Bull. Korean Math. Soc. **59** (2022), No. 5, pp. 1119–1129
https://doi.org/10.4134/BKMS.b210622
pISSN: 1015-8634 / eISSN: 2234-3016

THE MATRIX REPRESENTATION OF A COMPOSITION OPERATOR ON THE HARDY SPACE

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Abstract. We formulate the matrix representation of a composition operator on the Hardy space of the unit disc with the symbol which is a Riemann map of the unit disc, with respect to a special orthonormal basis.

1. Introduction

Given a set X and a function $\varphi : X \to X$, the composition operator C_φ on a Banach space $\mathcal{H}$ on X with symbol φ is defined by $C_\varphi(f) = f \circ \varphi$ for $f \in \mathcal{H}$. One of important research areas related to composition operators is how the properties of the operator relates to those of the symbol. On the other hand, when the Banach space $\mathcal{H}$ is a Hilbert space, the composition operator is heavily dependent on the matrix representation of the operator with respect to orthonormal bases for the function space. However, except for very simple cases, computation of the matrix of an composition operator in an infinite dimensional Hilbert space is very complicated and difficult, so it has rarely been formulated so far. Suppose now that the base set is the unit disc U in the complex plane and the function space is the Hardy space $H^2(bU)$. In this category the composition operator C_φ with a holomorphic self map φ on U becomes a bounded operator on $H^2(bU)$. In particular, when φ is a Riemann map which maps the unit disc into itself, it characterizes automorphisms up to rotations and the associated composition operator becomes complex symmetric and vice versa with certain conditions. See [2] and [6] for references. So, from this point of view, it is very important to formulate the matrix of the composition operator with the symbol of the Riemann map. In this paper, we use a special orthonormal basis for the Hardy space which is a generalization of monomials z^n to compute the matrix representation of the composition operator with respect to the basis when the symbol is a Riemann self map of the unit disc.

Received August 22, 2021; Revised June 3, 2022; Accepted July 15, 2022.
2020 *Mathematics Subject Classification.* Primary 47B33, 47B02, 47B91, 30H10, 30C40.
Key words and phrases. Composition operator, matrix representation, Riemann map, Szegő kernel.

2. Some preliminaries

Throughout the paper, we denote by U the unit disc in the complex plane and we fix a point a in U, unless otherwise specified. Let $L^2(bU)$ be the space of square integrable functions on the boundary bU of the unit disc U with the usual inner product $\langle u, v\rangle = \int_{bU} u\overline{v}\, ds$, where ds is the differential element of arc length on bU. And let $H^2(bU)$ be the classical Hardy space which is the space of holomorphic functions on U with L^2-boundary values in bU.

For a holomorphic self map φ of U, the composition operator C_φ with the symbol φ is defined by $C_\varphi(v) = v \circ \varphi$ for $v \in H^2(bU)$. It is well known that C_φ is a bounded linear operator on the Hardy space $H^2(bU)$. See [5] for details. The properties of the operator C_φ depends heavily on the symbol φ. In this paper, from now on, it is assumed that the symbol φ is the Riemann map

$$f_a(z) = \frac{z-a}{1-\overline{a}z}.$$

The reason is that f_a is the only function that characterizes the automorphisms of the unit disk except for rotations. The reason more important than this is that it is the only symbol that makes C_{f_a} a complex symmetric operator under suitable conditions. See [2] for this matter.

For a positive integer m, we define the function v_m by

$$v_m(z) = \frac{\sqrt{1-|a|^2}}{\sqrt{2\pi}} \frac{(z-a)^{m-1}}{(1-\overline{a}z)^m}$$

which is holomorphic in a neighborhood of $\overline{U}$. It is well known that the class of v_m, $m = 1, 2, 3, \ldots$, forms an orthonormal basis for the Hardy space $h^2(bU)$. See [1] for general cases. The author also proved that the set $\{v_m \,|\, m = 0, \pm1, \pm2, \ldots\}$ forms an orthonormal basis for $L^2(bU)$. See [3] and [4] for details.

