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SECOND MAIN THEOREM FOR HOLOMORPHIC CURVES INTO ALGEBRAIC VARIETIES WITH THE MOVING TARGETS ON AN ANGULAR DOMAIN

Jiali Chen and Qingcai Zhang

Abstract. In this paper, we will prove the second main theorem for holomorphic curves intersecting the moving hypersurfaces in subgeneral position with index on an angular domain. Our results are an extension of the previous second main theorems for holomorphic curves with moving targets on an angular domain.

1. Introduction

In 1925, Nevanlinna [8] established the second main theorem for meromorphic function on complex plane. In 1933, H. Cartan [1] proved the second main theorem for linearly nondegenerate holomorphic curves from complex plane into complex projective space intersecting hyperplanes in general position. After that, second main theorems have been established for holomorphic curves into complex projective spaces intersecting fixed or moving targets [9,11,14,15]. Ru [12] proved a second main theorem for algebraically nondegenerate holomorphic curves into $\mathbb{P}^n(\mathbb{C})$ intersecting fixed hypersurfaces in 2004. S. D. Quang [10] obtained a second main theorem with truncated counting functions for algebraically nondegenerate meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting a set of slowly moving hypersurfaces in N-subgeneral position. In 2009, Ru [13] extended the second main theorem for holomorphic mappings into complex projective variety intersecting fixed hypersurfaces in general position. Recently, Dethloff and Tan [4] further researched the case for holomorphic curves into projective variety with moving hypersurfaces and proved the following second main theorem.

Theorem 1.1 ([4]). *Let $V \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible (possibly singular) variety of dimension ℓ, and let f be a nonconstant holomorphic map of $\mathbb{C}$ into V. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a set of homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ in*

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general position with $\deg Q_j = d_j \geq 1$*. Assume that* f *is algebraically non-degenerate over* $\mathcal{K}_{\mathcal{Q}}$*. Then for any* $\epsilon > 0$,

$$\| \ (q - \ell - 1 - \varepsilon) T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} N_f(r, Q_j).$$

In this paper, a notation " $\|$ " in the inequality means that the inequality holds for $r \in (1, \infty)$ outside a set with measure finite.

Ji-Yan-Yu [7] introduced the new concept of subgeneral position with index to improve the second main theorem interesting moving hypersurface targets. According to [7], Xie-Cao [18] gave a similar definition for moving hypersurfaces in N-subgenerate position with index κ.

Definition 1 ([18])**.** Let V be an algebraic subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $\{D_1, \ldots, D_q\}$ be a family of moving hypersurfaces which coefficients are defined on angular domain into $\mathbb{P}^n(\mathbb{C})$. Let N and κ be two position integers such that $N \geq \dim V \geq \kappa$.

(a) The hypersurfaces $\{D_1, \ldots, D_q\}$ are said to be in general position (or say in weakly general position) in V if there exists $z \in \bar{\Omega}(\alpha, \beta)$ for any subset $I \subset \{1, \ldots, q\}$ with $\sharp I \leq \dim V + 1$,

$$\operatorname{codim}\left(\bigcap_{i \in I} D_i(z) \cap V\right) \geq \sharp I.$$

(b) The hypersurfaces $\{D_1, \ldots, D_q\}$ are said to be in N-subgeneral position in V if there exists $z \in \bar{\Omega}(\alpha, \beta)$ for any subset $I \subset \{1, \ldots, q\}$ with $\sharp I \leq N + 1$,

$$\dim\left(\bigcap_{i \in I} D_i(z) \cap V\right) \leq N - \sharp I.$$

(c) The hypersurfaces $\{D_1, \ldots, D_q\}$ are said to be in N-subgeneral position with index κ in V if $D_1, \ldots, D_q$ are in N-subgeneral position and if there exists $z \in \bar{\Omega}(\alpha, \beta)$ for any subset $I \subset \{1, \ldots, q\}$ with $\sharp I \leq \kappa$,

$$\operatorname{codim}\left(\bigcap_{i \in I} D_i(z) \cap V\right) \geq \sharp I.$$

Here we set $\dim \emptyset = -\infty$.

In 2019, Xie-Cao [18] combined the methods and the above definition to obtain the following result which extends the second main theorem with moving hypersurfaces in subgeneral position due to S. D. Quang [10].

Theorem 1.2 ([18])**.** *Let* $V \subset \mathbb{P}^n(\mathbb{C})$ *be an irreducible algebraic subvariety of dimension* ℓ*. Let* $f : \mathbb{C}^m \to V$ *be a nonconstant holomorphic map. Let* $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ *be a set of slowly moving hypersurfaces in* N*-subgeneral* *position with index κ in V with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\epsilon > 0$,*

$$\| \left(q - (1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}})(\ell+1) - \epsilon \right) T_f(r)$$
$$\leq \sum_{j=1}^{q} \frac{1}{d_j} N_f(r, Q_j) + o(T_f(r)).$$

For $0 \leq \alpha < \beta \leq 2\pi$, by $\Omega(\alpha, \beta)$ we denote the angular domain $\Omega(\alpha, \beta) := \{z : \alpha < \arg z < \beta\}$ and by $\bar{\Omega}(\alpha, \beta)$ its closure. The behavior of a function meromorphic in an angle has been investigated in many references, such that [5, 6, 17, 19]. In 2015, J. H. Zheng [20] established the value distribution of holomorphic curves on an angular domain from the point of view of potential theory and established the first and second fundamental theorem corresponding to those theorems of Ahlfors-Shimizu, Nevanlinna and Tsuji on meromorphic functions in an angular domain. In 2017, N. V. Thin [16] proved some fundamental theorems for holomorphic curves on $\bar{\Omega}(\alpha, \beta)$ intersecting finite set of fixed hyperplanes in general position and finite set of fixed hypersurfaces in general position on complex projective variety with the level of truncation. In 2018, the author [2] proved this result to the following.

Theorem 1.3 ([2]). *Let $V \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible (possibly singular) variety of dimension ℓ. Let $f : \bar{\Omega}(\alpha, \beta) \to V$ be a non-constant holomorphic map. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a set of slowly moving hypersurfaces in general position in V with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\epsilon > 0$,*

$$\| \ (q - \ell - 1 - \epsilon) S_{\alpha\beta, f}(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} C_{\alpha\beta, f}(r, Q_j) + R_{\alpha,\beta}(r, f),$$

$R_{\alpha,\beta}(r, f)$ is the error term with the estimate

$$R_{\alpha,\beta}(r, f) \leq K(\log^+ S_{\alpha-\varepsilon, \beta+\varepsilon; f}(r) + \log^+ r + 1),$$

where K is a constant depending on ε.

Thus, it is natural to ask how about using the concept of subgeneral position with index to second main theorems on an angular domain. Motivated by this problem, the main purpose of this paper is to adopt their methods [4, 10, 18], and obtain the second theorem for holomorphic curves on an angular domain intersecting moving hypersurfaces targets in N-subgeneral position with index κ, which is an improvement and an extension of the above theorems.

Now, we state our main theorems which are an improvement and extension of the results of Zheng [20] concerning moving hypersurfaces targets on an angular domain. Theorem 1.3 is just the following result for the special case whenever $N = \dim V$ and $\kappa = 1$.

Theorem 1.4. *Let $V \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible algebraic subvariety of dimension ℓ. Let $f : \bar{\Omega}(\alpha, \beta) \to V$ be a non-constant holomorphic map. Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a set of slowly moving hypersurfaces in N-subgeneral position with index κ in V with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\epsilon > 0$,*

$$\| \left(q - (1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}})(\ell+1) - \epsilon \right) S_{\alpha\beta,f}(r)$$
$$\leq \sum_{j=1}^{q} \frac{1}{d_j} C_{\alpha\beta,f}(r, Q_j) + R_{\alpha,\beta}(r,f).$$

When $V = \mathbb{P}^n(\mathbb{C})$, we have the following second main theorem with truncated counting function.

