Bull. Korean Math. Soc. **59** (2022), No. 5, pp. 1237–1246
https://doi.org/10.4134/BKMS.b210694
pISSN: 1015-8634 / eISSN: 2234-3016

N-PURE IDEALS AND MID RINGS

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Abstract. In this paper, we introduce the concept of N-pure ideal as a generalization of pure ideal. Using this concept, a new and interesting type of rings is presented, we call it a mid ring. Also, we provide new characterizations for von Neumann regular and zero-dimensional rings. Moreover, some results about mp-ring are given. Finally, a characterization for mid rings is provided. Then it is shown that the class of mid rings is strictly between the class of reduced mp-rings (p.f. rings) and the class of mp-rings.

1. Introduction

In this paper, all rings are commutative with identity. This paper is devoted to study interesting class of ideals which are called N-pure ideals. This notion is a generalization of pure ideal. The theme that encouraged us to study N-pure ideals is that we can use this concept to characterize some different rings. Moreover, this tool lets us introduce a new type of rings which is completely different with mp-rings and reduced mp-rings. An ideal I of a ring R is said to be a pure ideal if the ring map $R \to R/I$ is flat, or equivalently for each $a \in I$ there exists $b \in I$ such that $a(1-b) = 0$. Pure ideal is an important tool in the study of some areas of ring theory. Pure notion was studied in some works, e.g. [1], [2], [3], [4] and [6].

In Section 2 we study the notion of N-pure ideal. At first, some basic properties of N-pure ideals are provided. Also, it is shown that the class of N-pure ideals is strictly greater than the class of pure ideals. These two class are the same if and only if the ring is reduced, see Proposition 2.3. In the following, we provide a characterization for N-pure ideals, see Theorem 2.6.

In Section 3 using the concepts pure and N-pure, we provide characterizations for some well known rings. Recall that a ring R is said to be an mp-ring if each prime ideal of R contains a unique minimal prime ideal of R. These rings are studied in [1] extensively. We describe such ring in terms of N-pure ideals, see Theorem 3.3. Also, a new characterization is given for zero-dimensional

Received September 19, 2021; Accepted December 6, 2021.
2020 *Mathematics Subject Classification.* 13A15, 13C05, 13C10, 13C15, 16E50.
Key words and phrases. Mid ring, mp-ring, p.f. ring, von Neumann regular ring, zero-dimensional ring, N-pure ideal, pure ideal.

rings, see Theorem 3.4. Then, von Neumann regular rings are characterized in terms of pure ideals, see Corollary 3.5.

Finally, in Section 4 a new ring is introduced which it is called a mid ring. We obtain some results about this ring. As an important result of this section, we provide a characterization for mid rings, see Theorem 4.7. Especially, we prove that a ring R is a mid ring if and only if $R_{\mathfrak{p}}$ is a primary ring for every prime ideal $\mathfrak{p}$ of R. Also, a new characterizations for p.p. rings is given, see Theorem 4.11 and Corollary 4.12. In the following, the N-pure prime ideals of mid rings are identified, see Theorem 4.14. Finally, we prove that the class of mid rings is strictly between the class of reduced mp-rings and the class of mp-rings, see Remark 4.13, Theorem 4.15 and Example 4.16.

Now we recall some notions which are used in this paper. By p.p. ring we mean a ring in which each principal ideal is projective. A ring R is said to be a p.f. ring if for each $a \in R$, $\mathrm{Ann}(a)$ is a pure ideal. Clearly every p.p. ring is a p.f. ring. A ring R is a Gpf-ring if for every $a \in R$ there exists $n \geq 1$ such that $\mathrm{Ann}(a^n)$ is a pure ideal. We denote nil-radical (Jacobson radical) of a ring R by $\mathfrak{N}$ (resp. $\mathfrak{J}$). Also, nil-radical (Jacobson radical) of ring $R_{\mathfrak{p}}$ is denoted by $\mathfrak{N}(R_{\mathfrak{p}})$ (resp. $\mathfrak{J}(R_{\mathfrak{p}})$) for a prime ideal $\mathfrak{p}$ of R.

2. N-pure ideals

We begin this section with the following definition.

Definition 2.1. An ideal I of a ring R is called *N-pure* if for every $a \in I$ there exists $b \in I$ such that $a(1-b) \in \mathfrak{N}$.

Remark 2.2. It is clear that every pure ideal is N-pure. Moreover, if I is a pure ideal of ring R, then its radical is an N-pure ideal. In particular, the nil-radical of R is an N-pure ideal. Therefore, if R is a non-reduced ring, it is straightforward to check that $\mathfrak{N}$ is an N-pure ideal which is not pure. The following result describes reduced rings in terms of pure and N-pure ideals.

