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ESTIMATES FOR THE RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER TYPE OPERATORS ON THE HEISENBERG GROUP

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Abstract. We consider the Schrödinger type operator $\mathcal{L} = (-\Delta_{\mathbb{H}^n})^2 + V^2$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the non-negative potential V belongs to the reverse Hölder class RH_s for $s \geq Q/2$ and $Q \geq 6$. We shall establish the (L^p, L^q) estimates for the Riesz transforms $T_{\alpha,\beta,j} = V^{2\alpha}\nabla^j_{\mathbb{H}^n}\mathcal{L}^{-\beta}$, $j = 0, 1, 2, 3$, where $\nabla_{\mathbb{H}^n}$ is the gradient operator on $\mathbb{H}^n$, $0 < \alpha \leq 1-j/4$, $j/4 < \beta \leq 1$, and $\beta-\alpha \geq j/4$.

1. Introduction

In this paper, we consider the Schrödinger type operator $\mathcal{L} = (-\Delta_{\mathbb{H}^n})^2 + V^2$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the non-negative potential V belongs to the reverse Hölder class RH_s for $s \geq Q/2$, $Q \geq 6$, and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Let us recall some basic facts about the Heisenberg group $\mathbb{H}^n$. By [7], the Heisenberg group $\mathbb{H}^n$ is a Lie group with the underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and the multiplication

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2x'y - 2xy').$$

A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n, \quad T = \frac{\partial}{\partial t}.$$

All non-trivial commutation relations are given by $[X_j, Y_j] = -4T$, $j = 1, 2, \ldots, n$. Then the sub-Laplacian $\Delta_{\mathbb{H}^n}$ and the gradient operator $\nabla_{\mathbb{H}^n}$ are defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j + Y_j), \qquad \nabla_{\mathbb{H}^n} = (X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$

respectively.

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The dilations on $\mathbb{H}^n$ have the form

$$\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We denote the measure of any measurable set E by $|E|$. Then $|\delta_\lambda E| = \lambda^Q |E|$.

We define a homogeneous norm on $\mathbb{H}^n$ by

$$|g| = ((|x|^2 + |y|^2)^2 + |t|^2)^{\frac{1}{4}}, \quad g = (x,y,t) \in \mathbb{H}^n.$$

This norm satisfies the triangle inequality and leads to a left-invariant distant $d(g,h) = |g^{-1}h|$. The ball of radius r centered at g is denoted by

$$B(g,r) = \{h \in \mathbb{H}^n : |g^{-1}h| < r\},$$

whose volume is given by $|B(g,r)| = c_n r^Q$, where c_n is a constant that only depends on n.

For $1 < s < \infty$, a non-negative locally L^s-integrable function V on $\mathbb{H}^n$ is said to belong to the reverse Hölder class RH_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(g)^s dg\right)^{1/s} \le \frac{C}{|B|}\int_B V(g)dg$$

holds for every ball $B \subset \mathbb{H}^n$.

It is well known that if $V \in RH_s$ for some $s > 1$, then $V(g)dg$ is a doubling measure.

The remarkable feature about the class RH_s is its self-improvement [6]; that is, if $V \in RH_s$ for some $s > 1$, then there exists $\epsilon > 0$ such that $V \in RH_{s+\epsilon}$.

Assume that $V \in RH_s$ for some $s \ge Q/2$. For $g \in \mathbb{H}^n$, the definition of the auxiliary function $\rho(g)$ is as follows:

$$\rho(g) \doteq \frac{1}{m(g,V)} = \sup_{r>0}\Big\{r : \frac{1}{r^{Q-2}}\int_{B(g,r)} V(h)dh \le 1\Big\}.$$

Recently, the boundedness of some transforms related to Schrödinger operators and Schrödinger type operators with non-negative potentials on the abstract settings have been received a great deal of attention. See for example [2–4, 10, 14]. In particular, Liu, Huang and Xie in [4] showed that the operator $V^2\mathcal{L}^{-1}$ is bounded on $L^p(\mathbb{H}^n)$, Liu in [15] established the $L^p(\mathbb{H}^n)$ estimates for Riesz transform $\nabla^4_{\mathbb{H}^n}\mathcal{L}^{-1}$, Liu and Xie in [5] obtained the boundedness of $L^p(\mathbb{H}^n)$ and weak $L^1(\mathbb{H}^n)$ for Riesz transform $\nabla^2_{\mathbb{H}^n}\mathcal{L}^{-1/2}$.

