions on u_n, v_n $(n \geq 1)$ we will not only show that F_n defined in (4) is a self-reciprocal polynomial with positive coefficients whose roots are unimodular, but also that the roots of F_n are uniformly distributed along the unit circle as $n \to \infty$.

Theorem 1.1. *For each* $n \in \mathbb{N}$ *the polynomial defined by* (4) *is a self-reciprocal polynomial of degree* $2n$ *with positive coefficients if* $c_0 > 0$, $u_i > 0$ *and* $v_i \geq 0$ *for* $i = 1, \dots, n$. *If* $c_0, u_1, \dots, u_n, v_1, \dots, v_n \in \mathbb{Z}$, *then* $F_n \in \mathbb{Z}[x]$.

Furthermore, for each $n \in \mathbb{N}$ *all* $2n$ *roots of* F_n *are unimodular. More precisely, for* $n \geq 2$ *they are of the form* $e^{\pm i\psi_1}, \dots, e^{\pm i\psi_n}$, *with arguments*

$$0 < \psi_1 < \cdots < \psi_n < \pi$$

satisfying

$$\psi_{2k-1} \in \Big(\frac{\pi(2k-1)}{n+1}, \frac{2\pi k}{n+1}\Big) \tag{5}$$

for $k = 1, \dots, \lfloor (n+1)/2 \rfloor$ *and*

$$\psi_{2k} \in \Big(\frac{2\pi k}{n+1}, \frac{\pi(2k+1)}{n+1}\Big) \tag{6}$$

for $k = 1, \dots, \lfloor n/2 \rfloor$.

Note that for $n = 1$ the polynomial F_1 may have a double root, and so condition $n \geq 2$ is necessary. Indeed, by (4), we find that

$$F_1(x) = u_1 c_0 (x+1)^2 + v_1(x^2+1).$$

It is a self-reciprocal quadratic polynomial in $\mathbb{R}[x]$ with nonpositive discriminant, so it has two unimodular roots. However, in the case when $v_1 = 0$ it has a double root at $x = -1$.

The proof of the first assertion of the theorem (without claiming positivity of the coefficients of F_n) is straightforward. Fix $n \geq 1$ and suppose F_{n-1} is a self-reciprocal polynomial of degree $2n-2$. Then, the first summand on the right hand side of (4) is a degree $2n$ self-reciprocal polynomial by (1). The second summand, $v_n(1 + x^2 + \cdots + x^{2n})$, is either a self-reciprocal polynomial of degree $2n$ or zero (when $v_n = 0$), which implies the first claim of the theorem by induction on n. Also, if $F_{n-1} \in \mathbb{Z}[x]$ and $u_n, v_n \in \mathbb{Z}$, then $F_n \in \mathbb{Z}[x]$ by (4). This proves the second assertion of the theorem, namely, $F_n \in \mathbb{Z}[x]$.

In all what follows we will prove that the coefficients of F_n are all positive and also that the roots of F_n are unimodular with arguments as indicated in (5) and (6). There are indeed n arguments in $(0, \pi)$, since $\lfloor (n+1)2 \rfloor + \lfloor n/2 \rfloor = n$.

In the next section we first prove a useful lemma and then state the remaining part of Theorem 1.1 in terms of cosine polynomials (see Theorem 2.2). In Section 3 we will complete the proof of Theorem 2.2 and so that of Theorem 1.1.

2. Reduction of the problem to cosine trigonometric polynomials

Throughout, for $n \geq 0$ let $\mathcal{T}_n$ be the set of cosine trigonometric polynomials

$$a_n \cos(nx) + a_{n-1} \cos((n-1)x) + \cdots + a_0$$

with positive coefficients $a_n, a_{n-1}, \dots, a_0$. Set

$$\phi_n(x) = \begin{cases} \frac{\sin((n+1)x)}{\sin(x)} & \text{if} \quad x \neq \pi k,\ k \in \mathbb{Z}, \\ (n+1)(-1)^{nk} & \text{if} \quad x = \pi k,\ k \in \mathbb{Z}. \end{cases} \tag{7}$$

By L'Hôpital's rule, the function ϕ_n is continuous for each $n \geq 0$. Likewise, for $n \geq j \geq 0$ we define the function

$$\phi_{n,j}(x) = \begin{cases} \frac{\cos((n+1)x)-\cos(jx)}{\cos(x)-1} & \text{if} \quad x \neq 2\pi k,\ k \in \mathbb{Z}, \\ (n+1)^2 - j^2 & \text{if} \quad x = 2\pi k,\ k \in \mathbb{Z}. \end{cases} \tag{8}$$

Applying L'Hôpital's rule twice, we see that the function $\phi_{n,j}$ is continuous.