Proposition 2.3. *Let R be a ring. Then R is a reduced ring if and only if every N-pure ideal is a pure ideal.*

Proof. Let I be an N-pure ideal and $a \in I$. Then there exists $b \in I$ and such that $a(1-b) \in \mathfrak{N}$. Thus $a(1-b) = 0$. Therefore, I is a pure ideal. The converse follows from Remark 2.2. $\square$

Proposition 2.4. *Let I be an ideal of a ring R. Then I is an N-pure ideal if and only if for each $a \in I$ there exist $n \geq 1$ and $b \in I$ such that $a^n(1-b) = 0$.*

Proof. Let I be an N-pure ideal and $a \in I$. Then there exists $c \in I$ such that $a(1-c) \in \mathfrak{N}$. Thus there exists $n \geq 1$ such that $(a(1-c))^n = 0$ and so $a^n(1-b) = 0$ for some $b \in I$. Conversely, if $a^n(1-b) = 0$, then we have $a(1-b) \in \mathfrak{N}$ and so I is N-pure. $\square$

Lemma 2.5. *Let I be an ideal of a ring R. Then I is an N-pure ideal if and only if $(I+\mathfrak{N})/\mathfrak{N}$ is a pure ideal.*

Proof. Assume that I is an N-pure ideal. If $a \in I$, then there exists $b \in I$ such that $a(1-b) \in \mathfrak{N}$. Thus $(I+\mathfrak{N})/\mathfrak{N}$ is a pure ideal. The converse case is straightforward. □

Recall that a subset E of $\mathrm{Spec}(R)$ is said to be stable under the generalization if for any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R with $\mathfrak{p} \subset \mathfrak{q}$, if $\mathfrak{q} \in E$, then $\mathfrak{p} \in E$. The following result provides a characterization for N-pure ideals.

Theorem 2.6. *Let I be an ideal of a ring R. Then the following conditions are equivalent.*

(i) *I is an N-pure ideal.*
(ii) *For every $a_1, \dots, a_n \in I$ there exist $b \in I$ and $t \geq 1$ such that $a_k^t = a_k^t b$ for all $k = 1, \dots, n$.*
(iii) *For every $a \in I$ there exists $t \geq 1$ such that* $\mathrm{Ann}(a^t) + I = R$.
(iv) $\sqrt{I} = \{a \in R \mid \exists n \geq 1, \mathrm{Ann}(a^n) + I = R\}$.
(v) *$\sqrt{I}$ is an N-pure ideal.*
(vi) *There exists a unique pure ideal J such that $\sqrt{I} = \sqrt{J}$.*

Proof. (i)⇒(ii): Since I is N-pure, then there exist $b_k \in I$ and $t_k \geq 1$ such that $a_k^{t_k} = a_k^{t_k} b_k$. Setting $t := \max\{t_k \mid 1 \leq k \leq n\}$ and let $b \in I$ where $1 - b = \prod_{k=1}^{n}(1-b_k)$. Clearly, we have $b \in I$ and $a_k^t = a_k^t b$ and so the assertion is proved.

(ii)⇒(iii): Let $a \in I$. Then by assumption there exist $b \in I$ and $t \geq 1$ such that $a^t = a^t b$. Thus we have $1 - b \in \mathrm{Ann}(a^t)$ and so $\mathrm{Ann}(a^t) + I = R$.

(iii)⇒(iv): Let $a \in \sqrt{I}$. Then there exists $m \geq 1$ such that $a^m \in I$ and so by hypothesis, we have $\mathrm{Ann}(a^t) + I = R$ for some $t \geq 1$. Conversely, if $\mathrm{Ann}(a^n) + I = R$ for some $n \geq 1$, then we have $a^n \in I$ and so $a \in \sqrt{I}$.

(iv)⇒(v): Let $a \in \sqrt{I}$. Then there exists $n \geq 1$ such that $\mathrm{Ann}(a^n) + I = R$. Thus we have $c + d = 1$ for some $c \in \mathrm{Ann}(a^n)$ and $d \in I$. Hence $a^n(1-d) = 0$ and so $\sqrt{I}$ is an N-pure ideal.

(v)⇒(vi): Let $\mathfrak{p} \in V(I)$ and $\mathfrak{q} \subset \mathfrak{p}$. If $\mathfrak{q} \notin V(I)$, then there exists $a \in I \setminus \mathfrak{q}$. Since $\sqrt{I}$ is N-pure, then there are $b \in \sqrt{I}$ and $n \geq 1$ such that $a^n(1-b) = 0$. Hence, $1 - b \in \mathfrak{p}$ and so $1 \in \mathfrak{p}$ which is a contradiction. Therefore, $V(I)$ is stable under the generalization. Now by [5, Theorem 3.2], there exists a unique pure ideal J such that $\sqrt{I} = \sqrt{J}$.