Theorem 1.5. *Let $f : \bar{\Omega}(\alpha, \beta) \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map. Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a set of slowly moving hypersurfaces in N-subgeneral position with index κ with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\epsilon > 0$,*

$$\| \left(q - (1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})(n+1) - \epsilon \right) S_{\alpha\beta,f}(r)$$
$$\leq \sum_{j=1}^{q} \frac{1}{d_j} C^{[M_0]}_{\alpha\beta,f}(r, Q_j) + R_{\alpha,\beta}(r,f),$$

where

$$M_0 := \binom{M+n}{n} p_0^{\binom{M+n}{n}\left(\binom{M+n}{n}^{-1}\right)\binom{q}{n}^{-2}} - 1,$$

with

$$M := (n+1)d + 2(1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})(n+1)^3 I(\epsilon^{-1})d,$$

$d := \operatorname{lcm}(d_1, \dots, d_q)$ is the least common multiple of all $\{d_j\}$, and

$$p_0 := \left[\frac{\binom{M+n}{n}\left(\binom{M+n}{n} - 1\right)\binom{q}{n} - 1}{\log(1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})} \right]^2.$$

Here, by $I(x)$ we denote the smallest integer which is not less than x.

By Theorem 1.5, we easily deduce the following corollary whenever $\kappa = 1$.

Corollary 1.6. *Let $f : \bar{\Omega}(\alpha, \beta) \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map. Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a set of slowly moving hypersurfaces in N-subgeneral* *position with* $\deg Q_j = d_j \geq 1$. *Assume that* f *is algebraically non-degenerate over* $\mathcal{K}_{\mathcal{Q}}$. *Then for any* $\epsilon > 0$,

$$\begin{aligned}&\| \ (q-(N-n+1)(n+1)-\epsilon)S_{\alpha\beta,f}(r)\\ &\leq \sum_{j=1}^{q}\frac{1}{d_j}C^{[M_0]}_{\alpha\beta,f}(r,Q_j)+R_{\alpha,\beta}(r,f),\end{aligned}$$

where

$$M_0 := \binom{M+n}{n} p_0^{\binom{M+n}{n}\left(\binom{M+n}{n}^{-1}\right)\binom{q}{n}^{-2}} - 1,$$

with

$$M := (n+1)d + 2(N-n+1)(n+1)^3 I(\epsilon^{-1})d,$$

$d := \mathrm{lcm}(d_1, \ldots, d_q)$ *is the least common multiple of all* $\{d_j\}$, *and*

$$p_0 := \left[\frac{\binom{M+n}{n}\left(\binom{M+n}{n}-1\right)\binom{q}{n}-1}{\log(1+\frac{\epsilon}{3(n+1)(N-n+1)})}\right]^2.$$

If $d_j = 1$, note that the hyperplans are in general position when $N = n$ and $\kappa = 1$. The following result can be obtained immediately.

Corollary 1.7. *Let* $f : \bar{\Omega}(\alpha,\beta) \to \mathbb{P}^n(\mathbb{C})$ *be a non-constant holomorphic map. Let* $\mathcal{H} = \{H_1, \ldots, H_q\}$ *be a set of slowly moving hyperplanes in general position. Assume that* f *is algebraically non-degenerate over* $\mathcal{K}_{\mathcal{H}}$. *Then for any* $\epsilon > 0$,

$$\| \ (q-n-1-\epsilon)S_{\alpha\beta,f}(r) \leq \sum_{j=1}^{q} C_{\alpha\beta,f}(r,H_j) + R_{\alpha,\beta}(r,f).$$

The remainder of this paper is organized as follows. In the next section, we will introduce some basic notions and auxiliary results from Nevanlinna theory on an angular domain. In Section 3 and Section 4, we give the proofs of Theorem 1.3 and Theorem 1.5. The methods and techniques to prove the main theorems by Dethloff-Tan [4], Quang [10], Ji-Yan-Yu [7] and Xie-Cao [18] are used in this paper.

2. Preliminaries and lemmas

We consider the set

$$\Omega(\alpha,\beta;r) = \Omega(\alpha,\beta) \cap \{1 <| z |< r\}.$$

Let f be a meromorphic function on the angle $\bar{\Omega}(\alpha,\beta;r)$, $0 < \beta - \alpha \leq 2\pi$, $1 \leq r < \infty$. We recall that

$$A_{\alpha\beta}(r,f) = \frac{k}{\pi}\int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right)[\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|]\frac{dt}{t};$$

$$B_{\alpha\beta}(r,f) = \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+|f(re^{i\varphi})|\cdot\sin(k(\varphi-\alpha))d\varphi;$$

$$\begin{aligned}C_{\alpha\beta}(r,f) &= 2k\int_1^r c_{\alpha\beta}(r,f)(\frac{1}{t^k}+\frac{t^k}{r^{2k}})\frac{dt}{t}\\ &= 2\sum_{1\leq\rho_n\leq r,\alpha\leq\psi_n\leq\beta}(\frac{1}{\rho_n^k}-\frac{\rho_n^k}{r^{2k}})\sin k(\psi_n-\alpha),\end{aligned}$$

where $c_{\alpha\beta}(r,f)=\sum_{1\leq\rho_n\leq r,\alpha\leq\psi_n\leq\beta}\sin k(\psi_n-\alpha)$, and $\rho_n e^{i\varphi_n}$ are poles of $f(z)$ counting with multiplicity. We denote $S_{\alpha\beta,f}(r)$ by the angular Nevanlinna characteristics on $\bar\Omega(\alpha,\beta;r)$ and define it as follows:

$$S_{\alpha\beta,f}(r) = A_{\alpha\beta}(r,f)+B_{\alpha\beta}(r,f)+C_{\alpha\beta}(r,f).$$

Now we introduce the Nevanlinna characteristic, counting function and proximity function of holomorphic curve in an angular domain.

Let $f:\bar\Omega(\alpha,\beta)\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve. Let $\mathbf{f}=(f_0:\cdots:f_n)$ be a reduced representation of f, where $f_0,\ldots,f_n$ are holomorphic functions and without common zeros in $\bar\Omega(\alpha,\beta)$. Let Q be a homogeneous polynomial of degree of d in the variables $x_0,\ldots,x_n$ with coefficients which are holomorphic functions without common zero on $\bar\Omega(\alpha,\beta)$. Assume that $Q(\mathbf{f})\not\equiv 0$. The counting function $C_{\alpha\beta,f}(r,Q)$ of f with respect to Q is defined as

$$C_{\alpha\beta,f}(r,Q) = 2\sum_{1\leq\rho_n\leq r,\alpha\leq\psi_n\leq\beta}(\frac{1}{\rho_n^k}-\frac{\rho_n^k}{r^{2k}})\sin k(\psi_n-\alpha),$$

where the $\rho_n e^{i\varphi_n}$ are zeros of $Q(\mathbf{f})$ in $\bar\Omega(\alpha,\beta)$ counting with multiplicity.

Let δ be a positive integer, the truncated counting function of f is defined by

$$C^{[\delta]}_{\alpha\beta,f}(r,Q) = 2\sum_{\substack{1\leq\rho_n\leq r,\alpha\leq\psi_n\leq\beta,\\ \min\{\mathrm{ord}_{Q(\mathbf{f})}(\rho_n e^{i\varphi_n}),\delta\}}}(\frac{1}{\rho_n^k}-\frac{\rho_n^k}{r^{2k}})\sin k(\psi_n-\alpha),$$

where any zero of multiplicity greater than δ of $Q(\mathbf{f})$ in $\bar\Omega(\alpha,\beta)$ is "truncated" and counted as if it only had multiplicity δ.

The angular proximity function of f with respect to Q is defined as following:

$$A_{\alpha\beta,f}(r,Q) = \frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\log\frac{\|\mathbf{f}(te^{i\alpha})\|^d\|\mathbf{f}(te^{i\beta})\|^d}{|Q(\mathbf{f})(te^{i\alpha})Q(\mathbf{f})(te^{i\beta})|}\frac{dt}{t}$$

and

$$B_{\alpha\beta,f}(r,Q) = \frac{2k}{\pi r^k}\int_\alpha^\beta\log\frac{\|\mathbf{f}(re^{i\varphi})\|^d}{|Q(\mathbf{f})(re^{i\varphi})|}\sin(k(\varphi-\alpha))d\varphi,$$

where $\|\mathbf{f}(z)\|=\max\{|f_0(z)|,\ldots,|f_n(z)|\}$ and d is the degree of Q.

Lemma 2.1 (First Main Theorem, [16, Theorem 1])**.**

$$dS_{\alpha\beta,f}(r) = A_{\alpha\beta,f}(r,Q)+B_{\alpha\beta,f}(r,Q)+C_{\alpha\beta,f}(r,Q).$$

We say that a meromorphic function φ on $\bar{\Omega}(\alpha,\beta)$ is "small" with respect to f if $S_{\alpha\beta,\varphi}(r) = o(S_{\alpha\beta,f}(r))$ as $r \to \infty$.