Now, we will prove the following lemma:

Lemma 2.1. *For $n > j \geq 0$ the functions ϕ_n and $\phi_{n,j}$ are cosine trigonometric polynomials of degree n. Moreover, ϕ_n has nonnegative coefficients and $\phi_{n,j} \in \mathcal{T}_n$.*

Proof. Note that for $x \neq \pi k$, $k \in \mathbb{Z}$,

$$\frac{\sin((n+1)x)}{\sin(x)} = e^{inx} + e^{i(n-2)x} + \dots + e^{-i(n-2)x} + e^{-inx}, \tag{9}$$

since both sides of this identity are equal to $\frac{e^{i(n+1)x} - e^{-i(n+1)x}}{e^{ix} - e^{-ix}}$. The right hand side of (9) can be written as

$$2\cos(nx) + 2\cos((n-2)x) + \dots + 2\cos(x) \tag{10}$$

for n odd, and

$$2\cos(nx) + 2\cos((n-2)x) + \dots + 2\cos(2x) + 1 \tag{11}$$

for n even.

At $x = \pi k$ each of $(n+1)/2$ summands of (10) is equal to $2(-1)^k$, since n is odd. So the sum on the right hand side of (9) is $(n+1)(-1)^k$, which equals $(n+1)(-1)^{nk}$. Similarly, at $x = \pi k$ each of $n/2$ first summands of (11) is equal to 2, since n is even. Thus, the sum on the right hand side of (9) is $2 \cdot (n/2) + 1 = n + 1 = (n+1)(-1)^{nk}$. Therefore, in view of (7) and (9) we obtain

$$\phi_n(x) = e^{inx} + e^{i(n-2)x} + \dots + e^{-i(n-2)x} + e^{-inx} \tag{12}$$

for each $n \geq 0$ and each $x \in \mathbb{R}$. By (10), (11), this implies that

$$\phi_n(x) = 2\cos(nx) + 2\cos((n-2)x) + \dots + 2\cos(x)$$

for n odd, and

$$\phi_n(x) = 2\cos(nx) + 2\cos((n-2)x) + \dots + 2\cos(2x) + 1$$

for n even. In both cases, n even or odd, ϕ_n is a cosine trigonometric polynomial of degree n with nonnegative coefficients.

In order to prove that $\phi_{n,j}$ is in $\mathcal{T}_n$, we fix two positive integers m and s satisfying $m+s=2n$. Consider the product of two sums

$$e^{imx/2}+e^{i(m-2)x/2}+\cdots+e^{-i(m-2)x/2}+e^{-imx/2}=\phi_m(x/2)$$

and

$$e^{isx/2}+e^{i(s-2)x/2}+\cdots+e^{-i(s-2)x/2}+e^{-isx/2}=\phi_s(x/2)$$

(see (12)). Note that the coefficient for each $e^{\pm i\ell x}$, where $0\le \ell\le n$, will be positive, and the coefficient for $e^{i\ell x}$, $1\le \ell\le n$, is the same as that for $e^{-i\ell x}$. Consequently,

$$\phi_m(x/2)\phi_s(x/2)\in\mathcal{T}_n \tag{13}$$

for any positive integers m,s satisfying $m+s=2n$. Observe that, by (9) and (12), for $x\ne 2\pi k$, $k\in\mathbb{Z}$, we have

$$\begin{aligned}\phi_m(x/2)\phi_s(x/2)&=\frac{\sin((m+1)x/2)\sin((s+1)x/2)}{\sin^2(x/2)}\\&=\frac{\cos((m+s+2)x/2)-\cos((m-s)x/2)}{\cos(x)-1}.\end{aligned}$$

In particular, selecting $m=n+j$ and $s=n-j$, we find that

$$\frac{\cos((n+1)x)-\cos(jx)}{\cos(x)-1}=\phi_{n+j}(x/2)\phi_{n-j}(x/2).$$

At $x=2\pi k$, $k\in\mathbb{Z}$, by (7), we get

$$\begin{aligned}\phi_{n+j}(\pi k)\phi_{n-j}(\pi k)&=(n+j+1)(-1)^{(n+j)k}(n-j+1)(-1)^{(n-j)k}\\&=(n+j+1)(n-j+1)=(n+1)^2-j^2.\end{aligned}$$

