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UNIQUE RANGE SETS WITHOUT FUJIMOTO'S HYPOTHESIS

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Abstract. This paper studies the uniqueness of two non-constant meromorphic functions when they share a finite set. Moreover, we will give an existence of unique range sets for meromorphic functions that are the zero sets of some polynomials that do not necessarily satisfy the Fujimoto's hypothesis ([6]).

1. Introduction

We use $M(\mathbb{C})$ to denote the set of all meromorphic functions in $\mathbb{C}$. Let $S \subset \mathbb{C} \cup \{\infty\}$ be a non-empty set with distinct elements. Further suppose that $f,\ g$ be two non-constant meromorphic (resp. entire) functions. We set

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where a zero of $f - a$ with multiplicity m counts m times in $E_f(S)$. If $E_f(S) = E_g(S)$, then we say that f and g share the set S CM.

If $E_f(S) = E_g(S)$ implies $f \equiv g$, then the set S is called a *unique range set* for meromorphic (resp. entire) functions, in short, URSM (resp. URSE).

The first example of a unique range set was given by F. Gross and C. C. Yang ([7]). They proved that if two non-constant entire functions f and g share the set $S = \{z \in \mathbb{C} : e^z + z = 0\}$ CM, then $f \equiv g$. Since then, many efforts have been made to construct new unique range sets with cardinalities as small as possible (see Chapter 10 of [9]).

So far, the smallest URSM has 11 elements which was constructed by G. Frank and M. Reinders ([5]). That URSM is the zero set of the following polynomial.

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c,$$

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where $n \geq 11$ and $c(\neq 0, 1)$ is any complex number.

To characterize the unique range sets, in 2000, H. Fujimoto ([6]) made a major breakthrough by observing that almost all *unique range sets are the zero sets of some polynomials which satisfy an injectivity condition* (which is known as Fujimoto's hypothesis). To state his result, we recall some well-known definitions.

Let $P(z)$ be a non-constant monic polynomial in $\mathbb{C}[z]$. The polynomial $P(z)$ is called a *uniqueness polynomial* for meromorphic (resp. entire) functions, in short, UPM (resp. UPE) if the condition $P(f) \equiv P(g)$ implies $f \equiv g$ where f and g are any two non-constant meromorphic (resp. entire) functions.

Also, the polynomial $P(z)$ is called a *strong uniqueness polynomial* for meromorphic (resp. entire) functions, in short, SUPM (resp. SUPE) if the condition $P(f) \equiv cP(g)$ implies $f \equiv g$ where f and g are any two non-constant meromorphic (resp. entire) functions and c is any non-zero complex number.

Thus strong uniqueness polynomials are uniqueness polynomials but the converse is not true, in general. For example, we consider the polynomial $P(z) = az + b$ $(a \neq 0)$. Then for any non-constant meromorphic function (resp. entire) g if we take $f := cg - \frac{b}{a}(1 - c)$ $(c \neq 0, 1)$, then we see that $P(f) = cP(g)$ but $f \neq g$.

Let $P(z)$ be a polynomial such that its derivative $P'(z)$ has k distinct zeros $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. The polynomial $P(z)$ is said to satisfy "condition H" ([6]) (which is known as Fujimoto's hypothesis) if

$$P(d_{l_s}) \neq P(d_{l_t}) \ (1 \leq l_s < l_t \leq k). \tag{1.1}$$

Now, we state Fujimoto's ([6]) result.

Theorem 1.1 ([6]). *Let $P(z)$ be a strong uniqueness polynomial of the form $P(z) = (z - a_1)(z - a_2) \cdots (z - a_n)$ $(a_i \neq a_j)$ satisfying the condition* (1.1). *Moreover, either $k \geq 3$ or $k = 2$ and $\min\{q_1, q_2\} \geq 2$. If $S = \{a_1, a_2, \ldots, a_n\}$, then S is a URSM* (*resp. URSE*) *whenever $n \geq 2k + 7$* (*resp. $n \geq 2k + 3$*).

But, in 2011, T. T. H. An ([1]) constructed a URSM that is the zero set of a polynomial which is not necessarily satisfying the Fujimoto's hypothesis (1.1).

