Bull. Korean Math. Soc. **59** (2022), No. 5, pp. 1105–1117
https://doi.org/10.4134/BKMS.b210620
pISSN: 1015-8634 / eISSN: 2234-3016

SECOND MAIN THEOREM WITH WEIGHTED COUNTING FUNCTIONS AND UNIQUENESS THEOREM

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Abstract. In this paper, we obtain a second main theorem for holomorphic curves and moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ where the counting functions are truncated multiplicity and have different weights. As its application, we prove a uniqueness theorem for holomorphic curves of finite growth index sharing moving hyperplanes with different multiple values.

1. Introduction

In the recent paper [9], Ru-Sibony developed value distribution theory for a class of holomorphic curves where the source is a disc of radius R instead of $\mathbf{C}$. In doing so, they introduced the notion of the growth index, denoted by $c_{f,\omega}$, for a holomorphic curve.

Definition. Let M be a complex manifold and ω be a positive $(1,1)$ form of finite volume on M. Let $0 < R \leq +\infty$ and $f : \Delta(R) \to M$ be a holomorphic curve. Recall that the characteristic function of f with respect to w, for $0 < r < R$, as $T_{f,w}(r) = \int_0^r \frac{dt}{t} \int_{|z|<r} f^*w$. We define the growth index of f with respect to ω as

$$c_{f,\omega} =: \inf \left\{ c > 0 : \int_0^R \exp(c\ T_{f,\omega}(r)) dr = \infty \right\}.$$

When M is the complex projective space $\mathbf{P}^n(\mathbf{C})$, the positive $(1,1)$ form is the Fubini-Study form, i.e., $\omega = \omega_{FS}$. For a holomorphic curve $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$, denote by c_f the growth index of f with respect to ω_{FS}. For convenient, we set $c_f = +\infty$, if

$$\left\{ c > 0 : \int_0^R \exp\left(cT_f(r)\right) dr = +\infty \right\} = \emptyset.$$

Received August 21, 2021; Revised January 4, 2022; Accepted March 3, 2022.
2010 *Mathematics Subject Classification.* 32H30, 30D35.
Key words and phrases. Second main theorem, holomorphic curve, uniqueness theorem, moving hyperplane.

In the same paper [9], Ru-Sibony obtained the following second main theorem for the nondegenerate holomorphic curves from the disk.

Theorem A. *Let $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate holomorphic curve with $c_f < +\infty$ and $0 < R \le +\infty$. Then for any $\epsilon > 0$, the inequality*

$$(q-n-1)T_f(r) \le \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + \frac{n(n+1)}{2}(1+\epsilon)(c_f+\varepsilon)T_f(r) + O(\log T_f(r)) + \frac{n(n+1)}{2}\varepsilon \log r$$

holds for all $r \in (0,R)$ outside a set $E \subset (0,R)$ with $\int_E \exp((c_f+\epsilon)T_f(r))dr < \infty$.

Recently, S. D. Quang [6] established some new second main theorems for holomorphic curves from the disk with infinite growth index into $\mathbf{P}^n(\mathbf{C})$ and moving hyperplanes.

Theorem B. *Let $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ $(0 < R \le +\infty)$ be a holomorphic curve. Let $\{a_j\}_{j=1}^q$ $(q \ge 2n-k+2)$ be holomorphic curves of $\Delta(R)$ into $\mathbf{P}^n(\mathbf{C})^*$ in general position such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$. Assume that $\operatorname{rank}_{\mathcal{R}\{a_i\}}(f) = k+1$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0,R)$ with $\int_0^R \gamma(r)dr = \infty$. Then for every $\varepsilon > 0$, we have*

$$\Big\|_E \; T_f(r) \le \frac{n+2}{q-(n-k)} \sum_{j=1}^{q} N^{[k]}_{(f,a_j)}(r) + S(r) + \frac{k(k+2)(n+1)}{2(n+2)}((1+\varepsilon)\log\gamma(r) + \varepsilon \log r).$$

Here and subsequently, the notation "$\|_E \;\; \mathcal{P}$" means the assertion $\mathcal{P}$ holds for all $r \in (0,R)$ outside a set E with $\int_E \gamma(r)dr < \infty$,

$$S(r) := O\left(\log T_f(r) + \max_{1 \le i \le q} T_{a_i}(r)\right).$$

And $\operatorname{rank}_{\mathcal{R}}(f)$ is the rank of the set $\{f_0, \dots, f_n\}$ over the field $\mathcal{R}$ for a reduced representation $(f_0, \dots, f_n)$ of the mapping f, and $(f,g) = \sum_{i=0}^n g_i f_i$ for each holomorphic mapping $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})^*$ with a reduced representation $(g_0, \dots, g_n)$.