(vi)⇒(i): Let $a \in I$. Then there exists $t \geq 1$ such that $a^t \in J$. Thus there exists $b \in J$ such that $a^t(1-b) = 0$, since J is a pure ideal. On the other hand, we have $b^s \in I$ for some $s \geq 1$. Therefore, $a^t(1-b^s) = 0$ and so I is an N-pure ideal. □

Corollary 2.7. *Let I be an ideal of a ring R. Then I is an N-pure ideal if and only if I^n is an N-pure ideal for all $n \geq 1$.*

Theorem 2.8. *Let (I_k) be a family of N-pure ideals of a ring R. Then the following statements hold.*

(i) $\sum_k I_k$ *is an N-pure ideal.*
(ii) *If* I *and* J *are N-pure ideals of* R*, then* IJ *and* $I \cap J$ *are N-pure ideals.*

Proof. (i) Let $a = \sum_{j=1}^{t} a_j \in \sum_k I_k$, where $a_j \in I_j = I_{k_j}$. Then there exist $b_j \in I_j$ and $n_j \geq 1$ such that $a_j^{n_j}(1 - b_j) = 0$. Setting $1 - b = \prod_{j=1}^{t}(1 - b_j)$ and $n = \sum_{j=1}^{t} n_j$. It is easy to see that $b \in \sum_k I_k$. But we have $a^n(1 - b) = \big(\sum_{j=1}^{t} a_j\big)^n(1 - b) = 0$. Thus $\sum_k I_k$ is an N-pure ideal.

(ii) Since the product of two pure ideals is a pure ideal, then the assertions follow from Theorem 2.6(vi). □

Lemma 2.9. *Let* R *be a ring and* I *be an ideal of* R*. If for every* $x \in R$ *there exists* $n \geq 1$ *such that* $Rx^n \cap I = x^n I$*, then* I *is an N-pure ideal.*

Proof. Let $a \in I$. Then there exists $n \geq 1$ such that $Ra^n \bigcap I = a^n I$. So there exists $b \in I$ such that $a^n = a^n b$. Therefore, I is an N-pure ideal. □

3. Some characterizations of rings

We begin this section with the following remark.

Remark 3.1. Let R be a ring. If $\mathfrak{p}$ is a minimal prime ideal of R, then $\sqrt{\operatorname{Ker}\pi_{\mathfrak{p}}} = \mathfrak{p}$, because if $\operatorname{Ker}\pi_{\mathfrak{p}} \subseteq \mathfrak{q}$ and $\mathfrak{p} \not\subseteq \mathfrak{q}$, then there exists $a \in \mathfrak{p} \setminus \mathfrak{q}$. So there exists $s \in R \setminus \mathfrak{p}$ such that $sa \in N(R)$. Hence $a^n \in \operatorname{Ker}\pi_{\mathfrak{p}}$ for some $n \geq 1$ and so $a^n \in \mathfrak{q}$ which is a contradiction. Therefore we have $V(\operatorname{Ker}\pi_{\mathfrak{p}}) = V(\mathfrak{p})$.

Recall that a ring R is said to be a *NJ-ring* if its nil-radical and jacobson radical are the same ($\mathfrak{N} = \mathfrak{J}$). Also a ring R is called a *semiprimitive* ring if $\mathfrak{J} = 0$. The following result provides characterizations for such rings.

Proposition 3.2. *The following statements hold.*
(i) R *is a* NJ*-ring if and only if* $\mathfrak{J}$ *is an N-pure ideal.*
(ii) R *is a semiprimitive ring if and only if* $\mathfrak{J}$ *is a pure ideal.*

Proof. (i) By Remark 2.2, $\mathfrak{J}$ is an N-pure ideal of R. Conversely, if $a \in \mathfrak{J}$, then there exist $n \geq 1$ and $b \in \mathfrak{J}$ such that $a^n = a^n b$. Hence we have $a^n(1 - b) = 0$. Thus $a^n = 0$ and so $a \in \mathfrak{N}$. Then R is a NJ-ring.

(ii) It is clear. Let $a \in \mathfrak{J}$. Then there exists $b \in \mathfrak{J}$ such that $a(1 - b) = 0$. Hence $a = 0$ and so $\mathfrak{J} = 0$. Therefore R is a semiprimitive ring. □

In the following result, we provide a characterization for mp-rings.

Theorem 3.3. *Let* R *be a ring. Then the following statements are equivalent.*
(i) R *is an mp-ring.*
(ii) *If* $ab = 0$*, then there exists* $n \geq 1$ *such that* $\operatorname{Ann}(a^n) + \operatorname{Ann}(b^n) = R$.
(iii) *Every minimal prime ideal of* R *is an N-pure ideal.*
(iv) *For every minimal prime ideal* $\mathfrak{p}$ *of* R, $\operatorname{Ker}\pi_{\mathfrak{p}}$ *is an N-pure ideal.*
(v) *For every prime ideal* $\mathfrak{p}$ *of* R, $\operatorname{Ker}\pi_{\mathfrak{p}}$ *is an N-pure ideal.*

Proof. For (i)⇔(ii), see [1, Theorem 6.2].