Theorem 1.2 ([1]). *Let $P(z) = a_n z^n + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$ $(1 \leq m < n,\ a_i \in \mathbb{C},$ and $a_m \neq 0)$ be a polynomial of degree n with only simple zeros, and let S be its zero set. Further suppose that k is the number of distinct zeros of the derivative $P'(z)$ and $I = \{i : a_i \neq 0\}$, $\lambda = \min\{i : i \in I\}$, $J = \{i - \lambda : i \in I\}$. If $n \geq \max\{2k + 7, m + 4\}$, then the following statements are equivalent*:

i) *S is a URSM.*
ii) *P is a SUPM.*
iii) *S is affine rigid.*
iv) *The greatest common divisors of the indices respectively in I and J are both* 1.

Later, in 2012, using the concept of *weighted sharing* ([8]), A. Banerjee and I. Lahiri constructed a unique range set that is the zero set of a polynomial which is not necessarily satisfying the Fujimoto's hypothesis. To state the result of Banerjee and Lahiri, we need to recall the definition of *weighted set sharing*.

Let f and g be two non-constant meromorphic functions and l be any non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a; f)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a; f) = E_l(a; g)$, then we say that f *and* g *share the value* a *with weight* l.

For $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S, l) = \cup_{a \in S} E_l(a; f)$. If $E_f(S, l) = E_g(S, l)$, then we say that f and g share the set S with weight l, or simply f *and* g *share* (S, l).

If $E_f(S, l) = E_g(S, l)$ implies $f \equiv g$, then the set S is called a *unique range set for meromorphic* (*resp. entire*) *functions with weight* l, in short, URSM_l (resp. URSE_l).

Theorem 1.3 ([2]). *Let* $P(z) = a_n z^n + \sum_{j=2}^{m} a_j z^j + a_0$ *be a polynomial of degree* n*, where* $n - m \geq 3$ *and* $a_p a_m \neq 0$ *for some positive integer* p *with* $2 \leq p \leq m$ *and* $\gcd(p, 3) = 1$*. Suppose further that* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *be the set of all distinct zeros of* $P(z)$*. Let* k *be the number of distinct zeros of the derivative* $P'(z)$*. If* $n \geq 2k + 7$ (*resp.* $2k + 3$)*, then the following statements are equivalent*:

i) P *is a SUPM* (*resp. SUPE*).
ii) S *is a* URSM_2 (*resp.* URSE_2).
iii) S *is a URSM* (*resp. URSE*).
iv) P *is a UPM* (*resp. UPE*).

We have seen from Theorem 1.2 and Theorem 1.3 that the polynomial which generates a unique range set is a specific polynomial, i.e., the polynomial has a gap after n-th degree term (where n is the degree of the respective polynomial). The motivation of this short note is to construct a family of new unique range sets such that the corresponding unique range set generating polynomials is *not necessarily satisfying the Fujimoto's hypothesis* as well as *the generating polynomials have no "such" gap.*

2. Main results

Let

$$P(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{2.1}$$

be a monic polynomial of degree n in $\mathbb{C}[z]$ without multiple zeros. Let $P(z) - P(0)$ have m_1 simple zeros and m_2 multiple zeros. Further suppose that $P'(z)$ has k distinct zeros.

Theorem 2.1. *Let* $P(z)$ *be a monic polynomial defined by* (2.1) *with* $P(0) \neq 0$. *Suppose further that* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *is the set of all distinct zeros of* $P(z)$.

If $k \geq 2$, $m_1 + m_2 \geq 5$ (resp. $m_1 + m_2 \geq 3$) and $n \geq \max\{2k+7, m_1+m_2+3\}$ (resp. $n \geq \max\{2k+3, m_1+m_2+1\}$), then the following statements are equivalent:

i) *P is a SUPM (resp. SUPE).*
ii) *S is a URSM$_2$ (resp. URSE$_2$).*
iii) *S is a URSM (resp. URSE).*

Theorem 2.2. *Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree n in $\mathbb{C}[z]$ with $P(0) \neq 0$. If $P(z) - P(0)$ has m_1 simple zeros and m_2 multiple zeros, and $n \geq 2(m_1+m_2)+2$ (resp. $n \geq 2(m_1+m_2)+1$), then the following two statements are equivalent*:

i) *P is a SUPM (resp. SUPE).*
ii) *P is a UPM (resp. UPE).*

Proof of Theorem 2.1. Since, the two cases (ii)⇒(iii) and (iii)⇒(i) are straightforward, so we only prove that (i)⇒(ii).