Now we would like to compute the matrix representation $[C_{f_a}]$ of the composition operator C_{f_a} on the Hardy space with respect to the orthonormal basis $\{v_m \,|\, m = 1, 2, \ldots\}$. Observe that since the basis is orthonormal, for positive integers l and m, the (l, m)-th entry of the matrix $[C_{f_a}]$ is obtained by $[C_{f_a}]_{lm} = \langle C_{f_a}(v_m), v_l\rangle$. Before keeping on, we lists several properties for easy computation whose proofs are all trivial. Suppose that z is on the boundary of the unit disc. Then the following identities hold.

$$\overline{\left(\frac{1-\overline{a}z}{z-a}\right)} = \frac{z-a}{1-\overline{a}z}. \tag{1}$$

$$\overline{\left(\frac{1}{z-a}\right)} = \frac{z}{1-\overline{a}z}. \tag{2}$$

$$\overline{\left(\frac{1}{1-\overline{a}z}\right)} = \frac{z}{z-a}. \tag{3}$$

$$z\, ds = -i\, dz. \tag{4}$$

We are ready to compute the entry $[C_{f_a}]_{lm}$.

$$[C_{f_a}]_{lm} = \langle C_{f_a}(v_m), v_l\rangle = \langle v_m \circ f_a, v_l\rangle$$

$$= \frac{1-|a|^2}{2\pi}\int_{bU}\left[(f_a^{m-1}\circ f_a)(z)\right]\left(\frac{1}{1-\overline{a}z}\circ f_a(z)\right)\overline{f_a(z)^{l-1}}\overline{\left(\frac{1}{1-\overline{a}z}\right)}\,ds_z. \tag{5}$$

Observe that

$$f_a \circ f_a(z) = \frac{-2a+(1+|a|^2)z}{1+|a|^2-2\overline{a}z}.$$

It thus follows from (1) and (3) that the identity (5) is equal to

$$\frac{1-|a|^2}{2\pi}\int_{bU}(f^{m-1})(z)\,\frac{1-\overline{a}z}{1+|a|^2-2\overline{a}z}\,\left(\frac{1-\overline{a}z}{z-a}\right)^{l-1}\frac{z}{z-a}\,ds_z,$$

where $f(z) = \frac{-2a+(1+|a|^2)z}{1+|a|^2-2\overline{a}z}$. And then by (4), the above identity equals

$$\frac{1-|a|^2}{2\pi i}\int_{bU}(f^{m-1})(z)\,\frac{(1-\overline{a}z)^l}{1+|a|^2-2\overline{a}z}\,\frac{1}{(z-a)^l}\,dz.$$

Finally, by residue theorem we have proved the following proposition.

Proposition 2.1. *The matrix $[C_{f_a}]$ of the composition operator C_{f_a} on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m \mid m = 1, 2, \ldots\}$ has (l, m)-th entry*

$$[C_{f_a}]_{lm} = \frac{1-|a|^2}{(l-1)!}\,\frac{d^{l-1}}{dz^{l-1}}\left[f^{m-1}(z)\frac{1}{1+|a|^2-2\overline{a}z}(1-\overline{a}z)^l\right]\Bigg|_{z=a}, \tag{6}$$

where $f(z) = \frac{-2a+(1+|a|^2)z}{1+|a|^2-2\overline{a}z}$.

3. Necessary lemmas

In order to get our final formula for the value of the matrix $[C_{f_a}]$, we consider separately all derivatives of the functions inside of the bracket in the formula (6) in this section. Let

$$f(z) = \frac{-2a+(1+|a|^2)z}{1+|a|^2-2\overline{a}z}.$$

Lemma 3.1. $f^{(0)}(a) = -a$.

Proof.

$$f^{(0)}(a) = \frac{-2a+(1+|a|^2)a}{1+|a|^2-2|a|^2} = \frac{-a+a|a|^2}{1-|a|^2} = -a. \qquad \square$$

Lemma 3.2. $f^{(1)}(a) = 1$.