We denote by $\mathcal{M}$ (resp. $\mathcal{K}_f$) the field of all meromorphic functions (resp. small meromorphic functions with respect to f) on $\bar{\Omega}(\alpha,\beta)$.

For a positive integer d, we set

$$\mathcal{T}_d := \{(i_0,\dots,i_n) \in \mathbb{N}_0^{n+1} : i_0 + \cdots + i_n = d\}.$$

Let $\mathcal{Q} = \{Q_1,\dots,Q_q\}$ be a set of $q \geq n+1$ homogeneous polynomials in $\mathcal{K}_f[x_0,\dots,x_n]$, $\deg Q_j = d_j \geq 1$. We write

$$Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I \ (j = 1,\dots,q),$$

where $x^I = x_0^{i_0} \cdots x_n^{i_n}$ for $x = (x_0,\dots,x_n)$ and $I = (i_0,\dots,i_n)$. For each j, there exists a_{jI_j}, one of the coefficients in Q_j, such that $a_{jI_j} \not\equiv 0$. We fix a_{jI_j} and set $\tilde{a}_{jI} = \frac{a_{jI}}{a_{jI_j}}$ and

$$\tilde{Q}_j = \sum_{I \in \mathcal{T}_{d_j}} \tilde{a}_{jI} x^I \ (j = 1,\dots,q)$$

which is a homogenerous polynomial in $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]$. The moving hypersurfaces $\mathcal{Q} = \{Q_1,\dots,Q_q\}$ are said to be "slowly" with respect to f if $S_{\alpha\beta,\frac{a_{jI}}{a_{jI_j}}}(r)$ $= o(S_{\alpha\beta,f}(r))$, i.e., $\frac{a_{jI}}{a_{jI_j}} \in \mathcal{K}_f$.

Let $\mathcal{K}_{\mathcal{Q}}$ be the smallest subfield of meromorphic function field $\mathcal{M}$ which contains $\mathbb{C}$ and all $\frac{a_{jI_s}}{a_{jI_t}}$, where $a_{jI_t} \not\equiv 0,\ j \in \{1,\dots,q\},\ I_t, I_s \in \mathcal{T}_{d_j}$.

Denote by Ω_f the set of all non-negative functions $h : \Omega_{\alpha,\beta} \to [0,+\infty]$, which are of the form $\frac{|u_1|+\cdots+|u_k|}{|v_1|+\cdots+|v_l|}$, where $k,l \in \mathbb{N}, u_i, v_j \in \mathcal{K}_f \setminus \{0\}$. Then, if $h \in \Omega_f$ we have

$$\begin{aligned}&\frac{k}{\pi}\int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right)[\log^+ |h(te^{i\alpha})| + \log^+ |h(te^{i\beta})|]\frac{dt}{t} \\ &\quad + \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+ |h(re^{i\varphi})| \cdot \sin(k(\varphi - \alpha))d\varphi \\ &= o(S_{\alpha\beta,f}(r)).\end{aligned}$$

Lemma 2.2 ([4,10])**.** *Assume that $\mathcal{Q} = \{Q_1,\dots,Q_q\}$ is in N-subgeneral position with $\deg Q_j = d_j$ and d is the* lcm *of the Q_j's. Then for any $Q_{j_1},\dots,Q_{j_{N+1}}$ $\in \mathcal{Q}$, there exist functions $h_1, h_2 \in \Omega_f \backslash \{0\}$ such that,*

$$h_2 \cdot \|\mathbf{f}\|^d \leq \max_{i \in \{1,\dots,N+1\}} |Q_{j_i}(f_0,\dots,f_n)| \leq h_1 \cdot \|\mathbf{f}\|^d.$$

Lemma 2.3 ([18, Lemma 3.3])**.** *Let $\tilde{Q}_1,\dots,\tilde{Q}_{N+1}$ be homogeneous polynomials in $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]$ of the same degree $d \geq 1$ in (weakly) N-subgeneral position with index κ in V. For each point $a \in \bar{\Omega}(\alpha,\beta)$ satisfying the following conditions:*

(i) *the coefficients of $\tilde{Q}_1,\dots,\tilde{Q}_{N+1}$ are holomorphic at a,*

(ii) $\tilde{Q}_1(a), \ldots, \tilde{Q}_{N+1}(a)$ *have no non-trivial common zeros,*

(iii) $\dim V(a) = \ell$, *then there exist homogeneous polynomials* $\tilde{P}_1(a) = \tilde{Q}_1(a)$, $\ldots$, $\tilde{P}_\kappa(a) = \tilde{Q}_\kappa(a), \tilde{P}_{\kappa+1}(a), \ldots, \tilde{P}_{\ell+1}(a) \in \mathbb{C}[x_0, \ldots, x_n]$ *with*

$$\tilde{P}_t(a) = \sum_{j=\kappa+1}^{N-\ell+t} c_{tj}\tilde{Q}_j(a), \ c_{tj} \in \mathbb{C}, \ t \geq \kappa+1,$$

such that

$$\left(\bigcap_{t=1}^{\ell+1} \tilde{P}_t(a)\right) \cap V = \emptyset.$$

Let V be a subvariety in $\mathbb{P}^n(\mathbb{C})$ of dimension ℓ defined by the homogeneous ideal $\mathcal{I}(V) \subset \mathbb{C}[x_0, \ldots, x_n]$. Denote by $\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)$ the ideal in $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]$ generated by $\mathcal{I}(V)$. We say that f is algebraically nondegenerate over $\mathcal{K}_\mathcal{Q}$ if there is no homogeneous polynomial $Q \in \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n] \setminus \mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)$ such that $Q(f) \equiv 0$.

For a positive integer M, let $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_M$ be the vector space of homogeneous polynomials of degree M, and let $\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M := \mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V) \cap \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_M$.

The Hilbert polynomial $H_V(M)$ of V is defined by

$$H_V(M) := \dim_{\mathcal{K}_\mathcal{Q}} \frac{\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_M}{\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M}.$$

By the theory of Hilbert polynomials, we have the following fact:

$$H_V(M) = \frac{\deg V \cdot M^\ell}{\ell!} + O(M^{\ell-1}).$$

Definition 2 ([4, 12])**.** For each $I = (i_1, \ldots, i_\ell) \in \mathbb{N}_0^\ell$ and $M \in \mathbb{N}_0$ with $M \geq d\|I\|$, denote by $\mathcal{L}_\mathcal{M}^\mathcal{I}$ the set of all $\gamma \in \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_{M-d\|I\|}$ such that

$$P_{i,1}^{i_1} \cdots P_{i,\ell}^{i_\ell}\gamma - \sum_{E=(e_1,\ldots,e_\ell)>I} P_{i,1}^{e_1} \cdots P_{i,\ell}^{e_\ell}\gamma_E \in \mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M$$

for some $\gamma_E \in \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_{M-d\|E\|}$.

Denote by $\mathcal{L}^I$ the homogeneous ideal in $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]$ generated by

$$\cup_{M \geq d\|I\|} \mathcal{L}_\mathcal{M}^\mathcal{I}.$$

Remark 2.4 ([4,12]). (i) $\mathcal{L}_\mathcal{M}^\mathcal{I}$ is a $\mathcal{K}_\mathcal{Q}$-vector sub-space of $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_{M-d\|I\|}$, and $(\mathcal{I}(V), P_{i,1}, \ldots, P_{i,\ell})_{M-d\|I\|} \subset \mathcal{L}_\mathcal{M}^\mathcal{I}$, where $(\mathcal{I}_\mathcal{Q}(V), P_{i,1}, \ldots, P_{i,\ell})$ is the ideal in $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]$ generate by $\mathcal{I}_\mathcal{Q}(V) \cup \{P_{i,1}, \ldots, P_{i,\ell}\}$.

(ii) For any $\gamma \in \mathcal{L}_\mathcal{M}^\mathcal{I}$ and $P \in \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_k$, we have $\gamma \cdot P \in \mathcal{L}_{M+k}^I$.

(iii) $\mathcal{L}^I \cap \mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_{M-d\|I\|} = \mathcal{L}_\mathcal{M}^\mathcal{I}$.

(iv) $\frac{\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]}{\mathcal{L}^I}$ is graded module over the graded ring $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]$.