Let us consider the Riesz transforms

$$T_{\alpha,\beta,j} = V^{2\alpha}\nabla^j_{\mathbb{H}^n}\mathcal{L}^{-\beta}, \quad j = 0,1,2,3,$$

where $0 < \alpha \le 1 - j/4$, $j/4 < \beta \le 1$, and $\beta - \alpha \ge j/4$. In the setting of Euclidean, Sugano in [9] established estimates of the fundamental solution for $\mathcal{L}$ and showed some $L^p(\mathbb{R}^n)$ estimates for Schrödinger type operators, Wang in [11] obtained the $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ boundedness of operator $T_{\alpha,\beta,0} = V^{2\alpha}\mathcal{L}^{-\beta}$ for $0 < \alpha \le \beta \le 1$, the author of this article in [12] proved that the operator

$T_{\alpha,\beta,2} = V^{2\alpha}\nabla^2\mathcal{L}^{-\beta}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $0 < \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2$, and in [13] the author obtained the $L^p(\mathbb{R}^n)$ and weak $L^1(\mathbb{R}^n)$ estimate for $T_{0,\frac{1}{4},1}$. More boundedness results of these operators can be found in [1], [6] and [8].

In this paper, we concentrate on the boundedness estimates for the Riesz transforms $T_{\alpha,\beta,j}$. The following $(L^p(\mathbb{H}^n), L^q(\mathbb{H}^n))$ estimates are established.

Theorem 1.1. *For $j = 0, 1, 2, 3$, let $0 < \alpha \leq 1 - j/4,\ j/4 < \beta \leq 1, \beta - \alpha \geq j/4$. Suppose $V \in RH_s$ for $s \geq 2Q/(4-j)$. If $1 \leq p \leq \left(\frac{2\alpha}{s} + \frac{4(\beta-\alpha)-j}{Q}\right)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$, then there exists a constant C such that*

$$\|T_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \leq C\|f\|_{L^p(\mathbb{H}^n)}.$$

Theorem 1.2. *For $j = 1, 2, 3$, let $0 < \alpha \leq 1 - j/4,\ j/4 < \beta \leq 1, \beta - \alpha \geq j/4$. Suppose $V \in RH_s$ with $Q/2 \leq s < 2Q/(4-j)$. If $1 < p \leq \left(\frac{1}{p_\alpha} + \frac{4(\beta-\alpha)-j}{Q}\right)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$, then there exists a constant C such that*

$$\|T_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \leq C\|f\|_{L^p(\mathbb{H}^n)},$$

where $\frac{1}{p_\alpha} = \frac{2\alpha+2}{s} - \frac{4-j}{Q}$.

2. Some preliminaries

In this section we shall recall some results of the auxiliary function $\rho(g)$ and some estimates of fundamental solutions for $\mathcal{L} + \lambda$ on the Heisenberg group.

Assume that the potential V is non-negative and belongs to $RH_s, s \geq Q/2$. Auxiliary function $\rho(g)$ has the following properties, whose proofs are given in [2, 3].

Lemma 2.1. *If $r = \rho(g)$, then*

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh = 1.$$

Moreover, $\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \sim 1$ if and only if $r \sim \rho(g)$.