In 2015, Quang [4] initially introduced the second main theorem with weighted counting functions. Inspired by this idea and the technique shown in [4], we generalize Theorem B for the mappings and moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ to the case where the counting functions are truncated multiplicity and have different weights. The uniqueness theory for meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with shared moving targets is an interesting topic (see [1, 3, 5, 8] and the references given there). More recently, some uniqueness results for holomorphic curves of finite growth index sharing fixed hyperplanes have previously been studied in [10, 11]. As application of our general form of second main theorem, the second purpose of this article is to prove a uniqueness theorem for holomorphic curves of finite growth index sharing moving hyperplanes with different multiple values. For some related notions see Section 2.

Theorem 1.1 (Second Main Theorem). *Let* $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ $(0 < R \leq +\infty)$ *be a holomorphic curve. Let* $\{a_j\}_{j=1}^q$ $(q \geq 2n-k+2)$ *be holomorphic curves of* $\Delta(R)$ *into* $\mathbf{P}^n(\mathbf{C})^*$ *in general position such that* $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$. *Assume that* $\operatorname{rank}_{\mathcal{R}\{a_i\}}(f) = k+1$. *Let* $\gamma(r)$ *be a non-negative measurable function defined on* $(0, R)$ *with* $\int_0^R \gamma(r)dr = \infty$. *Let* $\lambda_1, \dots, \lambda_q$ *be* q *positive numbers with* $(2n-k+2)\max_{1\leq i\leq q}\lambda_i \leq \sum_{i=1}^q \lambda_i$. *Then for every* $\varepsilon > 0$ *and* $\eta \in \left[\max_{1\leq i\leq q}\lambda_i, \frac{\sum_{i=1}^q \lambda_i}{2n-k+2}\right]$, *we have*

$$\Big\|_E \ \frac{\sum_{j=1}^q \lambda_j - (n-k)\eta}{n+2}\left\{T_f(r) - \frac{k(n+1)}{2}\log\gamma(r)\right\}$$
$$\leq \sum_{j=1}^q \lambda_j N^{[k]}_{(f,a_j)}(r) + S(r) + \varepsilon\log(r\gamma(r)).$$

Letting $\lambda_1 = \cdots = \lambda_q = 1$ and $\eta = 1$ from Theorem 1.1, we get Theorem B in some sense. Letting $\eta = \frac{\sum_{i=1}^q \lambda_i}{2n-k+2}$, we have the following corollary.

Corollary 1.2. *Let* $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ $(0 < R \leq +\infty)$ *be a holomorphic curve. Let* $\{a_j\}_{j=1}^q$ $(q \geq 2n-k+2)$ *be holomorphic curves of* $\Delta(R)$ *into* $\mathbf{P}^n(\mathbf{C})^*$ *in general position such that* $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$. *Assume that* $\operatorname{rank}_{\mathcal{R}\{a_i\}}(f) = k+1$. *Let* $\gamma(r)$ *be a non-negative measurable function defined on* $(0, R)$ *with* $\int_0^R \gamma(r)dr = \infty$. *Let* $\lambda_1, \dots, \lambda_q$ *be* q *positive numbers with* $(2n-k+2)\max_{1\leq i\leq q}\lambda_i \leq \sum_{i=1}^q \lambda_i$. *Then for every* $\varepsilon > 0$, *we have*

$$\Big\|_E \ \frac{\sum_{j=1}^q \lambda_j}{2n-k+2}\left\{T_f(r) - \frac{k(n+1)}{2}\log\gamma(r)\right\}$$
$$\leq \sum_{j=1}^q \lambda_j N^{[k]}_{(f,a_j)}(r) + S(r) + \varepsilon\log(r\gamma(r)).$$

In addition, if we take $\lambda_1 = \cdots = \lambda_q = 1$ in Corollary 1.2, we get the following result.

Corollary 1.3. *Let* $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ $(0 < R \leq +\infty)$ *be a holomorphic curve. Let* $\{a_j\}_{j=1}^q$ $(q \geq 2n-k+2)$ *be holomorphic curves of* $\Delta(R)$ *into* $\mathbf{P}^n(\mathbf{C})^*$ *in general position such that* $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$. *Assume that* $\operatorname{rank}_{\mathcal{R}\{a_i\}}(f) = k+1$. *Let* $\gamma(r)$ *be a non-negative measurable function defined on* $(0, R)$ *with* $\int_0^R \gamma(r)dr = \infty$. *Then for every* $\varepsilon > 0$, *we have*

$$\Big\|_E \ \frac{q}{2n-k+2}\left\{T_f(r) - \frac{k(n+1)}{2}\log\gamma(r)\right\}$$

$$\leq \sum_{j=1}^{q} N^{[k]}_{(f,a_j)}(r) + S(r) + \varepsilon \log(r\gamma(r)).$$

Before stating our uniqueness theorem, we introduce a definition. Assume that a is a holomorphic curve of $\Delta(R)$ into $\mathbf{P}^n(\mathbf{C})^*$, and f is a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. If the mapping a is a small function with respect to f, that is $||_E\ T_a(r) = o(T_f(r))$, then a is said to be a slowly (with respect to f) moving hyperplane in $\mathbf{P}^n(\mathbf{C})$.