(ii)⇒(iii): Let $\mathfrak{p} \in \mathrm{Min}(R)$. If $a \in \mathfrak{p}$, then there exists $b \in R \setminus \mathfrak{p}$ such that $ab \in N(R)$ and so by hypothesis, there exists $n \geq 1$ such that $\mathrm{Ann}(a^n) + \mathrm{Ann}(b^n) = R$. Then there exist $c \in \mathrm{Ann}(a^n)$ and $d \in \mathrm{Ann}(b^n)$ such that $c + d = 1$. Thus we have $a^n(1 - d) = 0$ and so $\mathfrak{p}$ is an N-pure ideal, since $d \in \mathfrak{p}$.

(iii)⇔(iv): It follows from Theorem 2.6 and Remark 3.1.

(iv)⇒(i): Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct minimal prime ideals of R. Then by Lemma [1, Lemma 3.2], there exist $a \in R \setminus \mathfrak{p}$ and $b \in R \setminus \mathfrak{q}$ such that $ab = 0$. Hence $b \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$. Thus there exist $n \geq 1$ and $c \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$ such that $b^n(1 - c) = 0$, since $\mathrm{Ker}\,\pi_{\mathfrak{p}}$ is an N-pure ideal. So $1 - c \in \mathfrak{q}$. Therefore, $\mathfrak{p} + \mathfrak{q} = R$ and so R is an mp-ring by [1, Theorem 6.2(ii)].

(ii)⇒(v): Let $ab \in \sqrt{\mathrm{Ker}\,\pi_{\mathfrak{p}}}$ and $a \notin \sqrt{\mathrm{Ker}\,\pi_{\mathfrak{p}}}$. Then there exists $n \geq 1$ such that $a^n b^n \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$. Hence there exists $s \in R \setminus \mathfrak{p}$ such that $sa^n b^n = 0$. Therefore by hypothesis, there exists $m \geq 1$ such that $\mathrm{Ann}(a^{mn}) + \mathrm{Ann}(s^m b^{mn}) = R$. Thus $\mathrm{Ann}(s^m b^{mn}) \nsubseteq \mathfrak{p}$ and so there exists $t \in R \setminus \mathfrak{p}$ such that $ts^m b^{mn} = 0$. Then $b^{mn} \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$ and so $b \in \sqrt{\mathrm{Ker}\,\pi_{\mathfrak{p}}}$. Therefore $\sqrt{\mathrm{Ker}\,\pi_{\mathfrak{p}}}$ is a minimal prime ideal of R. Now, the assertion follows from Theorem 2.6 and (ii)⇒(iii).

(v)⇒(i): Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct minimal prime ideals of R. Then there exist $a \in R \setminus \mathfrak{p}$ and $b \in R \setminus \mathfrak{q}$ such that $ab = 0$. Thus $b \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$. So by assumption, there exist $n \geq 1$ and $c \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$ such that $b^n(1 - c) = 0$. Hence $1 - c \in \mathfrak{q}$ and so $\mathfrak{p} + \mathfrak{q} = R$. This means that R is an mp-ring. □

It is well known that a ring R is von Neumann regular if and only if it is a reduced zero-dimensional ring. Clearly, a ring R is a semiprimitive local ring if and only if it is a field. In the following result using N-pure concept, a new characterization for zero-dimensional rings is given.

Theorem 3.4. *Let R be a ring. Then the following statements are equivalent.*

(i) *R is a zero-dimensional ring.*

(ii) *Every ideal of R is an N-pure ideal.*

(iii) *Every principal ideal of R is an N-pure ideal.*

(iv) *Every maximal ideal of R is an N-pure ideal.*

(v) $\sqrt{\mathrm{Ker}\,\pi_{\mathfrak{m}}} = \mathfrak{m}$ *for all* $\mathfrak{m} \in \mathrm{Max}(R)$.

(vi) $R_{\mathfrak{p}}$ *is a NJ-ring for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

(vii) $R_{\mathfrak{m}}$ *is a NJ-ring for all* $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. (i)⇒(ii): We know that $R/\mathfrak{N}$ is a von Neumann regular ring. Then if I is an ideal of R and $a \in I$, then there exists $b \in R$ such that $(a - a^2 b) \in \mathfrak{N}$. Thus I is an N-pure ideal.

(ii)⇒(iii): There is nothing to prove.

(iii)⇒(iv): It is obvious.

(iv)⇒(v): Let $a \in \mathfrak{m}$. Then there exist $n \geq 1$ and $b \in \mathfrak{m}$ such that $a^n(1-b) = 0$. Thus $a^n \in \mathrm{Ker}\,\pi_{\mathfrak{m}}$ and so $\sqrt{\mathrm{Ker}\,\pi_{\mathfrak{m}}} = \mathfrak{m}$.

(v)⇒(vi): Let $a/s \in \mathfrak{J}(R_\mathfrak{p})$. Assume that $\mathfrak{m}$ is a maximal ideal which $\mathfrak{p} \subseteq \mathfrak{m}$. Thus $a \in \mathfrak{m}$ and so there exist a natural number $n \geq 1$ and $t \in R \setminus \mathfrak{m}$ such that $ta^n = 0$. Hence $a^n/s^n = 0$ in $R_\mathfrak{p}$. Therefore we have $\mathfrak{J}(R_\mathfrak{p}) = \mathfrak{N}(R_\mathfrak{p})$.