Lemma 2.2. *There exists a constant $l_0 > 0$ such that*

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \leq C\left(1 + \frac{r}{\rho(g)}\right)^{l_0}.$$

Lemma 2.3. *There exist constants $C > 0$ and $k_0 \geq 1$ such that, for any $g, h \in \mathbb{H}^n$*

$$\frac{1}{C}\left(1 + m(h,V)|g^{-1}h|\right)^{1/(1+k_0)} \leq 1 + m(g,V)|g^{-1}h| \leq C\left(1 + m(h,V)|g^{-1}h|\right)^{1+k_0}.$$

Lemma 2.4. *For $0 < r < R < \infty$,*

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \le C \left(\frac{R}{r}\right)^{Q/s-2} \frac{1}{R^{Q-2}} \int_{B(g,R)} V(h)dh.$$

Let $\Gamma_{\mathcal{L}}(g,h,\lambda)$ be the fundamental solution of $\mathcal{L}+\lambda$ for $\lambda \in [0,\infty)$. From the results in [15], we have:

Lemma 2.5. *Assume $V \in RH_{Q/2}$. For any positive integer N there exists a constant C_N such that*

$$|\Gamma_{\mathcal{L}}(g,h,\lambda)| \le \frac{C_N}{(1+\sqrt{\lambda}|g^{-1}h|^2)^N(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-4}}.$$

Lemma 2.6. *Let $V \in RH_{\frac{2Q}{4-j}}$, $j = 1,2,3$. Then for any positive integer N there exists a constant C_N such that,*

$$|\nabla^j_{\mathbb{H}^n,g}\Gamma_{\mathcal{L}}(g,h,\lambda)| \le \frac{C_N}{(1+\sqrt{\lambda}|g^{-1}h|^2)^N(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-4+j}}.$$

Proof. Fix $g_0, h_0 \in \mathbb{H}^n$, and put $R = |g_0^{-1}h_0|$. Assume that $(-\triangle_{\mathbb{H}^n})^2u + Vu + \lambda u = 0$ in $B(g_0,R)$. It follows from the proof of Lemma 13 in [15] we have

$$\begin{aligned}|\nabla^j_{\mathbb{H}^n,g}u(g_0)| &\le C\int_{B(g_0,R)} \frac{V(h)^2|u(h)|\mathrm{d}h}{|g_0^{-1}h|^{Q-4+j}} + \frac{1}{R^{Q+j}}\int_{B(g_0,R)}|u(h)|\mathrm{d}h \\ &\le C\sup_{B(g_0,R)}|u(g)|\left(\int_{B(g_0,R)}\frac{V(h)^2\mathrm{d}h}{|g_0^{-1}h|^{Q-4+j}} + \frac{1}{R^j}\right).\end{aligned}$$

Since $V \in RH_{\frac{2Q}{4-j}}$, it follows that $V \in RH_q$ for some $q > 2Q/(4-j)$. We choose t such that $2/q + 1/t = 1$. By the Hölder inequality and Lemma 2.2, it follows that

$$\begin{aligned}&\int_{B(g_0,R)}\frac{V(h)^2\mathrm{d}h}{|g_0^{-1}h|^{Q-4+j}} \\ &\le CR^Q\left(\frac{1}{R^Q}\int_{B(g_0,R)}V(h)^q\mathrm{d}h\right)^{2/q}\left(\frac{1}{R^Q}\int_{B(g_0,R)}\frac{\mathrm{d}h}{|g_0^{-1}h|^{(Q-4+j)t}}\right)^{1/t} \\ &\le CR^{Q-4}\left(\frac{1}{R^{Q-2}}\int_{B(g_0,R)}V(h)\mathrm{d}h\right)^2 R^{-Q+4-j} \\ &\le CR^{-j}(1+Rm(g,V))^{2l_0}.\end{aligned}$$

Then, using Lemma 2.5 we arrive at the desired estimate. □

Remark 2.7. It can be seen from Lemma 2.3 that Lemma 2.5 and Lemma 2.6 still hold if $m(g,V)$ is replaced by $m(h,V)$.

3. The proof of Theorem 1.1

We first give the estimates for the kernel of the operator $\mathcal{L}^{-\beta}$.

Lemma 3.1. *Assume $V \in RH_s$ for $s \geq Q/2$, and $0 < \beta \leq 1$. The kernel of the operator $\mathcal{L}^{-\beta}$ is denoted as $K_{\beta,0}$. Then, for any positive integer N there exists a constant C_N such that*

$$|K_{\beta,0}(g,h)| \leq \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-4\beta}}.$$

Moreover, the inequality above also holds with $m(g,V)$ replaced by $m(h,V)$.