Theorem 1.4 (Uniqueness Theorem)**.** *Let $f, g : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ be holomorphic curves of finite growth index $c_f, c_g < +\infty$. Let $\{a_i\}_{i=1}^q$ be slowly (with respect to f and g) moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position such that*

(i) $(f, a_i)^{-1}\{0\} \cap (f, a_j)^{-1}\{0\} = \emptyset$ $(1 \leq i < j \leq q)$,
(ii) $\nu^1_{(f,a_i),\leq m_i} = \nu^1_{(g,a_i),\leq m_i} (1 \leq i \leq q)$,
(iii) $f(z) = g(z)$ *for all* $z \in \bigcup_{i=1}^q \{z \in \Delta(R) : 0 < \nu_{(f,a_i)}(z) \leq m_i\}$.

Assume that $\operatorname{rank}_{\mathcal{R}}(f) = \operatorname{rank}_{\mathcal{R}}(g) = k+1,\ q \geq 2k(2n-k+1)+2$,

$$\sum_{i=1}^{q} \frac{k}{m_i + 1 - k} < \frac{q(q - 2k(2n-k+1) - 2)}{(q + 2k - 2)(2n - k + 1)},$$

$$\max_{1 \leq i \leq q} \frac{k}{m_i + 1 - k} \leq \frac{q}{2n - k + 1} - 1,$$

and

$$\min\{c_f, c_g\} < \frac{1}{k(n+1)} \left(\frac{q(q-2)(2n-k+2)}{(q+2k-2)\sum_{i=1}^{q} \frac{m_i+1}{m_i+1-k}} - 2n + k - 1 \right).$$

Then $f = g$.

In the case where $R = +\infty$, we have $c_f = c_g = 0$, see [9]. Thus our results also include the following unicity theorem for holomorphic curves on the whole complex plane.

Corollary 1.5. *Let $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be holomorphic curves. Let $\{a_i\}_{i=1}^q$ be slowly (with respect to f and g) moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position such that*

(i) $(f, a_i)^{-1}\{0\} \cap (f, a_j)^{-1}\{0\} = \emptyset$ $(1 \leq i < j \leq q)$,
(ii) $\nu^1_{(f,a_i),\leq m_i} = \nu^1_{(g,a_i),\leq m_i} (1 \leq i \leq q)$,
(iii) $f(z) = g(z)$ *for all* $z \in \bigcup_{i=1}^q \{z \in \mathbf{C} : 0 < \nu_{(f,a_i)}(z) \leq m_i\}$.

Assume that $\operatorname{rank}_{\mathcal{R}}(f) = \operatorname{rank}_{\mathcal{R}}(g) = k+1,\ q \geq 2k(2n-k+1)+2$,

$$\sum_{i=1}^{q} \frac{k}{m_i + 1 - k} < \frac{q(q - 2k(2n-k+1) - 2)}{(q + 2k - 2)(2n - k + 1)}$$

and

$$\max_{1 \leq i \leq q} \frac{k}{m_i + 1 - k} \leq \frac{q}{2n - k + 1} - 1.$$

Then $f = g$.

In particular, if we take $m_i = +\infty$ $(1 \leq i \leq q)$, we have:

Corollary 1.6. *Let* $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be holomorphic curves. Let* $\{a_i\}_{i=1}^q$ *be slowly* (*with respect to* f *and* g) *moving hyperplanes in* $\mathbf{P}^n(\mathbf{C})$ *in general position such that*

(i) $(f, a_i)^{-1}\{0\} \cap (f, a_j)^{-1}\{0\} = \emptyset$ $(1 \leq i < j \leq q)$,

(ii) $\nu^1_{(f,a_i)} = \nu^1_{(g,a_i)} (1 \leq i \leq q)$,

(iii) $f(z) = g(z)$ *for all* $z \in \bigcup_{i=1}^q (f, a_i)^{-1}\{0\}$.

If $\operatorname{rank}_{\mathcal{R}}(f) = \operatorname{rank}_{\mathcal{R}}(g) = k+1,\ q \geq 2k(2n-k+1)+2$, *then* $f = g$.

2. Preliminaries

In this section, we state some basic notions in value distribution for holomorphic curves. For more details we refer the reader to [2, 7].

Let D be a domain in $\mathbf{C}$, $f : D \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve and U be an open set in D. Any holomorphic curve $\tilde{f} : U \to \mathbf{C}^{n+1}$ such that $\mathbf{P}(\tilde{f}(z)) \equiv f(z)$ in U is called a *representation* of f on U, where $\mathbf{P} : \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{P}^n(\mathbf{C})$ is the standard projective map.

Definition. For an open subset U of D we call a representation $\tilde{f} = (f_0, \ldots, f_n)$ a *reduced representation* of f on U if $f_0, \ldots, f_n$ are holomorphic functions on U without common zeros.