Assume that $P(z)$ is a SUPM (resp. SUPE) and $E_f(S,2) = E_g(S,2)$. Now, we put

$$F(z) := \frac{1}{P(f(z))} \text{ and } G(z) := \frac{1}{P(g(z))}.$$

Let $S(r) : (0,\infty) \to \mathbb{R}$ be any function satisfying $S(r) = o(T(r,F) + T(r,G))$ for $r \to \infty$ outside a set of finite Lebesgue measure. Next we let

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.$$

First we assume that $H \not\equiv 0$. The lemma of logarithmic derivative gives

$$m(r,H) = S(r).$$

By construction of H, H has at most simple poles and poles of H can only occur at poles of F and G, and zeros of F' or G' (for details, see [3,4]). Since F and G share ∞ with weight 2, thus

$$\begin{aligned} N(r,\infty;H) \\ \leq \sum_{j=1}^{k} \big(\overline{N}(r,\lambda_j;f) + \overline{N}(r,\lambda_j;g)\big) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') \\ + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,\infty;F,G), \end{aligned} \tag{2.2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct zeros of $P'(z)$. (Here we write $\overline{N}_0(r,0;f')$ for the reduced counting function of zeros of f', which are not zeros of $\prod_{i=1}^{n}(f-\alpha_i)\prod_{j=1}^{k}(f-\lambda_j)$. Similarly $\overline{N}_0(r,0;g')$ is defined. Also we write $\overline{N}_*(r,\infty;F,G)$ to denote the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.)

Now the Laurent series expansion of H shows that H has a zero at every simple pole of F (hence, that of G). Thus using the first fundamental theorem, we conclude that

$$N(r,\infty;F|=1)\le N(r,0;H)\le N(r,\infty;H)+S(r), \tag{2.3}$$

where $N(r,\infty;F|=1)$ is the counting function of simple poles of F. Thus combining the inequalities (2.2) and (2.3), we obtain

$$\begin{aligned}
&\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')\\
\le\ &\sum_{j=1}^{k}\big(\overline{N}(r,\lambda_j;f)+\overline{N}(r,\lambda_j;g)\big)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&+\overline{N}(r,\infty;F|\ge 2)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+S(r)\\
\le\ &\sum_{j=1}^{k}\big(\overline{N}(r,\lambda_j;f)+\overline{N}(r,\lambda_j;g)\big)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&+\frac{1}{2}\{N(r,\infty;F)+N(r,\infty;G)\}+S(r).
\end{aligned}$$

The second fundamental theorem applied to f and g gives

$$\begin{aligned}
&(n+k-1)\,(T(r,f)+T(r,g))\\
\le\ &\overline{N}(r,\infty;f)+\sum_{i=1}^{n}\overline{N}(r,\alpha_i;f)+\sum_{j=1}^{k}\overline{N}(r,\lambda_j;f)-\overline{N}_0(r,0;f')+\overline{N}(r,\infty;g)\\
&+\sum_{i=1}^{n}\overline{N}(r,\alpha_i;g)+\sum_{j=1}^{k}\overline{N}(r,\lambda_j;g)-\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\
\le\ &\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\sum_{j=1}^{k}\big(\overline{N}(r,\lambda_j;f)+\overline{N}(r,\lambda_j;g)\big)\\
&+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r)\\
\le\ &2\big(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\big)+(2k+\frac{n}{2})\,(T(r,f)+T(r,g))+S(r),
\end{aligned}$$

which contradicts the assumption $n\ge 2k+7$ (resp. $n\ge 2k+3$). Thus from now we assume that $H\equiv 0$. Then by integration, we obtain

$$\frac{1}{P(f(z))}\equiv\frac{c_0}{P(g(z))}+c_1, \tag{2.4}$$

where c_0 is a non zero complex constant. Thus

$$T(r,f)=T(r,g)+O(1).$$

Now we distinguish two cases:

Case-I. Assume that $c_1\ne 0$. Then equation (2.4) can be written as

$$P(f)\equiv\frac{P(g)}{c_1P(g)+c_0}.$$

Thus

$$\overline{N}(r,-\frac{c_0}{c_1};P(g))\leq \overline{N}(r,\infty;P(f))=\overline{N}(r,\infty;f).$$

Since $P(z)-P(0)$ has m_1 simple zeros and m_2 multiple zeros, so we can assume

$$P(z)-P(0)=(z-b_1)(z-b_2)\cdots(z-b_{m_1})(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_{m_2})^{l_{m_2}},$$

where $l_i\geq 2$ for $1\leq i\leq m_2$. Moreover, $l_i<n$ as $P'(z)$ has at least two zeros. If $P(0)\neq -\frac{c_0}{c_1}$, then the first and second fundamental theorems to $P(g)$ give

$$\begin{aligned}
&nT(r,g)+O(1)\\
=&\,T\left(r,P(g)\right)\\
\leq&\,\overline{N}\left(r,\infty;P(g)\right)+\overline{N}\left(r,P(0);P(g)\right)+\overline{N}\left(r,-\frac{c_0}{c_1};P(g)\right)+S(r,P(g))\\
\leq&\,\overline{N}\left(r,\infty;g\right)+\overline{N}\left(r,\infty;f\right)+(m_1+m_2)T(r,g)+S(r,g),
\end{aligned}$$

which is impossible as $n\geq m_1+m_2+3$ (resp. $n\geq m_1+m_2+1$). Thus $P(0)=-\frac{c_0}{c_1}$. Hence

$$P(f)\equiv\frac{P(g)}{c_1(P(g)-P(0))}.$$

Thus every zero of $g-b_j$ $(1\leq j\leq m_1)$ has a multiplicity at least n, and every zero of $g-c_i$ $(1\leq i\leq m_2)$ has a multiplicity at least 2.

Thus applying the second fundamental theorem to g, we have

$$\begin{aligned}
&(m_1+m_2-1)T(r,g)\\
\leq&\,\overline{N}(r,\infty;g)+\sum_{j=1}^{m_1}\overline{N}(r,b_j;g)+\sum_{i=1}^{m_2}\overline{N}(r,c_i;g)+S(r,g)\\
\leq&\,\overline{N}(r,\infty;g)+\frac{1}{n}\sum_{j=1}^{m_1}N(r,b_j;g)+\frac{1}{2}\sum_{i=1}^{m_2}N(r,c_i;g)+S(r,g)\\
\leq&\,\overline{N}(r,\infty;g)+\frac{m_1+m_2}{2}T(r,g)+S(r,g),
\end{aligned}$$

which is impossible as $m_1+m_2\geq 5$ (resp. 3).

Case-II. Next we assume that $c_1=0$. Then equation (2.4) can be written as

$$P(g)\equiv c_0P(f).$$

Since P is a strong uniqueness polynomial, thus

$$f\equiv g.$$

This completes the proof. □

Proof of Theorem 2.2. Since strong uniqueness polynomials are uniqueness polynomials, so we only prove the case (ii)⇒(i). It is given that $P(z)$ is a uniqueness polynomial. Assume that

$$P(g)=c_0P(f),$$

where f and g are two non-constant meromorphic functions and c_0 is any non-zero complex constant. Thus $T(r,f) = T(r,g) + O(1)$. Now, if $c_0 \neq 1$, then

$$P(g) - P(0) \equiv c_0(P(f) - \frac{P(0)}{c_0}).$$

Thus using the first and second fundamental theorems to $P(f)$, we obtain

$$\begin{aligned} & nT(r,f) + O(1) \\ = & \, T(r, P(f)) \\ \leq & \, \overline{N}(r, \infty; P(f)) + \overline{N}(r, P(0); P(f)) + \overline{N}\left(r, \frac{P(0)}{c_0}; P(f)\right) + S(r,f) \\ \leq & \, \overline{N}(r, \infty; f) + 2(m_1 + m_2)T(r,f) + S(r,f), \end{aligned}$$

which contradicts our assumptions on n. Thus $c_0 = 1$, i.e.,

$$P(f) \equiv P(g).$$

Since $P(z)$ is a uniqueness polynomial, so $f \equiv g$. This completes the proof. □

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