(vi)⇒(vii): It is clear.

(vii)⇒(i): Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R which $\mathfrak{p} \subseteq \mathfrak{q}$. If $\mathfrak{m}$ is a maximal ideal of R containing $\mathfrak{q}$, then we have $\mathfrak{N}(R_\mathfrak{m}) \subseteq \mathfrak{p}R_\mathfrak{m} \subseteq \mathfrak{q}R_\mathfrak{m} \subseteq \mathfrak{m}R_\mathfrak{m} = \mathfrak{J}(R_\mathfrak{m})$. Hence $\mathfrak{p}R_\mathfrak{m} = \mathfrak{q}R_\mathfrak{m}$ and so R is a zero-dimensional ring. □

In the following result, using the previous theorem and pure notion, a characterization for von Neumann regular rings is given which some of its conditions are well known.

Corollary 3.5. *Let R be a ring. Then the following statements are equivalent.*

(i) *R is a von Neumann regular ring.*

(ii) *Every ideal of R is a pure ideal.*

(iii) *Every principal ideal of R is a pure ideal.*

(iv) *Every maximal ideal of R is a pure ideal.*

(v) $\operatorname{Ker} \pi_\mathfrak{m} = \mathfrak{m}$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

(vi) *$R_\mathfrak{p}$ is a semiprimitive ring for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

(vii) *$R_\mathfrak{m}$ is a semiprimitive ring for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. (i)⇒(ii): This follows from Proposition 2.3 and Theorem 3.4.

(ii)⇒(iii): There is nothing to prove.

(iii)⇒(iv): It is obvious.

(iv)⇒(v): Suppose $\mathfrak{m} \in \operatorname{Max}(R)$ and $a \in \mathfrak{m}$. Then by the hypothesis, there exists $b \in \mathfrak{m}$ such that $a(1-b) = 0$. Thus $a \in \operatorname{Ker} \pi_\mathfrak{m}$ and so $\operatorname{Ker} \pi_\mathfrak{m} = \mathfrak{m}$.

(v)⇒(vi): It is easy to see that R is reduced. Then by Theorem 3.4, $R_\mathfrak{p}$ is a semiprimitive ring.

(vi)⇒(vii): There is nothing to prove.

(vii)⇒(i): By Theorem 3.4, it suffices to show that R is reduced. Let $a \in \mathfrak{N}$. Then $a \in \mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. Hence by the hypothesis, there exists $b_\mathfrak{m} \in R \setminus \mathfrak{m}$ such that $b_\mathfrak{m} a = 0$. Thus $I = \big(b_\mathfrak{m} : \mathfrak{m} \in \operatorname{Max}(R)\big)$ is equal to R. Therefore, we have $1 = \sum_{i=1}^n r_i b_i$, where $b_i = b_{\mathfrak{m}_i}$ and $r_i \in R$. Hence $a = \sum_{i=1}^n r_i b_i a = 0$ and so R is a reduced ring. □

Recall that a proper ideal P of a ring R is said to be purely-prime if for pure ideals I and J of R with $IJ \subseteq P$ we have either $I \subseteq P$ or $J \subseteq P$ [6]. The set of all purely-prime ideals of a ring R is called the pure spectrum of R and is denoted by $\operatorname{Spp}(R)$.

Corollary 3.6. *Let R be a ring. Then R is a von Neumann regular ring if and only if the prime spectrum of R and the pure spectrum of it are the same.*

Proof. This is an immediate consequence of Corollary 3.5. □

4. Mid rings

In this section, we introduce a new class of rings and study some basic properties of it.

Definition 4.1. A ring R is called a *mid ring* if for every $a \in R$, $\mathrm{Ann}(a)$ is an N-pure ideal.

Proposition 4.2. *Let R be a mid ring. Then $R_{\mathfrak{p}}$ is a mid ring for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proof. Let $a/s \in R_{\mathfrak{p}}$ and $b/t \in \mathrm{Ann}(a/s)$. Then there exists $u \in R \backslash \mathfrak{p}$ such that $uba = 0$. Thus there exist $c \in \mathrm{Ann}(a)$ and $m \geq 1$ such that $(ub)^m(1-c) = 0$ and so we have $(ub)^m/(ut)^m(1-c) = 0$. Therefore, $\mathrm{Ann}(a/s)$ is N-pure and the claim is proved. □

Lemma 4.3. *Let R be a mid ring. If I is a pure ideal, then R/I is a mid ring.*

Proof. Let $a \in R \backslash I$. Then $\mathrm{Ann}(a)$ is an N-pure ideal. If $b + I \in \mathrm{Ann}(a+I)$, then we have $ab \in I$ and so there exists $c \in I$ such that $ab(1-c) = 0$. Thus there exist $d \in \mathrm{Ann}(a)$ and $m \geq 1$ such that $b^m(1-c)^m(1-d) = 0$. Therefore $b^m(1-d) \in I$ and so R/I is a mid ring. □

Remark 4.4. It is straightforward to see that if I_k is an ideal of ring R_k, $R = \prod_{k=1}^{n} R_k$ and $I = \prod_{k=1}^{n} I_k$, then I is an N-pure ideal if and only if every I_k is an N-pure ideal. On the other hand, for $a = (a_k) \in R$ we have $\mathrm{Ann}(a) = \prod_{k=1}^{n} \mathrm{Ann}(a_k)$. Then we can obtain the following result.