Remark 2.1. As is easily seen, if both $\tilde{f}_j : U_j \to \mathbf{C}^{n+1}$ are reduced representations of f for $j = 1, 2$ with $U_1 \cap U_2 \neq \emptyset$, then there is a holomorphic function $h(\neq 0) : U_1 \cap U_2 \to \mathbf{C}$ such that $\tilde{f}_2 = h\tilde{f}_1$ on $U_1 \cap U_2$.

Let $0 < R \leq +\infty$ and f be a holomorphic curve from the disc $\Delta(R)$ into the complex projective space $\mathbf{P}^n(\mathbf{C})$ and let

$$\tilde{f} = (f_1, \ldots, f_{n+1}) : \Delta(R) \to \mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$$

be a reduced representation of f, where n is a positive integer. We use the following notations:

$$\|\tilde{f}(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{\frac{1}{2}}.$$

The Cartan's characteristic function of f is defined as follows:

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log\|\tilde{f}(re^{\mathrm{i}\theta})\| \mathrm{d}\theta - \log\|\tilde{f}(0)\|,$$

where $0 < r < R$.

Remark 2.2. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

The Ahlfors' characteristic function of f is defined as follows:

$$T_{f,\omega_{FS}}(r) = \int_0^r \frac{\mathrm{d}t}{t} \int_{|z|<t} f^*\omega_{FS},$$

where $f^*\omega_{FS}$ is the pullback of Fubini-Study metric form ω_{FS} under the curve f.

Remark 2.3. It follows from Green-Jensen's formula that Ahlfors' characteristic function agrees with Cartan's characteristic function.

For a divisor ν on $\Delta(R)$ and for a positive integer M or $M = +\infty$, we define the counting function of ν by

$$\nu^{[M]}(z) = \min\{\nu(z), M\},\ n^{[M]}(t,\nu) = \sum_{|z|\le t} \nu^{[M]}(z),\ 0 < t < R.$$

Define

$$N_F^{[M]}(r,\nu) = \int_0^r \frac{n^{[M]}(t,\nu)}{t}\mathrm{d}t,\ 0 < r < R.$$

Let $F : \Delta(R) \to \mathbf{C}$ be a holomorphic function. Define

$$N_F(r) = N(r,\nu_F),\ N_F^{[M]}(r) = N^{[M]}(r,\nu_F),\ 0 < r < R.$$

For brevity we will omit the character $[M]$ if $M = +\infty$.

Let k, M be positive integers or $+\infty$. For a divisor ν on $\mathbf{C}$. Set

$$\nu_{\le k}^{[M]}(z) = \begin{cases} 0, & \text{if} \quad \nu(z) > k, \\ \nu^{[M]}(z), & \text{if} \quad \nu(z) \le k, \end{cases}$$

and

$$n_{\le k}^{[M]}(t) = \sum_{|z|\le t} \nu_{\le k}^{[M]}(z).$$

We define

$$N\left(r, \nu_{\le k}^{[M]}\right) = \int_1^r \frac{n_{\le k}^{[M]}(t)}{t^{2n-1}} dt \quad (r > 1).$$

Similarly, we define $n_{\ge k}^{[M]}(t)$ and $N\left(r, \nu_{\ge k}^{M}\right)$, and denote them by $N_{\le k}^{[M]}(r,\nu)$ and $N_{\ge k}^{[M]}(r,\nu)$, respectively.

Assume that a is a moving hyperplane of $\Delta(R)$ into $\mathbf{P}^n(\mathbf{C})$ (i.e., a holomorphic curve of $\Delta(R)$ into $\mathbf{P}^n(\mathbf{C})^*$), and f is a holomorphic curve of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$ with $(f,a) = \sum_{i=0}^n f_i a_i \not\equiv 0$. Then, using the zero divisor $\nu_{(f,a)}^0$ we define $N_{(f,a)}(r) := N\left(r, \nu_{(f,a)}^0\right)$. We note that $N_{(f,a)}(r)$ measures how many times f take value in the moving hyperplane a. Similarly, we have $N_{(f,a)}^{[M]}(r), N_{(f,a),\le k}^{[M]}(r), N_{(f,a),\ge k}^{[M]}(r)$, etc.

To prove our result, we need the following lemma due to Quang [6, Theorem 1.1, Eq. (2.10)].

Lemma 2.4. *Let* $f : \Delta(R) \to \mathbf{P}^n(\mathbf{C})$ $(0 < R \leq +\infty)$ *be a holomorphic curve. Let* $\{a_j\}_{j=1}^q$ $(q \geq 2n-k+2)$ *be holomorphic curves of* $\Delta(R)$ *into* $\mathbf{P}^n(\mathbf{C})^*$ *in general position such that* $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$*. Assume that* $\mathrm{rank}_{\mathcal{R}\{a_i\}}(f) = k+1$*. Then there exist a subset* $J \subset \{1, \ldots, 2n-k+2\}$ *with* $|J| = d+2$ $(\leq n+2)$ *and a positive integer* $n_0 \leq \frac{k(k+2)}{d+2}$ *such that*

$$\Big\|_E \; T_f(r) \leq \sum_{j \in J} N^{[n_0]}_{(f,a_j)}(r) + S(r) + \frac{n_0(d+1)}{2}((1+\varepsilon)\log\gamma(r) + \varepsilon \log r),$$

where $S(r) = O\left(\log T_f(r) + \max_{1 \leq i \leq q} T_{a_i}(r)\right)$.