Proof. Since

$$f^{(1)}(z) = (1-|a|^2)^2(1+|a|^2-2\overline{a}z)^{-2},$$

we have

$$f^{(1)}(a) = (1-|a|^2)^2(1+|a|^2-2|a|^2)^{-2} = 1. \qquad \square$$

Lemma 3.3. *For a positive integer k,*

$$f^{(k)}(a) = k!\, 2^{k-1}\, \overline{a}^{k-1}(1-|a|^2)^{-k+1}.$$

Proof. For $k=1$, by Lemma 3.2, the above formula holds. Since $f^{(2)}(z) = (1-|a|^2)^2(-2)(1+|a|^2-2\overline{a}z)^{-3}(-2\overline{a})$, it is easy to see from mathematical induction on the order of derivative that

$$f^{(k)}(z) = (-2)(-3)(-4)\cdots(-k)(1-|a|^2)^2(1+|a|^2-2\overline{a}z)^{-k-1}(-2\overline{a})^{k-1} \tag{7}$$

which proves the statement of the lemma. □

Remark 3.4. It thus follows from Lemmas 3.1, 3.2, 3.3 that for a *nonnegative* integer k,

$$f^{(k)}(a) = \delta_0^k(-a) + \chi_{\mathbb{N}}(k)k!\, 2^{k-1}\, \overline{a}^{k-1}(1-|a|^2)^{-k+1}, \tag{8}$$

where δ_i^j is the Kronecker delta defined by $\delta_i^j = 1$ for $i=j$, $\delta_i^j = 0$ for $i \neq j$ and $\chi_{\mathbb{N}}(k)$ is the characteristic function defined by $\chi_{\mathbb{N}}(k) = 1$ for $k \in \mathbb{N}$, $\chi_{\mathbb{N}}(k) = 0$ for $k \notin \mathbb{N}$.

We would like to compute the value of

$$(f^{m-1})^{(k)}(a) \text{ for integers } m \text{ and } k \text{ with } m \geq 1 \text{ and } k \geq 0.$$

Notice from Lemma 3.1 that

$$(f^{m-1})^{(0)}(a) = (-a)^{m-1}.$$

We use Leibniz product rule for higher derivatives of an arbitrary number of factors to obtain

$$\begin{aligned} &(f^{m-1})^{(k)}\Big|_{z=a} \\ &= \sum_{j_1+j_2+\cdots+j_{m-1}=k} \binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}\Big|_{z=a}. \end{aligned} \tag{9}$$

Here indices j_i run over all nonnegative integers. In order to compute the values of each term in the summation more effectively, we consider the n-tuples in the summation by partitioning according to the number of coordinates which are zeros. Let A denote the set of the $(m-1)$-tuples $(j_1, j_2, \ldots, j_{m-1})$ for which j_i are nonnegative integers satisfying $j_1+j_2+\cdots+j_{m-1} = k$ and let A^j denote the subset of A whose elements have only j zero coordinates for $j = 0, 1, \ldots, m-1$. If $\mathbf{j} = (j_1, j_2, \ldots, j_{m-1}) \in A^0$, i.e., for each i, $j_i \neq 0$, then each term in the summation of (9) can be from Lemmas 3.2 and 3.3 written as

$$\begin{aligned} &\binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}\Big|_{z=a} \\ &= \binom{k}{j_1, j_2, \ldots, j_{m-1}} (j_1! j_2! \cdots j_{m-1}!)(2^{j_1+j_2+\cdots+j_{m-1}-m+1}) \\ &\quad \cdot (\overline{a}^{j_1+j_2+\cdots+j_{m-1}-m+1})(1-|a|^2)^{-(j_1+j_2+\cdots+j_{m-1})+m-1} \end{aligned}$$

$$= k!2^{k-m+1}\overline{a}^{k-m+1}(1-|a|^2)^{-k+m-1},$$

which does not depend on indices $j_1, \ldots, j_{m-1}$. On the other hand, since each $\mathbf{j} \in A^0$ is a solution of the equation $j_1 + j_2 + \cdots + j_{m-1} = k$ with $j_i \geq 1$, the set A^0 has exactly

$$\binom{k-1}{k-m+1} \cdot \binom{m-1}{0}$$

elements. Here the reason this product is multiplied by the latter $\binom{m-1}{0}$ is because the positions $j_1, j_2, \ldots, j_{m-1}$ has no rooms for the zero. By selecting elements of A^0 from A, we have the identity

$$\sum_{\mathbf{j}\in A^0} \binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}\Big|_{z=a}$$
$$= \binom{k-1}{k-m+1} \cdot \binom{m-1}{0} k!2^{k-m+1}\overline{a}^{k-m+1}(1-|a|^2)^{-k+m-1}.$$