Now, taking into account (8) and (13) we conclude that

$$\phi_{n,j}(x)=\phi_{n+j}(x/2)\phi_{n-j}(x/2)\in\mathcal{T}_n.$$

This completes the proof of the lemma. □

Suppose $F_n\in\mathbb{R}[x]$ is a self-reciprocal polynomial of degree $2n$. Then,

$$U_n(x)=e^{-inx}F_n(e^{ix}) \tag{14}$$

is a cosine trigonometric polynomial of degree n. If U_n has n roots in $(0,\pi)$, say $0<\psi_1<\cdots<\psi_n<\pi$, then $e^{\pm i\psi_1},\ldots,e^{\pm i\psi_n}$ are the roots of F_n and vice versa. In particular, all $2n$ roots of F_n are unimodular if and only if U_n has $2n$ roots in $[0,2\pi)$. The coefficients of F_n are positive if and only if $U_n\in\mathcal{T}_n$.

Inserting e^{ix} instead of x into (4) and using (14), by the identities

$$\begin{aligned}\frac{e^{2(n+1)ix}+1}{(e^{ix}-1)^2}F_{n-1}(1)&=e^{inx}F_{n-1}(1)\frac{2\cos((n+1)x)}{2\cos(x)-2}\\&=e^{inx}U_{n-1}(0)\frac{\cos((n+1)x)}{\cos(x)-1},\end{aligned}$$

$$\frac{2e^{2ix}F_{n-1}(e^{ix})}{(e^{ix}-1)^2} = e^{inx}\frac{U_{n-1}(x)}{\cos(x)-1},$$

and

$$\frac{e^{2(n+1)ix}-1}{e^{2ix}-1} = e^{inx}\frac{\sin((n+1)x)}{\sin(x)},$$

we obtain

$$U_n(x) = u_n\frac{\cos((n+1)x)U_{n-1}(0)-U_{n-1}(x)}{\cos(x)-1} + v_n\frac{\sin((n+1)x)}{\sin(x)}$$

when $x \neq \pi k$, $k \in \mathbb{Z}$. Here, for $x = \pi k$, $k \in \mathbb{Z}$, the second summand is defined by (9), (12), while for $x = 2\pi k$, $k \in \mathbb{Z}$, the function

$$\frac{U_{n-1}(0)\cos((n+1)x)-U_{n-1}(x)}{\cos(x)-1}$$

is defined by continuity (see also (18) for an explicit expression in terms of (7), (8) and the coefficients of U_{n-1}).

The unimodularity of the roots of F_n for $n = 1$ has been explained below Theorem 1.1. The remaining parts of Theorem 1.1 follow from the next theorem.

Theorem 2.2. *Let* $(u_n)_{n=1}^{\infty}$ *be a sequence of positive real numbers, and let* $(v_n)_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers. Consider the sequence defined by* $U_0(x) = c_0 > 0$ *and*

$$U_n(x) = u_n\frac{U_{n-1}(0)\cos((n+1)x)-U_{n-1}(x)}{\cos(x)-1} + v_n\phi_n(x) \quad (15)$$

for $n = 1, 2, 3, \ldots$. *Then,*

$$U_n \in \mathcal{T}_n \quad (16)$$

for each $n \geq 0$. *Furthermore, for every* $n \geq 2$ *this cosine trigonometric polynomial* U_n *in the interval* $[-\pi, \pi)$ *has* $2n$ *roots* $\pm\psi_1, \ldots, \pm\psi_n$, *where* $0 < \psi_1 < \cdots < \psi_n < \pi$ *belong to the intervals as described in* (5) *and* (6).

3. Proof of Theorem 2.2

The claim (16) is trivial for $n = 0$. Fix $n \in \mathbb{N}$ and assume that $U_{n-1} \in \mathcal{T}_{n-1}$, that is,

$$U_{n-1}(x) = b_{n-1}\cos((n-1)x) + \cdots + b_1\cos(x) + b_0,$$

where $b_{n-1}, \ldots, b_1, b_0 > 0$.