Proof. When $\beta = 1$, it follows from Lemma 2.5 that

$$|K_{\beta,0}(g,h)| = |\Gamma_{\mathcal{L}}(g,h,0)| \leq \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-4}}.$$

For $0 < \beta < 1$, by the functional calculus, we may write

$$\mathcal{L}^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta} (\mathcal{L}+\lambda)^{-1} \mathrm{d}\lambda.$$

Let $f \in C_0^\infty(\mathbb{H}^n)$. Then

$$\mathcal{L}^{-\beta}(f)(g) = \int_{\mathbb{H}^n} K_{\beta,0}(g,h) f(h) \mathrm{d}h,$$

where

$$K_{\beta,0}(g,h) = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta} \Gamma_{\mathcal{L}}(g,h,\lambda) \mathrm{d}\lambda.$$

Note that

$$\int_0^\infty \frac{dh}{\lambda^\beta (1+\sqrt{\lambda}|g^{-1}h|^2)^N} \leq C|g^{-1}h|^{4\beta-4}$$

holds for $0 < \beta < 1$, then by Lemma 2.5 we get

$$|K_{\beta,0}(g,h)| \leq \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-4\beta}}.$$

□

Let us estimate the kernel function of the operators $\nabla_{\mathbb{H}^n}^j \mathcal{L}^{-\beta}$, $j = 1,2,3$. We denote the kernel functions of the operators $\nabla_{\mathbb{H}^n}^j \mathcal{L}^{-\beta}$ as $K_{\beta,j}$.

Lemma 3.2. *Assume that $V \in RH_s$ for $s \geq 2Q/(4-j)$, $j = 1,2,3$, and $j/4 < \beta \leq 1$. Then, for any positive integer N there exists a constant C_N such that*

$$|K_{\beta,j}(g,h)| \leq \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N} \frac{1}{|g^{-1}h|^{Q-(4\beta-j)}}.$$

Moreover, the inequality above also holds with $m(g,V)$ replaced by $m(h,V)$.

Proof. When $\beta = 1$, it follows from Lemma 2.6 that

$$|K_{\beta,j}(g,h)| = |\nabla^j_{\mathbb{H}^n,g}\Gamma_{\mathcal{L}}(g,h,0)| \le \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N}\frac{1}{|g^{-1}h|^{Q-(4-j)}}.$$

For $j/4 < \beta < 1$, by the functional calculus, we have

$$\mathcal{L}^{-\beta} = \frac{\sin \pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}(\mathcal{L}+\lambda)^{-1}d\lambda.$$

Let $f \in C_0^\infty(\mathbb{H}^n)$. It follows from $(\mathcal{L}_2+\lambda)^{-1}f(x) = \int_{\mathbb{H}^n}\Gamma_{\mathcal{L}_2}(x,z,\lambda)f(z)dz$ that

$$\nabla^j_{\mathbb{H}^n}\mathcal{L}^{-\beta}(f)(g) = \int_{\mathbb{H}^n} K_{\beta,j}(g,h)f(h)dh,$$

where

$$K_{\beta,j}(g,h) = \frac{\sin \pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}\nabla^j_{\mathbb{H}^n,g}\Gamma_{\mathcal{L}}(g,h,\lambda)d\lambda.$$

Then, by Lemma 2.6 and the inequality

$$\int_0^\infty \frac{dh}{\lambda^\beta(1+\sqrt{\lambda}|g^{-1}h|^2)^N} \le C|g^{-1}h|^{4\beta-4},$$

we can arrive at the desired result. □

Next, we give the maximal function estimates for $T^*_{\alpha,\beta,j},\ j = 0,1,2,3$.

Let $f \in L^1_{loc}(\mathbb{H}^n)$. For $0 \le \gamma < Q$, the fractional maximal operator is defined by

$$M_\gamma(f)(g) = \sup_{g\in B}\frac{1}{|B|^{1-\frac{\gamma}{Q}}}\int_B |f(h)|\mathrm{d}h,$$

where the supremum on the right side is taken over all ball $B \subset \mathbb{H}^n$ such that $g \in B$.