Next suppose that $\mathbf{j} = (j_1, j_2, \ldots, j_{m-1})$ belongs to A^1 with $j_i = 0$ for some $i = 1, \ldots, m-1$. Then by Lemmas 3.1, 3.2 and 3.3 each term in the summation of (9) is equal to

$$\binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}$$
$$= \binom{k}{j_1, j_2, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_{m-1}} f^{(0)} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{i-1})} f^{(j_{i+1})} \cdots f^{(j_{m-1})}$$
$$= \binom{k}{j_1, j_2, \ldots, j_{i-1}, 0, j_{i+1}, \ldots, j_{m-1}} (-a)(j_1!j_2!\cdots j_{i-1}!j_{i+1}!\cdots j_{m-1}!)$$
$$\cdot\, (2^{j_1+j_2+\cdots+j_{i-1}+j_{i+1}+\cdots+j_{m-1}-m+2})$$
$$\cdot\, (\overline{a}^{j_1+j_2+\cdots+j_{i-1}+j_{i+1}+\cdots+j_{m-1}-m+2})$$
$$\cdot\, (1-|a|^2)^{-(j_1+j_2+\cdots+j_{i-1}+j_{i+1}+\cdots+j_{m-1})+m-2}$$
$$= k!(-a)2^{k-m+2}\overline{a}^{k-m+2}(1-|a|^2)^{-k+m-2},$$

which does not depend on $j_1, \ldots, j_{m-1}$. Notice that each $\mathbf{j} \in A^1$ with $j_i = 0$ is a solution of the equation $j_1 + j_2 + \cdots + j_{i-1} + j_{i+1} \cdots + j_{m-1} = k$ with $j_p \geq 1$ for $p \neq i$ and there are $\binom{m-1}{1}$ cases for selecting $j_i = 0$ and hence the set A^1 has exactly

$$\binom{k-1}{k-m+2} \cdot \binom{m-1}{1}$$

elements. It thus follows that the sum of all terms from $(m-1)$-tuples of A^1 equals

$$\sum_{\mathbf{j}\in A^1} \binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}$$

$$= \binom{k-1}{k-m+2} \cdot \binom{m-1}{1} k!(-a)2^{k-m+2}\overline{a}^{k-m+2}(1-|a|^2)^{-k+m-2}.$$

Continuing this process through the subset A^{m-2}, we obtain

$$\sum_{\mathbf{j}\in A^{m-2}} \binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}$$
$$= \binom{k-1}{k-1} \cdot \binom{m-1}{m-2} k!(-a)^{m-2} 2^k \overline{a}^k (1-|a|^2)^{-k+1}.$$

By collecting all sums computed in the summation from $A^0, A^1, \ldots, A^{m-2}$ like

$$\sum_{i=0}^{m-2} \sum_{\mathbf{j}\in A^i} \binom{k}{j_1, j_2, \ldots, j_{m-1}} f^{(j_1)} f^{(j_2)} \cdots f^{(j_{m-1})}\Big|_{z=a},$$

we get the final value $(f^{m-1})^{(k)}(a)$ for $k \geq 1$.

Lemma 3.5. *For positive integers k and m,*

$$(f^{m-1})^{(k)}\Big|_{z=a}$$
$$= k! \sum_{i=0}^{m-2} \binom{k-1}{k-m+1+i} \cdot \binom{m-1}{i} (-a)^i 2^{k-m+1+i}\ \overline{a}^{k-m+1+i} \tag{10}$$
$$\cdot (1-|a|^2)^{-k+m-1-i}.$$

Next we need computation of the second factor of the formula (6).