3. Proofs

Proof of Theorem 1.1. We denote by $\mathcal{I}$ the set of all permutations of q-tuple $(1, \ldots, q)$. For each element $I = (i_1, \ldots, i_q) \in \mathcal{I}$, we set

$$N_I = \left\{ r \in \mathbf{R}^+ : N^{[k]}_{(f,a_{i_1})}(r) \leq \cdots \leq N^{[k]}_{(f,a_{i_q})}(r) \right\}.$$

Fix a permutation $I = (i_1, \ldots, i_q) \in \mathcal{I}$. Since $\eta \leq \frac{\sum_{i=1}^q \lambda_i}{2n-k+2} < \frac{\sum_{i=1}^q \lambda_i}{n-k}$ by assumption, $\sum_{j=1}^q \lambda_j - (n-k)\eta > 0$. Applying Lemma 2.4, there exists a subset $J_0 \subset \{1, \ldots, 2n-k+2\}$ with $|J_0| = n+2$ such that

$$\begin{aligned} \|_E \; & T_f(r) - \frac{k(n+1)}{2}\log\gamma(r) \\ & \leq \sum_{l \in J_0} N^{[k]}_{(f,a_{i_l})}(r) + S(r) + \frac{\varepsilon \log(r\gamma(r))}{\sum_{j=1}^q \lambda_j - (n-k)\eta}. \end{aligned} \tag{1}$$

Put $J_1 = \{1, \ldots, 2n-k+2\} \backslash J_0$ and

$$J_2 = \begin{cases} \{2n-k+3, \ldots, q\}, & \text{if } q > 2n-k+2, \\ \emptyset, & \text{if } q = 2n-k+2, \end{cases}$$

then $|J_1| = (2n-k+2) - |J_0| = n-k$. Hence, we observe from (1) that

$$\begin{aligned} & \|_E \left(\sum_{j=1}^q \lambda_j - (n-k)\eta \right) \left\{ T_f(r) - \frac{k(n+1)}{2}\log\gamma(r) \right\} \\ \leq & \left(\sum_{j=1}^q \lambda_j - (n-k)\eta \right) \sum_{l \in J_0} N^{[k]}_{(f,a_{i_l})}(r) + S(r) + \varepsilon \log(r\gamma(r)) \\ \leq & \left(\sum_{j \in J_0 \bigcup J_2} \lambda_{i_j} \right) \sum_{l \in J_0} N^{[k]}_{(f,a_{i_l})}(r) + \left(\sum_{j \in J_1} \lambda_{i_j} - (n-k)\eta \right) \sum_{l \in J_0} N^{[k]}_{(f,a_{i_l})}(r) \\ & + S(r) + \varepsilon \log(r\gamma(r)). \end{aligned}$$

Note that $\sum_{j\in J_1}\lambda_{i_j}-(n-k)\eta>0$ since $\eta\geq\max_{1\leq i\leq q}\{\lambda_i\}$. We then have

$$\begin{aligned}(2)\qquad &\|_E \left(\sum_{j=1}^{q}\lambda_j-(n-k)\eta\right)\left\{T_f(r)-\frac{k(n+1)}{2}\log\gamma(r)\right\}\\ &\leq \left(\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}\right)\sum_{l\in J_0}N^{[k]}_{(f,a_{i_l})}(r)+S(r)+\varepsilon\log(r\gamma(r)).\end{aligned}$$

It is easily seen that

$$\begin{aligned}(3)\quad &\left(\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}\right)\sum_{l\in J_0}N^{[k]}_{(f,a_{i_l})}(r)\\ &=|J_0|\left\{\sum_{l\in J_0}\lambda_{i_l}N^{[k]}_{(f,a_{i_l})}(r)+\frac{\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}}{|J_0|}\sum_{l\in J_0}N^{[k]}_{(f,a_{i_l})}(r)-\sum_{l\in J_0}\lambda_{i_l}N^{[k]}_{(f,a_{i_l})}(r)\right\}\\ &=(n+2)\left\{\sum_{l\in J_0}\lambda_{i_l}N^{[k]}_{(f,a_{i_l})}(r)+\sum_{l\in J_0}\left(\frac{\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}}{n+2}-\lambda_{i_l}\right)N^{[k]}_{(f,a_{i_l})}(r)\right\}.\end{aligned}$$