Set

$$m_M^I := \dim_{\mathcal{K}_\mathcal{Q}} \frac{\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_n]_{M-d\|I\|}}{\mathcal{L}_\mathcal{M}^\mathcal{I}}.$$

For each positive integer M, denote by τ_M the set of all $I := (i_1, \dots, i_\ell) \in \mathbb{N}_0^\ell$ with $M - d\|I\| \geq 0$. Let $\gamma_{I1}, \dots, \gamma_{Im_M^I} \in \mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_n]_{M-d\|I\|}$ such that they form a basis of the $\mathcal{K}_{\mathcal{Q}}$-vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]_{M-d\|I\|}}{\mathcal{L}_{\mathcal{M}}^{\mathcal{I}}}$.

Lemma 2.5 ([4,12])**.** *$\{[P_{i,1}^{i_1} \cdots P_{i,\ell}^{i_\ell} \cdot \gamma_{I1}], \dots, [P_{i,1}^{i_1} \cdots P_{i,\ell}^{i_\ell} \cdot \gamma_{Im_M^I}], I = (i_1, \dots, i_\ell) \in \tau_M\}$ is a basis of the $\mathcal{K}_{\mathcal{Q}}$-vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]_{M-d\|I\|}}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_M}$.*

Using Definition 2 and Lemma 2.5 by the arguments as [4], we have the following lemma.

Lemma 2.6 ([4, Lemma 2.11])**.** *For all $M \gg 0$ be an integer divisible by d, there are homogeneous polynomials $\phi_1, \dots, \phi_{H_V(M)}$ in $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_n]_M$ such that they form a basis of the $\mathcal{K}$-vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]_M}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_M}$, and*

$$\prod_{j=1}^{H_V(M)} \phi_j - (P_{i,1} \cdots P_{i,\ell})^{\frac{\deg V \cdot M^{\ell+1}}{d \cdot (\ell+1)!} - u(M)} \cdot P \in \mathcal{I}_{\mathcal{K}}(V)_{M \cdot H_V(M)},$$

where $u(M)$ is a function in M satisfying $u(M) \leq O(M^\ell)$ and P is a homogeneous polynomial of degree

$$M \cdot H_V(M) - \frac{\ell \cdot \deg V \cdot M^{\ell+1}}{(\ell+1)!} + \ell d \cdot u(M) = \frac{\deg V \cdot M^{\ell+1}}{(\ell+1)!} + O(M^\ell).$$

Lemma 2.7 ([16, Lemma 6])**.** *Let $f : \bar{\Omega}(\alpha, \beta) \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve and $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$\begin{aligned}
&\| \frac{k}{\pi} \int_1^r (\frac{1}{t^k} - \frac{t^k}{r^{2k}})(\max_K \log \prod_{k \in K} (\frac{\|\mathbf{f}\|}{|H_k(\mathbf{f})|}(te^{i\alpha})) \\
&+ \max_K \log \prod_{k \in K} (\frac{\|\mathbf{f}\|}{|H_k(\mathbf{f})|}(te^{i\beta}))) \frac{dt}{t} \\
&+ \frac{2k}{\pi r^k} \int_\alpha^\beta \max_K \log \prod_{k \in K} (\frac{\|\mathbf{f}\|}{|H_k(\mathbf{f})|}(re^{i\varphi})) \cdot \sin(k(\varphi - \alpha)) d\varphi \\
\leq\ & (n+1) S_{\alpha\beta,f}(r) - C_{\alpha\beta,W}(r, 0) + R_{\alpha,\beta}(r, f),
\end{aligned}$$

where W is the Wronskian of f.

3. Proof of Theorem 1.4

Replacing Q_j by $Q^{\frac{d}{d_j}}$, where d is the lcm of the $Q_j's$, we may assume that the polynomials $Q_1, \dots, Q_q$ have the same degree d. We denote by $\mathcal{I}$ the set of all permutations of the set $\{1, \dots, q\}$.

For a fixed point $z \in \bar{\Omega}(\alpha, \beta) \setminus \cup_{i=1}^q \tilde{Q}_i(\mathbf{f})^{-1}\{0\}$, we may assume that there exists $I_i = (I_i(1), \dots, I_i(q)) \in \mathcal{I}$ such that

$$|\tilde{Q}_{I_i(1)}(\mathbf{f})(z)| \leq |\tilde{Q}_{I_i(2)}(\mathbf{f})(z)| \leq \cdots \leq |\tilde{Q}_{I_i(q)}(\mathbf{f})(z)|.$$

For $I_i \in \mathcal{I}$, by Lemma 2.3, we denote the hypersurfaces $P_{i,1}, \ldots, P_{i,\ell+1}$ with respect to the hypersurfaces $\tilde{Q}_{I_i(1)}, \ldots, \tilde{Q}_{I_i(N+1)}$. It is easy to see that there exists a positive function $h_1 \in \Omega_f$, for all $z \in \bar{\Omega}(\alpha,\beta) \setminus \cup_{i=1}^{q} \tilde{Q}_i(\mathbf{f})^{-1}\{0\}$, such that

$$(1)\quad |P_{i,t}(\mathbf{f})(z)| \leq h_1(z) \max_{\kappa+1 \leq j \leq N-\ell+t} |\tilde{Q}_{I_i(j)}(\mathbf{f})(z)| = h_1(z) |\tilde{Q}_{I_i(N-\ell+t)}(\mathbf{f})(z)|$$

for $\kappa + 1 \leq t \leq \ell + 1$.

Since $Q_1, \ldots, Q_q$ are in N-subgeneral position in V, by Lemma 2.2, there exists $h_2 \in \Omega_f$ for all $I_i \in \mathcal{I}$, such that

$$h_2(z)\|\mathbf{f}(z)\|^d \leq \max_{1 \leq j \leq N+1} |\tilde{Q}_{I_i(j)}(\mathbf{f})(z)| = |\tilde{Q}_{I_i(N+1)}(\mathbf{f})(z)|.$$

Therefore, combining this with (1), we get

$$\begin{aligned}
\prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|} &\leq \frac{1}{h_2^{q-N}(z)} \prod_{j=1}^{N} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{I_i(j)}(\mathbf{f})(z)|} \\
&= \frac{1}{h_2^{q-N}(z)} \prod_{j=1}^{\kappa} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{I_i(j)}(\mathbf{f})(z)|} \cdot \prod_{j=\kappa+1}^{N-\ell+\kappa} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{I_i(j)}(\mathbf{f})(z)|} \\
&\quad \cdot \prod_{j=N-\ell+\kappa+1}^{N} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{I_i(j)}(\mathbf{f})(z)|} \\
&\leq h_3(z) \prod_{j=1}^{\kappa} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \cdot \prod_{j=\kappa+1}^{N-\ell+\kappa} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{I_i(j)}(\mathbf{f})(z)|} \\
&\quad \cdot \prod_{j=\kappa+1}^{\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}, \qquad (2)
\end{aligned}$$

where $h_3 = \frac{1}{h_2^{q-N}(z)} \cdot h_1^{\ell-\kappa}(z)$. By Lemma 2.2, there exists a function $h_4 \in \Omega_f$ such that $|P_{i,j}(\mathbf{f})(z)| \leq h_4(z)\|\mathbf{f}(z)\|^d$.

If $N - \ell \leq \kappa$, by (2), we have

$$\begin{aligned}
\prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|} &\leq h_3(z) \prod_{j=1}^{\kappa} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \cdot \prod_{j=1}^{N-\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \cdot \prod_{j=\kappa+1}^{\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \\
&\leq h_3(z) \cdot h_4^{\kappa+\ell-N}(z) \prod_{j=1}^{\kappa} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \cdot \prod_{j=1}^{N-\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \\
&\quad \cdot \prod_{j=N-\ell+1}^{\kappa} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \cdot \prod_{j=\kappa+1}^{\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}
\end{aligned}$$

$$\leq h_5(z)\left(\prod_{j=1}^{\kappa}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^2\cdot\left(\prod_{j=\kappa+1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^2$$

$$= h_5(z)\left(\prod_{j=1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^2, \tag{3}$$

where $h_5(z) = h_3(z)\cdot h_4^{2\ell-N}(z)\in\Omega_f$.