Lemma 3.6. *Define*

$$h(z) = (1+|a|^2 - 2\overline{a}z)^{-1}, \ z \in U.$$

Then for a nonnegative integer k,

$$h^{(k)}(a) = k!\, 2^k\, \overline{a}^k (1-|a|^2)^{-k-1}.$$

Proof. For $k=0$, it is obvious. Since $h^{(1)}(z) = (-1)(1+|a|^2-2\overline{a}z)^{-2}(-2\overline{a})$ and $h^{(2)}(z) = (-1)(-2)(1+|a|^2-2\overline{a}z)^{-3}(-2\overline{a})^2$, it is easy to see from mathematical induction on the order of derivative that

$$h^{(k)}(z) = (-1)(-2)\cdots(-k)(1-|a|^2)^2(1+|a|^2-2\overline{a}z)^{-k-1}(-2\overline{a})^k, \tag{11}$$

which proves Lemma 3.6. □

Finally we consider the last factor of (6).

Lemma 3.7. *For a positive integer l, define*

$$g_l(z) = (1-\overline{a}z)^l, \ z \in U.$$

Then for a nonnegative integer k,

$$g_l^{(k)}(a) = (-1)^k \frac{l!}{(l-k)!} \overline{a}^k (1-|a|^2)^{l-k}.$$

Proof. For $k=0$, it is obvious. Since $g_l^{(1)}(z) = l(1-\overline{a}z)^{l-1}(-\overline{a})$ and $g_l^{(2)}(z) = l(l-1)(1-\overline{a}z)^{l-2}(-\overline{a})^2$, it is easy to see from mathematical induction on the order of derivative that

$$g_l^{(k)}(z) = l(l-1)\cdots(l-k+1)(1-\overline{a}z)^{l-k}(-\overline{a})^k, \tag{12}$$

which proves Lemma 3.7. □

We remark here that the term containing $n!$ in summation is regarded as non-existent, that is, the term is not added when $n!$ appears with $n<0$.

4. Computation of the matrix of a composition operator

In this main section, we would like to compute the matrix representation of the composition operator C_{f_a} on the Hardy space $H^2(bU)$ with symbol, the Riemann map $f_a(z) = (z-a)/(1-\overline{a}z)$ on the unit disc U with respect to the special orthonormal basis $\{v_m \mid m = 1, 2, \ldots\}$ defined by

$$v_m(z) = \sqrt{\frac{1-|a|^2}{2\pi}}\,\frac{(z-a)^{m-1}}{(1-\overline{a}z)^m}$$

for a given fixed point a in U. From the identity (6), the entry of the matrix $[C_{f_a}]$ is written as

$$[C_{f_a}]_{lm} = \frac{1-|a|^2}{(l-1)!}\,\frac{d^{l-1}}{dz^{l-1}}\left[f^{m-1}(z)h(z)g_l(z)\right]\Big|_{z=a}, \tag{13}$$

where

$$f(z) = \frac{-2a + (1+|a|^2)z}{1+|a|^2-2\overline{a}z},\ h(z) = (1+|a|^2-2\overline{a}z)^{-1},\ g_l(z) = (1-\overline{a}z)^l.$$

To begin, we need to consider the first column $[C_{f_a}]^1$ of the matrix $[C_{f_a}]$ because it does not contain the function f. The $(l,1)$-th entry of $[C_{f_a}]$ is given by

$$\begin{aligned}
[C_{f_a}]_{l1} &= \frac{1-|a|^2}{(l-1)!}\,\frac{d^{l-1}}{dz^{l-1}}(hg_l)\Big|_{z=a} \\
&= \frac{1-|a|^2}{(l-1)!}\sum_{j=0}^{l-1}\binom{l-1}{j,\,l-1-j}\,h^{(j)}g_l^{(l-1-j)}\Big|_{z=a} \\
&= \frac{1-|a|^2}{(l-1)!}\sum_{j=0}^{l-1}\binom{l-1}{j,\,l-1-j}2^j\,j!\,\overline{a}^j(1-|a|)^{-j-1} \\
&\qquad\qquad \cdot(-1)^{l-1-j}\frac{l!}{(j+1)!}\overline{a}^{l-1-j}(1-|a|^2)^{j+1} \\
&= (1-|a|^2)l!\left[\sum_{j=0}^{l-1}\frac{(-1)^{l-1-j}2^j}{(l-1-j)!(j+1)!}\right]\overline{a}^{l-1}.
\end{aligned}$$

Here we used Lemmas 3.6 and 3.7.