Next we estimate $\sum_{l\in J_0}\left(\frac{\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}}{n+2}-\lambda_{i_l}\right)N^{[k]}_{(f,a_{i_l})}(r)$ for $r\in N_I$. By the definition of N_I, we get

$$\begin{aligned}(4)\qquad &\sum_{l\in J_0}\left(\frac{\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}}{n+2}-\lambda_{i_l}\right)N^{[k]}_{(f,a_{i_l})}(r)\\ &\leq \sum_{l\in J_0}\left(\frac{\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}}{n+2}-\lambda_{i_l}\right)N^{[k]}_{(f,a_{i_{2n-k+2}})}(r)\\ &=\left(\sum_{j\in J_0\bigcup J_2}\lambda_{i_j}-\sum_{l\in J_0}\lambda_{i_l}\right)N^{[k]}_{(f,a_{i_{2n-k+2}})}(r)\\ &=\left(\sum_{j=2n-k+3}^{q}\lambda_{i_j}\right)N^{[k]}_{(f,a_{i_{2n-k+2}})}(r)\\ &\leq \sum_{j=2n-k+3}^{q}\lambda_{i_j}N^{[k]}_{(f,a_{i_j})}(r).\end{aligned}$$

Therefore, combining (1), (2), (3) and (4), for all $r\in N_I$, we have

$$\|_E \left(\sum_{j=1}^{q}\lambda_j-(n-k)\eta\right)\left\{T_f(r)-\frac{k(n+1)}{2}\log\gamma(r)\right\}$$

$$\le (n+2)\left(\sum_{l\in J_0}\lambda_{i_l}N^{[k]}_{(f,a_{i_l})}(r)+\sum_{j=2n-k+3}^{q}\lambda_{i_j}N^{[k]}_{(f,a_{i_{2n-k+2}})}(r)\right)$$
$$+S(r)+\varepsilon\log(r\gamma(r))$$
$$\le (n+2)\sum_{j=1}^{q}\lambda_j N^{[k]}_{(f,a_j)}(r)+S(r)+\varepsilon\log(r\gamma(r)).$$

Hence, for $r\in N_I$, we have

$$(5)\qquad \|_E\,\frac{\sum_{j=1}^{q}\lambda_j-(n-k)\eta}{n+2}\left\{T_f(r)-\frac{k(n+1)}{2}\log\gamma(r)\right\}$$
$$\le \sum_{j=1}^{q}\lambda_j N^{[k]}_{(f,a_j)}(r)+S(r)+\varepsilon\log(r\gamma(r)).$$

We see that $\bigcup_{I\in\mathcal{I}}N_I=(0,R)$ and then the inequality (5) holds for every $r\in(0,R)$ outside a subset E with $\int_E\gamma(r)dr=+\infty$. Hence, the theorem is proved. □

Proof of Theorem 1.4. We assume, to the contrary, that $f\not\equiv g$. By changing indices, if necessary, we may assume that

$$\underbrace{\frac{(f,a_1)}{(g,a_1)}\equiv\frac{(f,a_2)}{(g,a_2)}\equiv\cdots\equiv\frac{(f,a_{k_1})}{(g,a_{k_1})}}_{\text{group }1}\not\equiv\underbrace{\frac{(f,a_{k_1+1})}{(g,a_{k_1+1})}\equiv\cdots\equiv\frac{(f,a_{k_2})}{(g,a_{k_2})}}_{\text{group }2}$$
$$\not\equiv\underbrace{\frac{(f,a_{k_2+1})}{(g,a_{k_2+1})}\equiv\cdots\equiv\frac{(f,a_{k_3})}{(g,a_{k_3})}}_{\text{group }3}\not\equiv\cdots\not\equiv\underbrace{\frac{(f,a_{k_{s-1}+1})}{(g,a_{k_{s-1}+1})}\equiv\cdots\equiv\frac{(f,a_{k_s})}{(g,a_{k_s})}}_{\text{group }s},$$

where $k_s=q$. The hypothesis of "in general position" implies that the number of each group does not exceed n.

We define the map $\sigma:\{1,\dots,q\}\to\{1,\dots,q\}$ by

$$\sigma(j)\;=\;\begin{cases} j+n, & \text{if } j+n\le q,\\ j+n-q, & \text{if } j+n>q.\end{cases}$$

It is easy to see that σ is bijective and $|\sigma(j)-j|\ge n$ for each $1\le j\le q$ (note $q>2n$). Hence $\frac{(f,a_j)}{(g,a_j)}$ and $\frac{(f,a_{\sigma(j)})}{(g,a_{\sigma(j)})}$ belong to distinct groups for each $1\le j\le q$. Set

$$P_j=(f,a_j)(g,a_{\sigma(j)})-(g,a_j)(f,a_{\sigma(j)})\ (1\le j\le q).$$

Since $f\not\equiv g$, we get that $P_j\not\equiv 0$. And hence $P:=\prod_{j=1}^{q}P_j\not\equiv 0$.