If $N-\ell\geq\kappa$, we have

$$\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}$$

$$\leq h_3(z)\prod_{j=1}^{\kappa}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\cdot\left(\prod_{j=1}^{\kappa}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{\frac{N-\ell}{\kappa}}\cdot\prod_{j=\kappa+1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}$$

$$= h_3(z)\left(\prod_{j=1}^{\kappa}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{1+\frac{N-\ell}{\kappa}}\cdot\prod_{j=\kappa+1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}$$

$$\leq h_6(z)\left(\prod_{j=1}^{\kappa}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{1+\frac{N-\ell}{\kappa}}\cdot\left(\prod_{j=\kappa+1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{1+\frac{N-\ell}{\kappa}}$$

$$= h_6(z)\left(\prod_{j=1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{1+\frac{N-\ell}{\kappa}}, \tag{4}$$

where $h_6(z) = h_3(z)\cdot h_4^{\frac{(\ell-\kappa)(N-\ell)}{\kappa}}(z)\in\Omega_f$.

Thus by (3) and (4), we can obtain

$$\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}\leq h\left(\prod_{j=1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right)^{1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}},$$

where $h=\max\{h_5,h_6\}\in\Omega_f$.

Then, we have

$$\log\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}$$

$$\leq \log h+\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)\log\left(\prod_{j=1}^{\ell}\frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|}\right). \tag{5}$$

Let $M\gg 0$ be an integer divisible by d. By Lemma 2.6, there are homogeneous polynomials $\phi_1,\dots,\phi_{H_V(M)}$ in $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]$ and there are functions

$u(M), v(M) \leq O(M^\ell)$ such that they form a basis of the $\mathcal{K}_\mathcal{Q}$-vector space $\frac{\mathcal{K}_\mathcal{Q}[x_0,\ldots,x_n]}{\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M}$, and

$$\prod_{j=1}^{H_V(M)} \phi_j - (P_{i,1}\cdots P_{i,\ell})^{\frac{\deg V\cdot M^{\ell+1}}{d\cdot(\ell+1)!}-u(M)} \cdot P \in \mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_{M\cdot H_V(M)},$$

where P is a homogeneous polynomial of degree $\frac{\deg V\cdot M^{\ell+1}}{(\ell+1)!}+v(M)$. On the other hand, for any $Q\in\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_{M\cdot H_V(M)}$, we have $Q(\mathbf{f})\equiv 0$. Therefore

$$\prod_{j=1}^{H_V(M)} \phi_j(\mathbf{f}) = (P_{i,1}(\mathbf{f})\cdots P_{i,\ell}(\mathbf{f}))^{\frac{\deg V\cdot M^{\ell+1}}{d\cdot(\ell+1)!}-u(M)} \cdot P(\mathbf{f}).$$

Since the coefficients of P are small functions, there exists $g\in\Omega_f$ such that

$$|P(\mathbf{f})| \leq \|\mathbf{f}\|^{\deg P}\cdot g = \|\mathbf{f}\|^{\frac{\deg V\cdot M^{\ell+1}}{(\ell+1)!}+v(M)}\cdot g.$$

Therefore,

$$\begin{aligned}\log(\prod_{j=1}^{H_V(M)} |\phi_j(\mathbf{f})|) \leq &\left(\frac{\deg V\cdot M^{\ell+1}}{d\cdot(\ell+1)!}-u(M)\right)\log|P_{i,1}(\mathbf{f})\cdots P_{i,\ell}(\mathbf{f})| \\ &+\log^+ g + \left(\frac{\deg V\cdot M^{\ell+1}}{(\ell+1)!}+v(M)\right)\log\|\mathbf{f}\|.\end{aligned}$$

This implies that there are functions $\omega_1(M), \omega_2(M) \leq O(\frac{1}{M})$ such that

$$\begin{aligned}\log|P_{i,1}(\mathbf{f})\cdots P_{i,\ell}(\mathbf{f})| \geq &\left(\frac{d\cdot(\ell+1)!}{\deg V\cdot M^{\ell+1}}-\frac{\omega_1(M)}{M^{\ell+1}}\right)\cdot\log(\prod_{j=1}^{H_V(M)}|\phi_j(\mathbf{f})|) \\ &-\frac{1}{M^{\ell+1}}\log^+\tilde{g} - (d+\omega_2(M))\log\|\mathbf{f}\| \end{aligned} \tag{6}$$

for some $\tilde{g}\in\Omega_f$.

We fix homogeneous polynomials $\Phi_1,\ldots,\Phi_{H_V(M)}\in\mathcal{K}_\mathcal{Q}[x_0,\ldots,x_n]_M$ such that they form a basis of the $\mathcal{K}_\mathcal{Q}$-vector space $\frac{\mathcal{K}_\mathcal{Q}[x_0,\ldots,x_n]_M}{\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M}$. Then there exist linear homogeneous polynomials $L_1,\ldots,L_{H_V(M)}\in\mathcal{K}_\mathcal{Q}[y_1,\ldots,y_{H_V(M)}]$ such that they are linearly independent over $\mathcal{K}_\mathcal{Q}$ and

$$\phi_j - L_j(\Phi_1,\ldots,\Phi_{H_V(M)}) \in \mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M \quad \text{for all } j\in\{1,\ldots,H_V(M)\}. \tag{7}$$

There exists a meromorphic function φ such that $C_{\alpha\beta}(r,\varphi)=o(S_{\alpha\beta,f}(r))$, $C_{\alpha\beta}(r,\frac{1}{\varphi})=o(S_{\alpha\beta,f}(r))$ and $\frac{\Phi_1(\mathbf{f})}{\varphi},\ldots,\frac{\Phi_{H_V(M)}(\mathbf{f})}{\varphi}$ are holomorphic and have no common zeros. Let $F:\bar{\Omega}(\alpha,\beta)\to\mathbb{P}^{H_V(M)-1}(\mathbb{C})$ be the holomorphic map with the reduced representation $F:=(\frac{\Phi_1(\mathbf{f})}{\varphi}:\cdots:\frac{\Phi_{H_V(M)}(\mathbf{f})}{\varphi})$. Since f is algebraically non-degenerate over $\mathcal{K}_\mathcal{Q}$, and since the polynomials $\Phi_1,\ldots,\Phi_{H_V(M)}$ form a basis of $\frac{\mathcal{K}_\mathcal{Q}[x_0,\ldots,x_n]_M}{\mathcal{I}_{\mathcal{K}_\mathcal{Q}}(V)_M}$, we get that F is linearly non-degenerate over $\mathcal{K}_\mathcal{Q}$.

As a corollary, F is linearly non-degenerate over the field over $\mathbb{C}$ generated by all coefficients of $L_j^{'}s$. We can see that

$$S_{\alpha\beta,F}(r) \leq M \cdot S_{\alpha\beta,f}(r) + o(S_{\alpha\beta,f}(r)). \tag{8}$$

In order to simplify the writing of the following series of inequalities, put $A(M) := \frac{d\cdot(\ell+1)!}{\deg V \cdot M^{\ell+1}} - \frac{\omega_1(M)}{M^{\ell+1}}$. By (7), for all $j \in \{1, \dots, H_V(M)\}$ we have

$$\log|\phi_j(\mathbf{f})| = \log|L_j(F)| + \log|\varphi|.$$

Hence, by (6), we get

$$\begin{aligned}
&\log(|P_{i,1}(\mathbf{f})| \cdots |P_{i,\ell}(\mathbf{f})|)\\
&\geq A(M) \cdot \left(H_V(M) \cdot \log|\varphi| + \log(\prod_{j=1}^{H_V(M)} |L_j(F)|) \right)\\
&\quad - \frac{1}{M^{\ell+1}} \log^+ \tilde{g} - (d + \omega_2(M)) \log\|\mathbf{f}\|\\
&\geq A(M) \cdot \log\left(\prod_{j=1}^{H_V(M)} |L_j(F)| \right) + A(M) \cdot H_V(M) \cdot \log|\varphi|\\
&\quad - \log^+ \tilde{g} - (d + \omega_2(M)) \log\|\mathbf{f}\|.
\end{aligned} \tag{9}$$

Then, by (9), we have

$$\begin{aligned}
\log\left(\prod_{j=1}^{\ell} \frac{\|\mathbf{f}(z)\|^d}{|P_{i,j}(\mathbf{f})(z)|} \right) \leq\ &\log\|\mathbf{f}\|^{\ell d} - A(M) \cdot \log\left(\prod_{j=1}^{H_V(M)} |L_j(F)| \right) + \log^+ \tilde{g}\\
&- A(M) \cdot H_V(M) \cdot \log|\varphi| + (d + \omega_2(M)) \log\|\mathbf{f}\|\\
=\ &((\ell+1)d + \omega_2(M)) \log\|\mathbf{f}\| - A(M) \cdot H_V(M) \log\|F\|\\
&+ A(M) \cdot \log\left(\prod_{j=1}^{H_V(M)} \frac{\|F\|}{|L_j(F)|} \right)\\
&- A(M) \cdot H_V(M) \cdot \log|\varphi| + \log^+ \tilde{g}.
\end{aligned} \tag{10}$$