The second column of the matrix is given by

$$
\begin{aligned}
[C_{f_a}]_{l2} &= \frac{1-|a|^2}{(l-1)!}\, \frac{d^{l-1}}{dz^{l-1}}(fhg_l)\Big|_{z=a} \\
&= \frac{1-|a|^2}{(l-1)!} \sum_{j_1+j_2+j_3=l-1} \binom{l-1}{j_1, j_2, j_3} f^{(j_1)} h^{(j_2)} g_l^{(j_3)}\Big|_{z=a}.
\end{aligned}
$$

We divide the sum into two parts, one for $j_1 = 0$ and the other for $j_1 \geq 1$. Then the above identity yields

$$
\begin{aligned}
&\frac{1-|a|^2}{(l-1)!}\Bigg[\sum_{j_2+j_3=l-1} \binom{l-1}{0, j_2, j_3} (-a) 2^{j_2}\, j_2!\, \overline{a}^{j_2} (1-|a|^2)^{-j_2-1} \\
&\qquad\qquad \cdot (-1)^{j_3} \frac{l!}{(l-j_3)!} \overline{a}^{j_3} (1-|a|^2)^{l-j_3} \\
&\quad + \frac{1-|a|^2}{(l-1)!} \sum_{\substack{j_1+j_2+j_3=l-1 \\ j_1\neq 0}} \binom{l-1}{j_1, j_2, j_3} j_1! 2^{j_1-1} \overline{a}^{j_1-1} (1-|a|^2)^{-j_1+1} \\
&\quad \cdot 2^{j_2} j_2! \overline{a}^{j_2} (1-|a|^2)^{-j_2-1} (-1)^{j_3} \frac{l!}{(l-j_3)!} \overline{a}^{j_3} (1-|a|^2)^{l-j_3} \Bigg].
\end{aligned}
$$

Now apply the identity $j_2+j_3 = l-1$ for the first sum of the above expression and replace the index j_2 by a new index j for convenience. Use the identity $j_1 + j_2 + j_3 = l - 1$ to the second sum. It then follows that the above value is equal to

$$
\begin{aligned}
&(1-|a|^2) l! \left[\sum_{j=0}^{l-1} \frac{(-1)^{l-1-j} 2^j}{(l-1-j)!(j+1)!} \right] (-a) \overline{a}^{l-1} \\
&+ (1-|a|^2) l! \sum_{\substack{j_1+j_2+j_3=l-1 \\ j_1\neq 0}} \frac{(-1)^{j_3} 2^{l-2-j_3}}{(l-j_3)! j_3!} \overline{a}^{l-2} (1-|a|^2).
\end{aligned}
$$

For the second sum of the above expression, we first consider a solution (j_1, j_2) with $j_1, j_2 \geq 1$ of the equation $j_1 + j_2 = l - 1 - j_3$ for fixed j_3 and allow for j_2 to be zero. Thus we have $\binom{l-2-j_3}{l-3-j_3} + 1$ cases for each fixed j_3 and hence by replacing j_3 by j, the above identity which is the same as the entry $[C_{f_a}]_{l2}$ is given by

$$
\begin{aligned}
[C_{f_a}]_{l2} &= (1-|a|^2) l! \left[\sum_{j=0}^{l-1} \frac{(-1)^{l-1-j} 2^j}{(l-1-j)!(j+1)!} \right] (-a) \overline{a}^{l-1} \\
&\quad + (1-|a|^2) l! \sum_{j=0}^{l-1} \left[\binom{l-2-j}{l-3-j} + 1 \right] \frac{(-1)^j 2^{l-2-j}}{(l-j)! j!} \overline{a}^{l-2} (1-|a|^2).
\end{aligned}
$$