Proposition 4.5. *Let $R = \prod_k R_k$. If R is a mid ring, then every R_k is a mid ring. If index set is finite, then the converse holds.*

Proof. Let $a_{k_0} \in R_{k_0}$. Setting $a = (a_k)$, where $a_k = a_{k_0}$ if $k = k_0$ and $a_k = 0$ if $k \neq k_0$. Then there exist $b = (b_k) \in \mathrm{Ann}(a)$ and $n \geq 1$ such that $a^n(1-b) = 0$. Thus $a_{k_0}^n(1 - b_{k_0}) = 0$, where $b_{k_0} \in \mathrm{Ann}(a_{k_0})$. Then $\mathrm{Ann}(a_{k_0})$ is an N-pure ideal of R_{k_0} and so R_{k_0} is a mid ring. The last assertion follows easily from Remark 4.4. □

Lemma 4.6. *Let R be a ring. Then $\mathrm{Ker}\,\pi_{\mathfrak{p}}$ is a primary ideal for each $\mathfrak{p} \in \mathrm{Min}(R)$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of R. If $ab \in \mathrm{Ker}\,\pi_{\mathfrak{p}}$ and $a \notin \mathrm{Ker}\,\pi_{\mathfrak{p}}$, then there exists $c \in R \setminus \mathfrak{p}$ such that $abc = 0$. Since $\mathrm{Ann}(a) \subseteq \mathfrak{p}$, then $bc \in \mathfrak{p}$ and so $b \in \mathfrak{p}$. Thus $\mathrm{Ker}\,\pi_{\mathfrak{p}}$ is a primary ideal by Remark 3.1. □

Recall that a ring is a primary ring if its zero ideal is a primary ideal. The following result provides a characterization for mid rings.

Theorem 4.7. *Let R be a ring. Then the following are equivalent.*

(i) *R is a mid ring.*

(ii) *If $ab = 0$, then there exists $n \geq 1$ such that $\mathrm{Ann}(a) + \mathrm{Ann}(b^n) = R$.*

(iii) $R_{\mathfrak{p}}$ *is a primary ring for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.
(iv) $R_{\mathfrak{m}}$ *is a primary ring for all* $\mathfrak{m} \in \operatorname{Max}(R)$.
(v) $\operatorname{Ker} \pi_{\mathfrak{p}}$ *is a pure ideal for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.
(vi) $\operatorname{Ker} \pi_{\mathfrak{p}}$ *is a pure ideal for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(vii) $\operatorname{Ker} \pi_{\mathfrak{p}} = \operatorname{Ker} \pi_{\mathfrak{q}}$ *for prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *with* $\mathfrak{p} \subseteq \mathfrak{q}$.
(viii) $\operatorname{Ker} \pi_{\mathfrak{p}}$ *is a primary ideal for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.
(ix) $\operatorname{Ker} \pi_{\mathfrak{m}}$ *is a primary ideal for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. (i)$\Rightarrow$(ii): Let $ab = 0$. Then by hypothesis, $\operatorname{Ann}(a)$ is an N-pure ideal and so there are $n \geq 1$ and $c \in \operatorname{Ann}(a)$ such that $b(1-c) \in \mathfrak{N}$. Thus we have $\operatorname{Ann}(a) + \operatorname{Ann}(b^n) = R$.

(ii)$\Rightarrow$(iii): Let $(a/s)(b/s') = 0$ and $a/s \neq 0$ in $R_{\mathfrak{p}}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tab = 0$. Thus there exists $n \geq 1$ such that $\operatorname{Ann}(a) + \operatorname{Ann}((tb)^n) = R$. Therefore $\operatorname{Ann}((tb)^n) \not\subseteq \mathfrak{p}$ and so b/s' is a nilpotent in $R_{\mathfrak{p}}$.

(iii)$\Rightarrow$(iv): There is nothing to prove.

(iv)$\Rightarrow$(ii): Let $ab = 0$ for some $a, b \in R$. Suppose that $\operatorname{Ann}(a) + \operatorname{Ann}(b^n) \neq R$ for all $n \geq 1$. Setting $I := \sum_{n \geq 1} \big(\operatorname{Ann}(a) + \operatorname{Ann}(b^n)\big)$, then there exists a maximal ideal $\mathfrak{m}$ such that $I \subseteq \mathfrak{m}$. Hence we have $a/1 \neq 0$ and $b/1 \not\subseteq \mathfrak{N}(R_{\mathfrak{m}})$ which is a contradiction.