Then, combining (5) and (10), we obtain

$$\begin{aligned}
&\log \prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}\\
\leq\ &\left(1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}}\right) ((\ell+1)d + \omega_2(M)) \log\|\mathbf{f}\|\\
&- \left(1 + \frac{N-\ell}{\max\{1, \min\{N-\ell, \kappa\}\}}\right) A(M) \cdot H_V(M) \log\|F\|
\end{aligned}$$

$$+ A(M)\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)\cdot\log\left(\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}\right)$$
$$-\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)\cdot A(M)\cdot H_V(M)\cdot\log|\varphi|$$
$$+\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)\log^+\tilde{g}+\log h. \tag{11}$$

By applying Lemma 2.7 to the holomorphic map $F:\bar{\Omega}(\alpha,\beta)\to\mathbb{P}^{H_V(M)-1}(\mathbb{C})$ and the system of linear polynomials $L_1,\dots,L_{H_V(M)}\in\mathcal{K}_{\mathcal{Q}}[y_1,\dots,y_{H_V(M)}]$, we get:

$$\begin{aligned}
\| \ & \frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\log(\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}(te^{i\alpha}))\frac{dt}{t}\\
&+\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\log(\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}(te^{i\beta}))\frac{dt}{t}\\
&+\frac{2k}{\pi r^k}\int_\alpha^\beta\log(\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}(re^{i\varphi}))\cdot\sin(k(\varphi-\alpha))d\varphi\\
\leq\ & \frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\max_K\log\prod_{k\in K}(\frac{\|F\|}{|L_j(F)|}(te^{i\alpha}))\frac{dt}{t}\\
&+\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\max_K\log\prod_{k\in K}(\frac{\|F\|}{|L_j(F)|}(te^{i\beta}))\frac{dt}{t}\\
&+\frac{2k}{\pi r^k}\int_\alpha^\beta\max_K\log\prod_{k\in K}(\frac{\|F\|}{|L_j(F)|}(re^{i\varphi}))\cdot\sin(k(\varphi-\alpha))d\varphi\\
\leq\ & H_V(M)S_{\alpha\beta,F}(r)+R_{\alpha,\beta}(r,F),
\end{aligned}\tag{12}$$

where $\max_K$ is taken over all subsets of the system of linear polynomials $L_1,\dots,L_{H_V(M)}\in\mathcal{K}_{\mathcal{Q}}[y_1,\dots,y_{H_V(M)}]$ such that these linear polynomials are linearly independent over $\mathcal{K}_{\mathcal{Q}}$. By integrating (11), we obtain

$$\begin{aligned}
&\|\sum_{j=1}^q A_{\alpha\beta,f}(r,Q_j)+\sum_{j=1}^q B_{\alpha\beta,f}(r,Q_j)\\
\leq\ & d\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)(\ell+1)S_{\alpha\beta,f}(r)+o(S_{\alpha\beta,f}(r))\\
&-\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)A(M)\cdot H_V(M)(C_{\alpha\beta}(r,\varphi)-C_{\alpha\beta}(r,\frac{1}{\varphi}))\\
&+\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)A(M)
\end{aligned}$$

$$\Big[\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})(\log\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}(te^{i\alpha})+\log\prod_{j=1}^{H_V}\frac{\|F\|}{|L_j(F)|}(te^{i\beta})\frac{dt}{t}$$
$$+\frac{2k}{\pi r^k}\int_\alpha^\beta\log\prod_{j=1}^{H_V(M)}\frac{\|F\|}{|L_j(F)|}(re^{i\varphi})\cdot\sin(k(\varphi-\alpha))d\varphi\Big]$$
$$-A(M)\cdot H_V(M)\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)S_{\alpha\beta,F}(r).$$

Combining above inequality with (8) and (12), we have (using that $C_{\alpha\beta}(r,\varphi)=o(S_{\alpha\beta,f}(r)); C_{\alpha\beta}(r,\frac{1}{\varphi})=o(S_{\alpha\beta,f}(r))$, that $A(M)\cdot H_V(M)\le O(\frac{1}{M})$ and that $\tilde{h}\in\Omega_f$)

$$\begin{aligned}
&\|\sum_{j=1}^q A_{\alpha\beta,f}(r,Q_j)+\sum_{j=1}^q B_{\alpha\beta,f}(r,Q_j)\\
&\le d\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)(\ell+1)S_{\alpha\beta,f}(r)\\
&\quad+A(M)\cdot H_V(M)\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)S_{\alpha\beta,F}(r)\\
&\quad-A(M)\cdot H_V(M)\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)S_{\alpha\beta,F}(r)+R_{\alpha,\beta}(r,f)\\
&\le d\left(1+\frac{N-\ell}{\max\{1,\min\{N-\ell,\kappa\}\}}\right)(\ell+1)S_{\alpha\beta,f}(r)+R_{\alpha,\beta}(r,f).
\end{aligned}$$

Therefore, by the first main theorem, Theorem 1.3 is proved.

4. Proof of Theorem 1.5

For $I_{i_0}\in\mathcal{I}$, by Lemma 2.3, we denote the hypersurfaces $P_{i_0,1},\dots,P_{i_0,n+1}$ with respect to the hypersurfaces $\tilde{Q}_{I_{i_0}(1)},\dots,\tilde{Q}_{I_{i_0}(N+1)}$.

Similar to the argument of (5), we have

$$\begin{aligned}
&\log\prod_{j=1}^q\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}\\
&\le\log h+\left(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}\right)\log\left(\prod_{j=1}^n\frac{\|\mathbf{f}(z)\|^d}{|P_{i_0,j}(\mathbf{f})(z)|}\right).
\end{aligned}\tag{13}$$

Now, for a positive integer M, we denote by V_M the vector subspace of $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]$ which consists of all homogeneous polynomials of degree M and zero polynomial. Denote by $(P_{i_0,1},\dots,P_{i_0,n})$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_n]$ generated by $P_{i_0,1},\dots,P_{i_0,n}$.

We first introduce the following lemma.

Lemma 4.1 ([3, Proposition 3.3]). *Let $\{P_i\}_{i=1}^q (q \geq n+1)$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \ldots, x_n]$ in weakly general position. Then for any nonnegative integer M and for any $J := \{j_1, \ldots, j_n\} \subset \{1, \ldots, q\}$, the dimension of vector space $\frac{V_M}{(P_{j_1}, \ldots, P_{j_n}) \cap V_M}$ is equal to the number of n-tuples $(s_1, \ldots, s_n) \in \mathbb{N}_0^n$ such that $s_1 + \cdots + s_n \leq M$ and $0 \leq s_1, \ldots, s_n \leq d-1$. In particular, for all $M \geq n(d-1)$, we have*

$$\dim \frac{V_M}{(P_{j_1}, \ldots, P_{j_n}) \cap V_M} = d^n.$$

Next, we continue to prove the theorem.

We consider M divisible by d. For each $(i) = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ with $\sigma(i) = \sum_{s=1}^n i_s \leq \frac{M}{d}$, we set

$$W_{(i)}^{I_{i_0}} = \sum_{(j)=(j_1,\ldots,j_n) \geq (i)} P_{i_0,1}^{j_1} \cdots P_{i_0,n}^{j_n} \cdot V_{M - d\sigma(j)}.$$

Then we see that $W_{(0,\ldots,0)}^{I_{i_0}} = V_M$ and $W_{(i)}^{I_{i_0}} \supset W_{(j)}^{I_{i_0}}$ if $(i) < (j)$ in the lexicographic order. So, $W_{(i)}^{I_{i_0}}$ is a filtration of V_M.