Lemma 3.3. *Suppose $V \in RH_s$ for $s \ge Q/2,\ 0 < \alpha \le \beta \le 1$. Let $T^*_{\alpha,\beta,0}$ be the adjoint operator of $T_{\alpha,\beta,0}$. Then*

$$|T^*_{\alpha,\beta,0}(f)(g)| \le C\{M_{\gamma q_0}(|f|^{q_0})(g)\}^{\frac{1}{q_0}}$$

for some $1 < q_0 < q_1$, where $\frac{1}{q_1} = 1 - \frac{2\alpha}{s}$ and $\gamma = 4(\beta-\alpha)$.

Proof. Let $r = \rho(g)$. Then by Lemma 3.1 and Lemma 2.3 we have

$$\begin{aligned}|T^*_{\alpha,\beta,0}(f)(g)| &= \int_{\mathbb{H}^n} |K_{\beta,0}(h,g)|V(h)^{2\alpha}|f(h)|\mathrm{d}h\\ &\le C\int_{\mathbb{H}^n}\frac{V(h)^{2\alpha}}{(1+m(g,V)|g^{-1}h|)^N}\frac{|f(h)|}{|g^{-1}h|^{Q-4\beta}}\mathrm{d}h\\ &\le C\sum_{k=-\infty}^{\infty}\frac{(2^kr)^{4\beta}}{(1+2^k)^N}\frac{1}{(2^kr)^Q}\int_{|g^{-1}h|\le 2^kr} V(h)^{2\alpha}|f(h)|\mathrm{d}h.\end{aligned}$$

Since $1/q_1 + (2\alpha)/s = 1$ and $V \in RH_s, s \geq Q/2$, we have

$$\frac{1}{(2^k r)^Q}\int_{|g^{-1}h|\leq 2^k r} V(h)^{2\alpha}|f(h)|\mathrm{d}h$$
$$\leq C\left(\frac{1}{(2^k r)^Q}\int_{|g^{-1}h|\leq 2^k r} V(h)\mathrm{d}h\right)^{2\alpha}\left(\frac{1}{(2^k r)^Q}\int_{|g^{-1}h|\leq 2^k r}|f(h)|^{q_1}\mathrm{d}h\right)^{1/q_1}.$$

For $k \geq 1$, by Lemma 2.2,

$$\begin{aligned}\left(\frac{1}{(2^k r)^Q}\int_{|g^{-1}h|\leq 2^k r} V(h)\mathrm{d}h\right)^{2\alpha} &\leq C(2^k r)^{-4\alpha}(1+2^k)^{2l_0\alpha}\\ &\leq C(2^k r)^{-4\alpha}2^{2l_0\alpha k}.\end{aligned}$$

For $k \leq 0$, by Lemma 2.4, we have

$$\begin{aligned}&\left(\frac{1}{(2^k r)^Q}\int_{|g^{-1}h|\leq 2^k r} V(h)\mathrm{d}h\right)^{2\alpha}\\ &\leq C(2^k r)^{-4\alpha}\left(\frac{r}{2^k r}\right)^{2\alpha(Q/s-2)}\left(\frac{1}{r^{Q-2}}\int_{|g^{-1}h|\leq r} V(h)\mathrm{d}h\right)^{2\alpha}\\ &\leq C(2^k r)^{-4\alpha}2^{2\alpha k(2-Q/s)}.\end{aligned}$$

Taking $N > 2l_0\alpha$, then

$$\begin{aligned}|T^*_{\alpha,\beta,0}(h)| &\leq C\left(\sum_{k=-\infty}^{0}2^{2\alpha k(2-Q/s)}+\sum_{k=1}^{\infty}2^{k(2l_0\alpha-N)}\right)\{M_{\gamma q_1}(|f|^{q_1})(g)\}^{\frac{1}{q_1}}\\ &\leq C\{M_{\gamma q_1}(|f|^{q_1})(g)\}^{\frac{1}{q_1}}.\end{aligned}$$

By the self-improvement of class RH_s we know that there exists some $1 < q_0 < q_1$ such that

$$|T^*_{\alpha,\beta,0}(f)(h)| \leq C\{M_{\gamma q_0}(|f|^{q_0})(g)\}^{\frac{1}{q_0}}. \qquad \square$$

Using the same method, we can obtain the following result.