(ii)$\Rightarrow$(i): Let $a \in R$ and $b \in \operatorname{Ann}(a)$. Then there exists $n \geq 1$ such that $\operatorname{Ann}(a) + \operatorname{Ann}(b^n) = R$. Hence there exists $c \in \operatorname{Ann}(a)$ such that $b(1-c) \in \mathfrak{N}$ and so R is a mid ring.

(ii)$\Rightarrow$(v): Let $\mathfrak{p}$ be a prime ideal of R and $a \in \operatorname{Ker} \pi_{\mathfrak{p}}$. Then there exists $b \in R \setminus \mathfrak{p}$ such that $ab = 0$. Thus by hypothesis there is $n \geq 1$ such that $\operatorname{Ann}(a) + \operatorname{Ann}(b^n) = R$. Therefore we have $c + d = 1$ for some $c \in \operatorname{Ann}(a)$ and $d \in \operatorname{Ann}(b^n)$. Hence $d \in \operatorname{Ker} \pi_{\mathfrak{p}}$ and $a(1-d) = 0$. Then $\operatorname{Ker} \pi_{\mathfrak{p}}$ is a pure ideal of R.

(v)$\Rightarrow$(vi): There is nothing to prove.

(vi)$\Rightarrow$(vii): Let $\mathfrak{m}$ be a maximal ideal of R containing minimal prime ideal $\mathfrak{p}$. Then we have $\operatorname{Ker} \pi_{\mathfrak{m}} \subseteq \operatorname{Ker} \pi_{\mathfrak{p}}$. Let $a \in \operatorname{Ker} \pi_{\mathfrak{p}}$. Since $\operatorname{Ker} \pi_{\mathfrak{p}}$ is a pure ideal, there exists $b \in \operatorname{Ker} \pi_{\mathfrak{p}}$ such that $a(1-b) = 0$. Thus $1 - b \notin \mathfrak{m}$ and so $a \in \operatorname{Ker} \pi_{\mathfrak{m}}$. This yields that $\operatorname{Ker} \pi_{\mathfrak{m}} = \operatorname{Ker} \pi_{\mathfrak{p}}$.

(vii)$\Rightarrow$(viii): It follows from Lemma 4.6.

(viii)$\Rightarrow$(ix): It is obvious.

(ix)$\Rightarrow$(ii): Assume that $ab = 0$. We claim that $\operatorname{Ann}(a) + \operatorname{Ann}(b^m) = R$ for some $m \geq 1$. Otherwise, there exists $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\sum_{n \geq 1} \big(\operatorname{Ann}(a) + \operatorname{Ann}(b^n)\big) \subseteq \mathfrak{m}$. Thus $a \notin \operatorname{Ker} \pi_{\mathfrak{m}}$ and $b \notin \sqrt{\operatorname{Ker} \pi_{\mathfrak{m}}}$ which is a contradiction. $\square$

Remark 4.8. It is easy to see that if R is a primary ring, then $R_{\mathfrak{p}}$ is a primary ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Then by Theorem 4.7, every primary ring is a mid ring.

Now let R be a *Gpf*-ring. Thus for each $a \in R$ there exists $n \geq 1$, whenever $a^n b = 0$, then $\operatorname{Ann}(a^n) + \operatorname{Ann}(b) = R$. Because if $a \in R$, then there exists $n \geq 1$ such that $\operatorname{Ann}(a^n)$ is a pure ideal. Thus easily we have $\operatorname{Ann}(a^n) + \operatorname{Ann}(b) = R$ for each $b \in \operatorname{Ann}(a^n)$.

Now we can obtain the next result.

Lemma 4.9. *If R is a Gpf-ring, then $R_{\mathfrak{p}}$ is a primary ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. Assume $(a/s)(b/t) = 0$ and $a/s \neq 0$ in $R_{\mathfrak{p}}$. Then there exists $u \in R \setminus \mathfrak{p}$ such that $uab = 0$. But $\operatorname{Ann}(u^n b^n)$ is a pure ideal for some $n \geq 1$. Then by Remark 4.8, $\operatorname{Ann}(a)+\operatorname{Ann}(u^n b^n) = R$. Thus $\operatorname{Ann}(u^n b^n) \not\subseteq \mathfrak{p}$, since $\operatorname{Ann}(a) \subseteq \mathfrak{p}$. Therefore $b^n/t^n = 0$ and so $R_{\mathfrak{p}}$ is a primary ring. □

Proposition 4.10. *Every Gpf-ring is a mid ring.*

Proof. It follows from Theorem 4.7 and Lemma 4.9. □

By $Q(R)$ we mean the totally ring fractions of R.

Theorem 4.11. *Let R be a ring. Then the following are equivalent.*
(i) *R is a p.p. ring.*
(ii) *R is a mid ring and $Q(R)$ is a von Neumann regular ring.*

Proof. (i)⇒(ii): Let $a \in R$. Then $\operatorname{Ann}(a)$ is projective and so there exists idempotent $e \in R$ such that $\operatorname{Ann}(a) = Re$. Now, $a + e$ is a non-zero divisor. Then $Q(R)$ is a von Neumann regular ring.