Fix a number M large enough (chosen later). Set $u_M = \dim V_M = \binom{M+n}{n}$. We assume that

$$V_M = W_{(i)_1}^{I_{i_0}} \supset W_{(i)_2}^{I_{i_0}} \supset \cdots \supset W_{(i)_K}^{I_{i_0}},$$

where $W_{(i)_{s+1}}^{I_{i_0}}$ follows $W_{(i)_s}^{I_{i_0}}$ in the ordering and $(i)_K = (\frac{M}{d}, 0, \ldots, 0)$. It is easy to see that K is the number of n-tuples $(i_1, \ldots, i_n)$ with $i_j \geq 0$ and $i_1 + \cdots + i_n \leq \frac{M}{d}$. Then we have

$$K = \binom{\frac{M}{d} + n}{n}.$$

For each $s \in \{1, \ldots, K-1\}$ we set $m_s^{I_{i_0}} = \dim \frac{W_{(i)_s}^{I_{i_0}}}{W_{(i)_{s+1}}^{I_{i_0}}}$, and set $m_K^{I_{i_0}} = 1$. Then by Lemma 4.1, $m_s^{I_{i_0}}$ does not depend on $\{P_{i_0,1}, \ldots, P_{i_0,n}\}$ and s, but on $\sigma((i)_s)$. We also note that

$$m_s^{I_{i_0}} = d^n$$

for all s with $M - d\sigma((i)_s) \geq nd$.

From the above filtration, we may choose a basis $\{\psi_1^{I_{i_0}}, \ldots, \psi_{u_M}^{I_{i_0}}\}$ of V_M such that

$$\{\psi^{I_{i_0}}_{u_M - (m_s^{I_{i_0}} + \cdots + m_K^{I_{i_0}}) + 1}, \ldots, \psi_{u_M}^{I_{i_0}}\}$$

is a basis of $W_{(i)_s}^{I_{i_0}}$. For each $s \in \{1, \ldots, K\}$ and $l \in \{u_M - (m_s^{I_{i_0}} + \cdots + m_K^{I_{i_0}}) + 1, \ldots, u_M - (m_{s+1}^{I_{i_0}} + \cdots + m_K^{I_{i_0}})\}$, we may write

$$\psi_l^{I_{i_0}} = P_{i_0,1}^{i_{1s}} \cdots P_{i_0,n}^{i_{ns}} h_l, \quad \text{where } (i_{1s}, \ldots, i_{ns}) = (i)_s, \quad h_l \in W_{M-d\sigma(i)}^{i_0}.$$

Then we have

$$\begin{aligned}|\psi_l^{I_{i_0}}(\mathbf{f})(z)| &= |P_{i_0,1}(\mathbf{f})(z)|^{i_{1s}}\cdots|P_{i_0,n}(\mathbf{f})(z)|^{i_{ns}}|h_l(\mathbf{f})(z)|\\ &\le c_l|P_{i_0,1}(\mathbf{f})(z)|^{i_{1s}}\cdots|P_{i_0,n}(\mathbf{f})(z)|^{i_{ns}}\|\mathbf{f}(z)\|^{M-d\sigma(i)}\\ &= c_l\left(\frac{|P_{i_0,1}(\mathbf{f})(z)|}{\|\mathbf{f}(z)\|^d}\right)^{i_{1s}}\cdots\left(\frac{|P_{i_0,n}(\mathbf{f})(z)|}{\|\mathbf{f}(z)\|^d}\right)^{i_{ns}}\|\mathbf{f}(z)\|^M,\end{aligned}$$

where $c_l\in\Omega_f$, which is independent of f and z. This implies that

$$\begin{aligned}&\log\prod_{l=1}^{u_M}|\psi_l^{I_{i_0}}(\mathbf{f})(z)|\\ &\le \sum_{s=1}^{K}m_s^{I_{i_0}}\left(i_{1s}\log\frac{|P_{i_0,1}(\mathbf{f})(z)|}{\|\mathbf{f}(z)\|^d}+\cdots+i_{ns}\log\frac{|P_{i_0,n}(\mathbf{f})(z)|}{\|\mathbf{f}(z)\|^d}\right)\\ &\quad+u_MM\log\|\mathbf{f}(z)\|+\log c_{I_{i_0}},\end{aligned}\tag{14}$$

where $c_{I_{i_0}}=\sum_{l=1}^{u_M}c_l\in\Omega_f$, which does not depend on f and z.

We see that

$$\sum_{s=1}^{K}m_s^{I_{i_0}}i_{ks}=\sum_{l=0}^{\frac{M}{d}}\sum_{s|\sigma((i)_s)=l}m(l)i_{ks}=\sum_{l=0}^{\frac{M}{d}}m(l)\sum_{s|\sigma(i_s)=l}i_{ks}.$$

Note that, by the symmetry $(i_1,\dots,i_n)\to(i_{\sigma_1},\dots,i_{\sigma_n})$ with $\sigma\in S(n)$, $\sum_{k|\sigma(i)=l}i_{ks}$ does not depend on k. We set

$$A=:\sum_{s=1}^{K}m_s^{I_{i_0}}i_{ks},\ \text{which is independent of } k \text{ and } I.$$

By (14), we obtain

$$\log\prod_{l=1}^{u_M}|\psi_l^{I_{i_0}}(\mathbf{f})(z)|\le A\left(\log\prod_{j=1}^{n}\frac{|P_{i_0,j}(\mathbf{f})(z)|}{\|\mathbf{f}(z)\|^d}\right)+u_MM\log\|\mathbf{f}(z)\|+\log c_{I_{i_0}},$$

that is

$$A\left(\log\prod_{j=1}^{n}\frac{\|\mathbf{f}(z)\|^d}{|P_{i_0,j}(\mathbf{f})(z)|}\right)\le\log\prod_{l=1}^{u_M}\frac{\|\mathbf{f}(z)\|^M}{|\psi_l^{I_{i_0}}(\mathbf{f})(z)|}+\log c_I.\tag{15}$$

Combining (13) and (15), we have

$$\log\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}\le\frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{A}\log\prod_{l=1}^{u_M}\frac{\|\mathbf{f}(z)\|^M}{|\psi_l^{I_{i_0}}(\mathbf{f})(z)|}+\log c_0,\tag{16}$$

where $c_0=h\prod_{I_{i_0}}(1+c_{I_{i_0}}^{(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})/A})\in\Omega_f$.

We now write

$$\psi_l^{I_{i_0}} = \sum_{J \in \mathcal{T}_M} c_{lJ}^{I_{i_0}} x^J \in V_M, \quad c_{lJ}^{I_{i_0}} \in \mathcal{K}_{\{Q_i\}},$$

where $\mathcal{T}_M$ is the set of all $(n+1)$-tuples $J = (j_0, \ldots, j_n)$ with $\sum_{s=0}^n j_s = M$, $x^J = x_0^{j_0} \cdots x_n^{j_n}$ and $l \in \{1, \ldots, u_M\}$. For each l, we fix an index $J_l^{I_{i_0}} \in J$ such that $c_{lJ_l^{I_{i_0}}}^{I_{i_0}} \not\equiv 0$. Define

$$\mu_{lJ}^{I_{i_0}} = \frac{c_{lJ}^{I_{i_0}}}{c_{lJ_l^{I_{i_0}}}^{I_{i_0}}}, \ J \in \mathcal{T}_M.$$

Set $\Phi = \{\mu_{lJ}^{I_{i_0}} : I_{i_0} \subset \{1, \ldots, q\}, \ \sharp I_{i_0} = n, \ 1 \le l \le M, \ J \in \mathcal{T}_M\}$. Note that $1 \in \Phi$. Let $B = \sharp\Phi$. We see that

$$\begin{aligned} B &\le u_M \binom{q}{n} \left(\binom{M+n}{n} - 1 \right) \\ &= \binom{M+n}{n} \left(\binom{M+n}{n} - 1 \right) \binom{q}{n}. \end{aligned}$$

For each positive integer l, we denote by $\mathcal{L}(\Phi(l))$ the linear span over $\mathbb{C}$ of the set

$$\Phi(l) = \{\gamma_1 \cdots \gamma_l : \gamma_i \in \Phi\}.$$

It is easy to see that

$$\dim \mathcal{L}(\Phi(l)) \le \sharp\Phi(l) \le \binom{B+l-1}{B-1}.$$

We may choose a positive integer p such that

$$p \le p_0 := \left[\frac{B-1}{\log(1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})})} \right]^2,$$

and

$$\frac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} \le 1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})}.$$

Indeed, if $\frac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} > 1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})}$ for all $p \le p_0$, we have

$$\dim \mathcal{L}(\Phi(p_0+1)) \ge (1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})})^{p_0}.$$

Therefore, we have

$$\log(1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})})$$

$$\begin{aligned}
&\le \frac{\log \dim \mathcal{L}(\Phi(p_0+1))}{p_0} \le \frac{\log \begin{pmatrix} B+p_0 \\ B-1 \end{pmatrix}}{p_0} \\
&= \frac{1}{p_0} \log \prod_{i=1}^{B-1} \frac{p_0+i+1}{i} < \frac{(B-1)\log(p_0+2)}{p_0} \\
&\le \frac{B-1}{\sqrt{p_0}} \le \frac{(B-1)\log(1+\frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})}{B-1} \\
&= \log(1+\frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}).
\end{aligned}$$

This is a contradiction.