Lemma 3.4. *Assume that* $V \in RH_s$ *for* $s \geq 2Q/(4-j),\ j = 1,2,3$. *Then*

$$|T^*_{\alpha,\beta,j}(f)(g)| \leq C\{M_{\gamma q_0}(|f|^{q_0})(g)\}^{\frac{1}{q_0}}$$

for some $1 < q_0 < q_1$, *where* $\frac{1}{q_1} = 1 - \frac{2\alpha}{s}$ *and* $\gamma = 4(\beta-\alpha) - j$.

Proof of Theorem 1.1. For $j = 0,1,2,3$, by Lemma 3.3, Lemma 3.4 and the $(L^p(\mathbb{H}^n), L^q(\mathbb{H}^n))$ boundedness of fractional maximal operator we get

$$\|T^*_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \leq C\|f\|_{L^p(\mathbb{H}^n)}$$

for $\left(\frac{s}{2\alpha}\right)' = q_1 \leq q < \frac{Q}{4(\beta-\alpha)-j}$, $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$.

By duality, we have

$$\|T_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \le C\|f\|_{L^p(\mathbb{H}^n)}$$

for $\frac{Q}{Q-(4(\beta-\alpha)-j)} < q \le \frac{s}{2\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$. These conditions are equivalent to

$$1 < p \le \frac{1}{\frac{2\alpha}{s} + \frac{4(\beta-\alpha)-j}{Q}} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}.$$

To complete the proof of Theorem 1.1, we also need to prove the inequality

$$\|T_{\alpha,\beta,j}(f)\|_{L^{\overline{q}}(\mathbb{H}^n)} \le C\|f\|_{L^1(\mathbb{H}^n)},$$

where $\overline{q} = \frac{Q}{Q-(4(\beta-\alpha)-j)}$.

Suppose $f \in L^1(\mathbb{H}^n)$, and let $r = \rho(g)$. Then

$$\begin{aligned}\nabla^j_{\mathbb{H}^n}\mathcal{L}^{-\beta}(f)(g) &= \int_{|g^{-1}h|<r} K_{\beta,j}(g,h)f(h)\mathrm{d}h + \int_{|g^{-1}h|\ge r} K_{\beta,j}(g,h)f(h)\mathrm{d}h \\ &= u_1(g) + u_2(g).\end{aligned}$$

By Lemma 3.1 and the Minkowski inequality,

$$\begin{aligned}&\left(\int_{\mathbb{H}^n} |V(g)^{2\alpha}u_1(g)|^{\overline{q}}\mathrm{d}g\right)^{1/\overline{q}} \\ = &\left(\int_{\mathbb{H}^n} V(g)^{2\alpha\overline{q}}\left(\int_{|g^{-1}h|<r}\frac{|f(h)|}{|g^{-1}h|^{Q-(4\beta-j)}}\mathrm{d}h\right)^{\overline{q}}\mathrm{d}g\right)^{1/\overline{q}} \\ \le &\, C\int_{\mathbb{H}^n}|f(h)|\left(\int_{|g^{-1}h|<r}\frac{V(g)^{2\alpha\overline{q}}}{|g^{-1}h|^{(Q-(4\beta-j))\overline{q}}}\mathrm{d}g\right)^{1/\overline{q}}\mathrm{d}h.\end{aligned}$$

It follows from $0 < \alpha \le 1 - j/4$, $j/4 < \beta \le 1$, $\beta - \alpha \ge j/4$, $Q \ge 6$ that $2\alpha\overline{q} < 2Q/(4-j) < s$. Let $t = s/(2\alpha\overline{q})$, by the Hölder inequality we have

$$\begin{aligned}&\left(\int_{|g^{-1}h|<r}\frac{V(g)^{2\alpha\overline{q}}}{|g^{-1}h|^{(Q-(4\beta-j))\overline{q}}}\mathrm{d}g\right)^{1/\overline{q}} \\ \le &\, C\left(\int_{|g^{-1}h|<r} V(g)^s\mathrm{d}g\right)^{2\alpha\overline{q}/s}\left(\int_{|g^{-1}h|<r}\frac{1}{|g^{-1}h|^{(Q-(4\beta-j))t'\overline{q}}}\mathrm{d}g\right)^{1/t'}.\end{aligned}$$