(ii)⇒(i): Let $a \in R$. Then $\operatorname{Ann}(a)$ is an N-pure ideal. Since $Q(R)$ is regular, then there exists a non-zero divisor $s \in R$ such that $sa = a^2$. Thus there exist $n \geq 1$ and $e \in \operatorname{Ann}(a)$ such that $(s - a)^n(1 - e) = 0$. Hence, we have $s^n(1 - e) = ac$ for some $c \in R$. Then e is an idempotent. But we have $\operatorname{Ann}(a) = Re$ and so R is a p.p. ring. □

Corollary 4.12. *Let R be a ring. Then the following are equivalent.*
(i) *R is a p.p. ring.*
(ii) *R is a Gpf-ring and $Q(R)$ is a von Neumann regular ring.*

Proof. It follows from Proposition 4.10 and Theorem 4.11. □

Remark 4.13. Clearly every zero-dimensional ring is a Gpp-ring. Let R be a zero-dimensional ring. Then for each $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n(1 - a^n b) = 0$. Then $\operatorname{Ann}(a^n) = R(1 - a^n b)$, where $1 - a^n b$ is an idempotent element of R. Thus $\mathbb{Z}/n\mathbb{Z}$ is a Gpp-ring and also is a mid ring for each positive integer n. Obviously, every p.f. ring is a mid ring. But the converse is not necessarily true. If $n = p_1^{k_1} \cdots p_t^{k_t}$, where some $k_j \geq 1$, then $\mathbb{Z}/n\mathbb{Z}$ is a mid ring by Proposition 4.10 which is not a p.f. ring.

Theorem 4.14. *Let R be a mid ring and $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p}$ is N-pure if and only if $\mathfrak{p} \in \operatorname{Min}(R)$.*

Proof. Let $\mathfrak{p}$ be an N-pure ideal of R. If $\mathfrak{q}$ is a prime ideal of R such that strictly contained in $\mathfrak{p}$, then there exists $a \in \mathfrak{p} \setminus \mathfrak{q}$. Hence there are $b \in \mathfrak{p}$ and $n \geq 1$ such that $a^n(1 - b) = 0$. Thus $1 \in \mathfrak{p}$ which is a contradiction. Therefore, $\mathfrak{p}$ is a minimal prime ideal of R. Conversely, let $\mathfrak{p}$ be a minimal prime ideal of R and $a \in \mathfrak{p}$. Using Theorem 2.6, it suffices to show that $\operatorname{Ann}(a^m) + \mathfrak{p} = R$

for some $m \geq 1$. Assume that $\mathrm{Ann}(a^n) + \mathfrak{p} \neq R$ for all $n \geq 1$. Setting $I := \sum_{n\geq 1} \big(\mathrm{Ann}(a^n) + \mathfrak{p}\big)$. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $I \subseteq \mathfrak{m}$. So $\mathrm{Ann}(a^n) \subseteq \mathfrak{m}$ for all $n \geq 1$. Hence $a/1 \notin \mathfrak{N}(R_\mathfrak{m}) = \mathfrak{p}R_\mathfrak{p}$ and so $a \notin \mathfrak{p}$ which is a contradiction. □

Theorem 4.15. *Every mid ring is an mp-ring.*

Proof. It follows from Theorems 3.3 and 4.14. □

Example 4.16. The converse of the above theorem is not necessarily true. As a specific example, let R be the polynomial ring $k[x, y, z]$ modulo $I = (x^3 - yz)$ where k is a field. If $\mathfrak{p} = (x, z)$, then we have $I \subseteq \mathfrak{p}$. But $\mathfrak{p}/I$ is a prime ideal of R, since $R/(\mathfrak{p}/I) \simeq k[y]$. Now we consider the ring $R_\mathfrak{m}$ where $\mathfrak{m} = (\overline{x}, \overline{y}, \overline{z})$. Then $\mathfrak{q} = (\overline{x}/1, \overline{z}^2/1)$ is a non-primary ideal of $R_\mathfrak{m}$ where $\overline{x} = x + I$. Because $(\overline{y}/1)(\overline{z}/1) = \overline{x}^3/1 \in \mathfrak{q}$ but $\overline{y}/1 \notin \sqrt{\mathfrak{q}}$ and $\overline{z}/1 \notin \mathfrak{q}$. Therefore $R_\mathfrak{m}/\mathfrak{q}$ is a local quasi-prime ring and hence mp-ring which is not a mid ring.

We can deduce from Remark 4.13 and Example 4.16, that the class of mid rings is strictly between the class of p.f. rings (reduced mp-rings) and the class of mp-rings. Indeed,

$$\mathbf{p.f.\ rings} = \mathbf{reduced\ mp\text{-}rings} \subset \mathbf{mid\ rings} \subset \mathbf{mp\text{-}rings}.$$

Acknowledgement. The author would like to thank the referee for careful reading of the present paper.

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