Fix an index i with $1\le i\le q$. It is easy to see for every $z\in\Delta(R)$,

$$\nu_{P_i}(z)\ge\;\min\left\{\nu_{(f,a_i),\le m_i}(z),\nu_{(g,a_i),\le m_i}(z)\right\}$$
$$+\min\left\{\nu_{\left(f,a_{\sigma(i)}\right),\le m_{\sigma(i)}}(z),\nu_{\left(g,a_{\sigma(i)}\right),\le m_{\sigma(i)}}(z)\right\}$$

$$+\sum_{\substack{v=1\\ v\neq i,(i)}}^{q}\nu^{[1]}_{(f,a_v),\le m_v}(z)$$

$$\geq \sum_{v=i,\sigma(i)}\left(\min\left\{k,\nu_{(f,a_v),\le m_v}\right\}+\min\left\{k,\nu_{(f,a_v),\le m_v}\right\}\right.$$

$$\left.-k\min\left\{1,\nu_{(f,a_v),\le m_v}\right\}\right)(z)+\sum_{\substack{v=1\\ v\neq i,(i)}}\nu^{[1]}_{(f,a_v),\le m_v}(z).$$

Integrating both sides of the above inequality, we have

$$N_{P_i}(r)\geq \sum_{v=i,\sigma(i)}\left(N^{[k]}_{(f,a_v),\le m_v}(r)+N^{[k]}_{(g,a_v),\le m_v}(r)-kN^{[1]}_{(f,a_v),\le m_v}(r)\right)$$

$$+\sum_{\substack{v=1\\ v\neq i,\sigma(i)}}^{q}N^{[1]}_{(f,a_v),\le m_v}(r)$$

$$=\sum_{v=i,\sigma(i)}\left(N^{[k]}_{(f,a_v),\le m_v}(r)+N^{[k]}_{(g,a_v),\le m_v}(r)\right)$$

$$+\sum_{v=1}^{q}N^{[1]}_{(f,a_v),\le m_v}(r)-\sum_{v=i,\sigma(i)}(k+1)N^{[1]}_{(f,a_v),\le m_v}(r)$$

for all $1\leq i\leq q$. Thus, by summing them up, we obtain

$$N_P(r)\geq 2\sum_{i=1}^{q}\left(N^{[k]}_{(f,a_i),\le m_i}(r)+N^{[k]}_{(g,a_i),\le m_i}(r)\right)$$
$$+\frac{q-2k-2}{2}\sum_{i=1}^{q}\left(N^{[1]}_{(f,a_i),\le m_i}(r)+N^{[1]}_{(g,a_i),\le m_i}(r)\right)$$
$$\geq 2\sum_{i=1}^{q}\left(N^{[k]}_{(f,a_i),\le m_i}(r)+N^{[k]}_{(g,a_i),\le m_i}(r)\right)$$
$$+\frac{q-2k-2}{2k}\sum_{i=1}^{q}\left(N^{[k]}_{(f,a_i),\le m_i}(r)+N^{[k]}_{(g,a_i),\le m_i}(r)\right)$$
$$=\frac{q+2k-2}{2k}\sum_{i=1}^{q}\left(N^{[k]}_{(f,a_i),\le m_i}(r)+N^{[k]}_{(g,a_i),\le m_i}(r)\right). \tag{6}$$

We check at once that

$$N^{[k]}_{(f,a_i),\le m_i}(r)\geq N^{[k]}_{(f,a_i)}(r)-\frac{k}{m_i+1}N_{(f,a_i),>m_i}(r)$$
$$\geq N^{[k]}_{(f,a_i)}(r)-\frac{k}{m_i+1}\left(N_{(f,a_i)}(r)-N^{[k]}_{(f,a_i),\le m_i}(r)\right),$$

which, together with First Main Theorem, implies

$$\begin{aligned}\text{(7)}\qquad \left(1-\frac{k}{m_i+1}\right)N^{[k]}_{(f,a_i),\le m_i}(r) &\ge N^{[k]}_{(f,a_i)}(r)-\frac{k}{m_i+1}N_{(f,a_i)}(r)\\ &\ge N^{[k]}_{(f,a_i)}(r)-\frac{k}{m_i+1}T_f(r).\end{aligned}$$

Let $\lambda_i=\frac{m_i+1}{m_i+1-k}$. Then $\frac{k}{m_i+1-k}=\lambda_i-1,\ 1\le i\le q$. Hence, we have

$$\text{(8)}\qquad \sum_{i=1}^{q}N^{[k]}_{(f,a_i),\le m_i}(r)\ge\sum_{i=1}^{q}\Big(\lambda_i N^{[k]}_{(f,a_i)}(r)-(\lambda_i-1)T_f(r)\Big)$$

and

$$\text{(9)}\qquad \sum_{i=1}^{q}N^{[k]}_{(g,a_i),\le m_i}(r)\ge\sum_{i=1}^{q}\Big(\lambda_i N^{[k]}_{(g,a_i)}(r)-(\lambda_i-1)T_g(r)\Big)$$

by (7).