Next using the identities (3.1), (3.6), (3.7), we compute the first row of the matrix as follows.

$$
\begin{aligned}
[C_{f_a}]_{1m} &= (1-|a|^2)\,(f^{m-1}hg_1)\big|_{z=a} \\
&= (1-|a|^2)[(-a)^{m-1}(1-|a|^2)^{-1}(1-|a|^2)] \\
&= (1-|a|^2)(-a)^{m-1}.
\end{aligned}
$$

Finally we compute the general form (l,m)-th entry of the matrix for $l \geq 2$ and $m \geq 3$.

$$
\begin{aligned}
[C_{f_a}]_{lm} &= \frac{1-|a|^2}{(l-1)!}\,\frac{d^{l-1}}{dz^{l-1}}\left[f^{m-1}hg_l\right]\Big|_{z=a} \\
&= \frac{1-|a|^2}{(l-1)!}\sum_{k+j_2+j_3=l-1}\binom{l-1}{k,j_2,j_3}(f^{m-1})^{(k)}h^{(j_2)}g_l^{(j_3)}\Big|_{z=a} \\
&= \frac{1-|a|^2}{(l-1)!}\sum_{j_2+j_3=l-1}\binom{l-1}{0,j_2,j_3}(f^{m-1})^{(0)}h^{(j_2)}g_l^{(j_3)}\Big|_{z=a} \qquad (14)\\
&\quad + \frac{1-|a|^2}{(l-1)!}\sum_{\substack{k+j_2+j_3=l-1\\ k\neq 0}}\binom{l-1}{k,j_2,j_3}(f^{m-1})^{(k)}h^{(j_2)}g_l^{(j_3)}\Big|_{z=a}.
\end{aligned}
$$

Here we used the index k for j_1 for the consistency with the formula (10). The first term of the equation (14) is from Lemmas 3.1, 3.6, and 3.7 equal to

$$
\begin{aligned}
&\frac{1-|a|^2}{(l-1)!}\sum_{j_2+j_3=l-1}\binom{l-1}{0,j_2,j_3} \\
&\qquad\cdot(-a)^{m-1}2^{j_2}j_2!\overline{a}^{j_2}(1-|a|^2)^{-j_2-1}(-1)^{j_3}\frac{l!}{(l-j_3)!}\overline{a}^{j_3}(1-|a|^2)^{l-j_3} \\
&= (1-|a|^2)l!\sum_{j_2=0}^{l-1}\frac{(-1)^{l-1-j_2}2^{j_2}}{(l-1-j_2)!(j_2+1)!}(-a)^{m-1}\overline{a}^{l-1}. \qquad (15)
\end{aligned}
$$

And the second term of (14) is from (10), Lemmas 3.6 and 3.7 equal to

$$
\begin{aligned}
&\frac{1-|a|^2}{(l-1)!}\sum_{\substack{k+j_2+j_3=l-1\\ k\neq 0}}\Bigg\{\binom{l-1}{k,j_2,j_3}\Bigg[k!\sum_{j=0}^{m-2}\binom{k-1}{k-m+1+j}\binom{m-1}{j} \\
&\qquad\cdot 2^{k-m+1+j}(-a)^j\overline{a}^{k-m+1+j}(1-|a|^2)^{-k+m-1-j}\Bigg] \\
&\qquad\cdot 2^{j_2}j_2!\overline{a}^{j_2}(1-|a|^2)^{-j_2-1}(-1)^{j_3}\frac{l!}{(l-j_3)!}\overline{a}^{j_3}(1-|a|^2)^{l-j_3}\Bigg\}
\end{aligned}
$$

$$= (1-|a|^2) \sum_{\substack{k+j_2+j_3=l-1 \\ k\neq 0}} \Bigg[\frac{1}{j_3!} \sum_{j=0}^{m-2} \binom{k-1}{k-m+1+j} \binom{m-1}{j} \cdot 2^{k-m+1+j+j_2} (-1)^{j_3} \frac{l!}{(l-j_3)!} (-a)^j \overline{a}^{k-m+1+j+j_2+j_3} \cdot (1-|a|^2)^{-k+m-1-j-j_2-1+l-j_3} \Bigg]$$