We fix a positive integer p satisfying the above condition. Put

$$s = \dim \mathcal{L}(\Phi(p)) \text{ and } t = \dim \mathcal{L}(\Phi(p+1)).$$

Let $b_1, \ldots, b_t$ be a $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p+1))$ such that $b_1, \ldots, b_s$ be a $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p))$.

For each $l \in 1, \ldots, u_M$, we set

$$\tilde{\psi}_l^{I_{i_0}} = \sum_{J \in \mathcal{T}_M} \mu_{lJ}^{I_{i_0}} x^J.$$

For each $J \in \mathcal{T}_M$, we consider homogeneous polynomials $\phi_J(x_0, \ldots, x_n) = x^J$. Let F be a meromorphic mapping of Ω into $\mathbb{P}^{tu_M-1}(\mathbb{C})$ with a reduced representation $F = (hb_i\phi_J(\mathbf{f}))_{1\le i\le t, J\in\mathcal{T}_M}$, where h is a nonzero meromorphic function on Ω. We see

$$S_{\alpha\beta,F}(r) = MS_{\alpha\beta,f}(r) + o(S_{\alpha\beta,f}(r))$$

and

$$\| \; C_{\alpha\beta}(r,h) + C_{\alpha\beta}(r,\frac{1}{h}) = o(S_{\alpha\beta,f}(r)).$$

Since f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, F is linearly nondegenerate over $\mathbb{C}$. We see that there exist nonzero functions $c_1, c_2 \in \Omega_f$ such that

$$c_1|h|\|\mathbf{f}\|^M \le \|F\| \le c_2|h|\|\mathbf{f}\|^M.$$

For each $l \in 1, \ldots, u_M, 1 \le i \le s$, we consider the linear form $L_{il}^{i_0}$ in x^J such that

$$hb_i\tilde{\psi}_l^{I_{i_0}}(\mathbf{f}) = L_{il}^{I_{i_0}}(F).$$

Since f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, it is easy to see that

$$\{b_i\tilde{\psi}_l^{I_{i_0}}(\mathbf{f}) : 1 \le i \le s, 1 \le l \le u_M\}$$

is linearly independent over $\mathbb{C}$, and so is $\{L_{il}^{I_{i_0}}(F) : 1 \le i \le s, 1 \le l \le u_M\}$.

For every point z which is not neither zero nor pole of any $hb_i\tilde{\psi}_l^{I_{i_0}}(\mathbf{f})$, we also see that

$$s\log\prod_{l=1}^{u_M}\frac{\|\mathbf{f}(z)\|^M}{|\tilde{\psi}_l^{I_{i_0}}(\mathbf{f})(z)|}\le\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F(z)\|}{|hb_i\tilde{\psi}_l^{I_{i_0}}(\mathbf{f})(z)|}+\log c_3$$
$$\le\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F(z)\|}{|L_{il}^{I_{i_0}}(F)(z)|}+\log c_4,$$

where c_3,c_4 are nonzero functions in Ω_f, which do not depend on f and I_{i_0}, but on $\{Q_i\}_{i=1}^q$.

Combining this inequality and (16), we obtain

$$\log\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_j(\mathbf{f})(z)|}\le\frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sA}\left(\max_{I_{i_0}}\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F(z)\|}{|L_{il}^{I_{i_0}}(F)(z)|}\right)$$
$$+\frac{1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}}}{sA}\log c_4+\log c_0. \tag{17}$$

Since F is linearly nondegenerate over $\mathbb{C}$, by Lemma 2.7, we have

$$\|\ \frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\left(\max_{I_{i_0}}\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F\|}{|L_{il}^{I_{i_0}}(F)|}(te^{i\alpha})\right)\frac{dt}{t}$$
$$+\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\left(\max_{I_{i_0}}\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F\|}{|L_{il}^{I_{i_0}}(F)|}(te^{i\beta})\right)\frac{dt}{t}$$
$$+\frac{2k}{\pi r^k}\int_\alpha^\beta\left(\max_{I_{i_0}}\log\prod_{\substack{1\le i\le s\\1\le l\le u_M}}\frac{\|F\|}{|L_{il}^{I_{i_0}}(F)|}(re^{i\varphi})\right)\cdot\sin(k(\varphi-\alpha))d\varphi$$
$$\le tu_MS_{\alpha\beta,F}(r)-C_{\alpha\beta,W}(r,0)+R_{\alpha,\beta}(r,f), \tag{18}$$

where W is the Wronskian of F.

Integrating both sides of (17) and using (18), we have

$$\|\ qdS_{\alpha\beta,f}(r)-\sum_{j=1}^{q}C_{\alpha\beta,f}(r,Q_j)$$
$$\le\frac{tu_M(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})}{sA}S_{\alpha\beta,F}(r)$$

$$
- \frac{1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}}}{sA} C_{\alpha\beta,W}(r,0) + R_{\alpha,\beta}(r,f). \tag{19}
$$

By using the method of Quang (to see [10]), we have the following inequality

$$
\sum_{j=1}^{q} C_{\alpha\beta,f}(r,Q_j) - \frac{1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}}}{sA} C_{\alpha\beta,W}(r,0) \le \sum_{j=1}^{q} C_{\alpha\beta,f}^{[tu_M-1]}(r,Q_j).
$$

Combining this with (19), we can get

$$
\begin{aligned}
&\| \left(q - \frac{Mtu_M(1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})}{dsA} \right) S_{\alpha\beta,f}(r) \\
&\le \sum_{j=1}^{q} \frac{1}{d} C_{\alpha\beta,f}^{[tu_M-1]}(r,Q_j) + R_{\alpha,\beta}(r,f).
\end{aligned} \tag{20}
$$

Similar to the estimation of Quang and Xie-Cao [10, 18], we have

$$
\frac{Mtu_M}{dsA} \le n+1 + \frac{\epsilon}{1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}}}. \tag{21}
$$

Combining (20) and (21), we obtain that

$$
\begin{aligned}
&\| \left(q - (1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})(n+1) - \epsilon \right) S_{\alpha\beta,f}(r) \\
&\le \sum_{j=1}^{q} \frac{1}{d} C_{\alpha\beta,f}^{[tu_M-1]}(r,Q_j) + R_{\alpha,\beta}(r,f).
\end{aligned}
$$

Here we note that:

$$
M := (n+1)d + 2(1 + \frac{N-n}{\max\{1, \min\{N-n, \kappa\}\}})(n+1)^3 I(\epsilon^{-1})d,
$$

$$
\begin{aligned}
p_0 &:= \left[\frac{B-1}{\log(1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})} \right]^2 \\
&\le \left[\frac{\binom{M+n}{n}\left(\binom{M+n}{n} - 1\right)\binom{q}{n} - 1}{\log(1 + \frac{\epsilon}{3(n+1)(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})})} \right]^2,
\end{aligned}
$$

$$
\begin{aligned}
tu_M - 1 &\le \binom{M+n}{n}\binom{B+p}{B-1} - 1 \le \binom{M+n}{n} p^{B-1} - 1 \\
&\le \binom{M+n}{n} p_0^{\binom{M+n}{n}\left(\binom{M+n}{n}-1\right)\binom{q}{n}-2} - 1 = M_0.
\end{aligned}
$$

Then we have

$$\begin{aligned}
&\| \left(q-(1+\frac{N-n}{\max\{1,\min\{N-n,\kappa\}\}})(n+1)-\epsilon\right) S_{\alpha\beta,f}(r)\\
\leq &\sum_{j=1}^{q}\frac{1}{d}C^{[M_0]}_{\alpha\beta,f}(r,Q_j)+R_{\alpha,\beta}(r,f).
\end{aligned}$$

The theorem is proved.

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Jiali Chen
Department of Mathematics
Renmin University of China
Beijing 100872, P. R. China
Email address: chenjl116@163.com

Qingcai Zhang
Department of Mathematics
Renmin University of China
Beijing 100872, P. R. China
Email address: zhangqcrd@163.com