By combining (6), (8) and (9), we have

$$\begin{aligned}\text{(10)}\qquad N_P(r) &\ge \frac{q+2k-2}{2k}\sum_{i=1}^{q}\Big(\lambda_i N^{[k]}_{(f,a_i)}(r)-(\lambda_i-1)T_f(r)\Big)\\ &\quad+\frac{q+2k-2}{2k}\sum_{i=1}^{q}\Big(\lambda_i N^{[k]}_{(g,a_i)}(r)-(\lambda_i-1)T_g(r)\Big)\\ &\ge \frac{q+2k-2}{2k}\sum_{i=1}^{q}\lambda_i N^{[k]}_{(f,a_i)}(r)+\frac{q+2k-2}{2k}\sum_{i=1}^{q}\lambda_i N^{[k]}_{(g,a_i)}(r)\\ &\quad-\frac{q+2k-2}{2k}\sum_{i=1}^{q}(\lambda_i-1)T(r),\end{aligned}$$

where $T(r)=T_f(r)+T_g(r)$.

Further notice that $\max_{1\le i\le q}\frac{k}{m_i+1-k}\le\frac{q}{2n-k+1}-1$ implies $(2n-k+2)\max_{1\le i\le q}\lambda_i\le\sum_{i=1}^{q}\lambda_i$. By (10) and Theorem 1.1 with $\eta=\frac{\sum_{i=1}^{q}\lambda_i}{2n-k+2}$ and $\gamma(r)=e^{(\min\{c_f,c_g\}+\varepsilon)(T_f(r)+T_g(r))}$, we have

$$\begin{aligned}\text{(11)}\quad N_P(r) &\ge \frac{q+2k-2}{2k}\left(\frac{\sum_{j=1}^{q}\lambda_j-(n-k)\eta}{n+2}\right)\{T(r)-k(n+1)\log\gamma(r)\}\\ &\quad-S(r)-2\varepsilon\log(r\gamma(r))-\frac{q+2k-2}{2k}\sum_{i=1}^{q}(\lambda_i-1)T(r)\\ &=\frac{q+2k-2}{2k}\cdot\frac{\sum_{j=1}^{q}\lambda_j}{2n-k+2}\{T(r)-k(n+1)\log\gamma(r)\}\\ &\quad-\frac{q+2k-2}{2k}\sum_{i=1}^{q}(\lambda_i-1)T(r)-S(r)-2\varepsilon\log(r\gamma(r)).\end{aligned}$$

On the other hand, by the Jensen formula, we have

$$
\begin{aligned}
(12) \qquad & N_P(r) \\
&= \int_{|z|=r} \log |P(re^{\sqrt{-1}\theta})| d\theta + O(1) \\
&= \sum_{i=1}^{q} \int_{|z|=r} \log \left| P_i(re^{\sqrt{-1}\theta}) \right| d\theta + O(1) \\
&\leq \sum_{i=1}^{q} \int_{|z|=r} \log \left(\left| (f, a_i) \left(re^{\sqrt{-1}\theta}\right) \right|^2 + \left| \left(f, a_{\sigma(i)}\right) \left(re^{\sqrt{-1}\theta}\right) \right|^2 \right)^{1/2} d\theta \\
&\quad + \sum_{i=1}^{q} \int_{|z|=r} \log \left(\left| (g, a_i) \left(re^{\sqrt{-1}\theta}\right) \right|^2 + \left| \left(g, a_{\sigma(i)}\right) \left(re^{\sqrt{-1}\theta}\right) \right|^2 \right)^{1/2} d\theta \\
&\quad + O(1) \\
&\leq q \left(T_f(r) + T_g(r)\right) + o \left(T_f(r) + T_g(r)\right) = qT(r) + o(T(r)).
\end{aligned}
$$

Now we derived from (11) and (12) that

$$
\begin{aligned}
\frac{2kq}{q+2k-2} T(r) + o(T(r)) \leq{}& \frac{\sum_{j=1}^{q} \lambda_j}{2n-k+2} \{T(r) - k(n+1) \log \gamma(r)\} \\
& - S(r) - \sum_{i=1}^{q} (\lambda_i - 1) T(r) - 2\varepsilon \log(r\gamma(r)).
\end{aligned}
$$

Letting $r \to R^-(r \notin E)$ and letting $\varepsilon \to 0^+$, we get

$$
\frac{2kq}{q+2k-2} \geq \frac{\sum_{j=1}^{q} \lambda_j}{2n-k+2} \left(1 - k(n+1) \min\{c_f, c_g\}\right) - \sum_{i=1}^{q} (\lambda_i - 1).
$$

Note that $q \geq 2k(2n-k+1) + 2$ and $\sum_{i=1}^{q} \frac{k}{m_i+1-k} < \frac{q(q-2k(2n-k+1)-2)}{(q+2k-2)(2n-k+1)}$, we get $\frac{q(q-2)(2n-k+2)}{(q+2k-2)\sum_{j=1}^{q} \lambda_j} - 2n + k - 1 > 0$, and therefore

$$
\min\{c_f, c_g\} \geq \frac{1}{k(n+1)} \left(\frac{q(q-2)(2n-k+2)}{(q+2k-2)\sum_{j=1}^{q} \lambda_j} - 2n + k - 1 \right).
$$

This is a contradiction. Then $f = g$. Hence, the theorem is proved. $\square$

Acknowledgement. The author would like to thank the referee for several useful comments and suggestions.

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