By Lemma 2.1, $v_n\phi_n(x)$ is a degree n cosine trigonometric polynomial with nonnegative coefficients (2, 0 and possibly 1 if n is even) or zero identically (if $v_n = 0$). So, in order to show that $U_n \in \mathcal{T}_n$ it suffices to prove that

$$\frac{U_{n-1}(0)\cos((n+1)x)-U_{n-1}(x)}{\cos(x)-1} \in \mathcal{T}_n. \quad (17)$$

Indeed, from $U_{n-1}(0)=\sum_{k=0}^{n-1} b_k$ it follows that

$$\frac{U_{n-1}(0)\cos((n+1)x)-U_{n-1}(x)}{\cos(x)-1} = \sum_{j=0}^{n-1} b_j \frac{\cos((n+1)x)-\cos(jx)}{\cos(x)-1} = \sum_{j=0}^{n-1} b_j \phi_{n,j}(x) \in \mathcal{T}_n$$

by (8) and Lemma 2.1. This finishes the proof of (17) and so that of (16). Note that, by (15), we have

$$U_n(x) = u_n \sum_{j=0}^{n-1} b_j \phi_{n,j}(x) + v_n \phi_n(x). \tag{18}$$

Next, we will investigate the cosine polynomial U_n in the interval $[0,\pi]$ for $n \geq 2$. Using (7), (8) and (18) at $x=0$ we derive that

$$U_n(0) = u_n \sum_{j=0}^{n-1} b_j((n+1)^2 - j^2) + v_n(n+1) > 0.$$

Set

$$y_k = \frac{2\pi k}{n+1} \quad \text{and} \quad z_k = \frac{\pi(2k+1)}{n+1} \quad \text{for} \quad k=0,1,\dots,n.$$

Then,

$$0 = y_0 < z_0 < y_1 < z_1 < \cdots < y_n < z_n < 2\pi.$$

We have just shown that $U_n(y_0)=U_n(0)>0$. We next claim that $U_n(y_k)<0$ for $k=1,2,\dots,n$ and $U_n(z_k)>0$ for $k=0,1,\dots,n$.

For $k=1,\dots,n$ it is clear that

$$\phi_n(y_k) = \frac{\sin((n+1)y_k)}{\sin(y_k)} = 0,$$

unless $y_k=\pi$. This is only possible if $n+1$ is even and $k=(n+1)/2$. Then, by (7) (with $k=1$), $\phi_n(\pi)=(n+1)(-1)^n=-n-1$. So, for each $k=1,\dots,n$ we have $v_n\psi_n(y_k)\leq 0$.

Therefore, in order to show that $U_n(y_k)<0$ for $k=1,\dots,n$, by (15), $\cos((n+1)y_k)=1$ and $\cos(y_k)-1<0$, it suffices to verify the inequality $U_{n-1}(0)>U_{n-1}(y_k)$. This is indeed the case in view of $n\geq 2$ and $U_{n-1}\in\mathcal{T}_{n-1}$, since then

$$U_{n-1}(0) = \sum_{j=0}^{n-1} b_j > \sum_{j=0}^{n-1} b_j \cos(jy_k) = U_{n-1}(y_k),$$

which is true by $b_{n-1},\dots,b_1,b_0>0$ and $\cos(y_k)<1$. Hence, $U_n(y_k)<0$ for $k=1,\dots,n$.

The proof of the inequality $U_n(z_k)>0$ for $k=0,1,\dots,n$ is similar. It is clear that $\phi_n(z_k)=0$, unless $z_k=\pi$. In that case, n is even and $k=n/2$. Then, by (7), $\phi_n(\pi)=n+1$. So, $v_n\psi_n(z_k)\geq 0$ for each $k=0,1,\dots,n$.

Now, as $\cos((n+1)z_k) = -1$, in order to show that $U_n(z_k) > 0$ for $k = 0, 1, \dots, n$, by (15) and $\cos(z_k) - 1 < 0$, it suffices to verify the inequality

$$U_{n-1}(0) + U_{n-1}(z_k) = \sum_{j=0}^{n-1} b_j(1 + \cos(jz_k)) > 0.$$

This is true in view of $b_{n-1}, \dots, b_1, b_0 > 0$.

The inequalities $U_n(y_k) < 0$, $k = 1, 2, \dots, n$, and $U_n(z_k) > 0$, $k = 0, 1, \dots, n$, imply that $U_n(x)$ has a root in each of the open intervals

$$(z_0, y_1), (y_1, z_1), (z_1, y_2), \dots, (z_{n-1}, y_n), (y_n, z_n).$$

Consequently, the intervals lying in $[0, \pi]$ that contain a root of U_n can be described as in (5) and (6). In particular, the interval $(0, \pi)$ contains precisely $\lfloor (n+1)2 \rfloor + \lfloor n/2 \rfloor = n$ roots of U_n. Finally, since $U_n \in \mathcal{T}_n$, any $\psi \in (0, \pi)$ is a root of U_n whenever $-\psi$ is a root of U_n.

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