$$= (1-|a|^2) l! \sum_{\substack{k+j_2+j_3=l-1 \\ k\neq 0}} \Bigg[\frac{(-1)^{j_3}}{j_3!(l-j_3)!} \sum_{j=0}^{m-2} \binom{k-1}{k-m+1+j} \binom{m-1}{j} \cdot 2^{l-m-j_3+j} (-a)^j \overline{a}^{l-m+j} (1-|a|^2)^{m-1-j} \Bigg]$$

$$= (1-|a|^2) l! \sum_{j_2=0}^{l-1} \sum_{\substack{k+j_3=l-1-j_2 \\ k\neq 0}} \frac{(-1)^{j_3}}{j_3!(l-j_3)!} \Bigg[\sum_{j=0}^{m-2} \binom{k-1}{k-m+1+j} \binom{m-1}{j} \cdot 2^{l-m-j_3+j} (-a)^j \overline{a}^{l-m+j} (1-|a|^2)^{m-1-j} \Bigg]. \tag{16}$$

Hence by replacing (14) by the sum of (15) and (16), we obtain the following final form of (l, m)-th entry of the matrix.

Theorem 4.1. *Let U be the unit disc and let a be in U. Then for given two positive integers l, m, the matrix $[C_{f_a}]$ of the composition operator C_{f_a} on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\{v_m \mid m = 1, 2, \ldots\}$ has (l, m)-th entry*

$$[C_{f_a}]_{lm} = (1-|a|^2) l! \Bigg\{ \Bigg[\sum_{i=0}^{l-1} \frac{(-1)^{l-1-i} 2^i}{(l-1-i)!(i+1)!} \Bigg] (-a)^{m-1} \overline{a}^{l-1} + \sum_{j_2=0}^{l-1} \sum_{\substack{k+j_3=l-1-j_2 \\ k\neq 0}} \frac{(-1)^{j_3}}{j_3!(l-j_3)!} \Bigg[\sum_{j=0}^{m-2} \binom{k-1}{k-m+1+j} \binom{m-1}{j} \cdot 2^{l-m-j_3+j} (-a)^j \overline{a}^{l-m+j} (1-|a|^2)^{m-1-j} \Bigg] \Bigg\}.$$

Remark 4.2. For an easy reference, the first few elements of the matrix are listed as follows:

$$[C_{f_a}] = (1-|a|^2) \begin{bmatrix} 1 & -a & (-a)^2 & \cdots \\ 0 & 1-|a|^2 & 2(-a)(1-|a|^2) & \cdots \\ \overline{a}^2 & (-a)\overline{a}^2 - 15\overline{a}(1-|a|^2) & (-a)^2\overline{a}^2 + (1-|a|^2)^2 + 4(-a)\overline{a}(1-|a|^2) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

References

[1] S. R. Bell, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), no. 4, 1337–1369. https://doi.org/10.1512/iumj.1995.44.2030

[2] P. S. Bourdon and S. Waleed Noor, *Complex symmetry of invertible composition operators*, J. Math. Anal. Appl. **429** (2015), no. 1, 105–110. https://doi.org/10.1016/j.jmaa.2015.04.008

[3] Y.-B. Chung, *Classification of Toeplitz operators on Hardy spaces of bounded domains in the plane*, Math. Notes **101** (2017), no. 3-4, 529–541. https://doi.org/10.1134/S0001434617030142

[4] Y.-B. Chung, *Special orthonormal basis for L^2 functions on the unit circle*, Bull. Korean Math. Soc. **54** (2017), no. 6, 2013–2027. https://doi.org/10.4134/BKMS.b160697

[5] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[6] S. K. Narayan, D. Sievewright, and M. Tjani, *Complex symmetric composition operators on weighted Hardy spaces*, Proc. Amer. Math. Soc. **148** (2020), no. 5, 2117–2127. https://doi.org/10.1090/proc/14909

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