By $V \in RH_s$ and Lemma 2.2,

$$\left(\int_{|g^{-1}h|<r} V(g)^s\mathrm{d}g\right)^{2\alpha\overline{q}/s} \le Cr^{-4\alpha\overline{q}+2Q\alpha\overline{q}/s}.$$

Notice $(Q-(4\beta-j))\overline{q}t'<Q$, so

$$\left(\int_{|g^{-1}h|<r}\frac{1}{|g^{-1}h|^{(Q-(4\beta-j))t'\overline{q}}}\mathrm{d}g\right)^{1/t'}\leq Cr^{-(Q-(4\beta-j))\overline{q}+Q/t'}.$$

Due to

$$\frac{2Q\alpha\overline{q}}{s}-4\alpha\overline{q}-(Q-(4\beta-j))\overline{q}+\frac{Q}{t'}=0,$$

we get

$$\int_{|g^{-1}h|<r}\frac{V(g)^{2\alpha\overline{q}}}{|g^{-1}h|^{(Q-(4\beta-j))\overline{q}}}\mathrm{d}g\leq Cr^{\frac{2Q\alpha\overline{q}}{s}-4\alpha\overline{q}-(Q-(4\beta-j))\overline{q}+\frac{Q}{t'}}\leq C.$$

So

$$\left(\int_{\mathbb{H}^n}V(g)^{2\alpha}|u_1(g)|^{\overline{q}}\mathrm{d}g\right)^{1/\overline{q}}\leq C\|f\|_{L^1(\mathbb{H}^n)}.$$

Note that

$$|u_2(g)|\leq\int_{|g^{-1}h|\geq r}\frac{|f(h)|\mathrm{d}h}{\left(1+\frac{|g^{-1}h|}{r}\right)^N|g^{-1}h|^{Q-(4\beta-j)}}.$$

Then, by the Minkowski inequality we get

$$\begin{aligned}&\left(\int_{\mathbb{H}^n}|V(g)^{2\alpha}u_2(g)|^{\overline{q}}\mathrm{d}g\right)^{1/\overline{q}}\\ =&\left(\int_{\mathbb{H}^n}V(g)^{2\alpha\overline{q}}\left(\int_{|g^{-1}h|\geq r}\frac{|f(h)|}{\left(1+\frac{|g^{-1}h|}{r}\right)^N|g^{-1}h|^{Q-(4\beta-j)}}\mathrm{d}h\right)^{\overline{q}}\mathrm{d}g\right)^{1/\overline{q}}\\ \leq&\, C\int_{\mathbb{H}^n}|f(h)|\left(\int_{|g^{-1}h|\geq r}\frac{V(g)^{2\alpha\overline{q}}}{\left(1+\frac{|g^{-1}h|}{r}\right)^{N\overline{q}}|g^{-1}h|^{(Q-(4\beta-j))\overline{q}}}\mathrm{d}g\right)^{1/\overline{q}}\mathrm{d}h.\end{aligned}$$

Since $V\in RH_s, s>2Q/(4-j)>2\alpha\overline{q}$, we have

$$\frac{1}{(2^jr)^Q}\int_{|g^{-1}h|\leq 2^jr}V(g)^{2\alpha\overline{q}}\mathrm{d}g\leq C2^{2\alpha l_0\overline{q}j}(2^jr)^{-\alpha\overline{q}}.$$

Note that $Q-(Q-(4\beta-j))\overline{q}-4\alpha\overline{q}=0$, taking $N>2\alpha l_0$, we get

$$\begin{aligned}&\int_{|g^{-1}h|\geq r}\frac{V(g)^{2\alpha\overline{q}}}{\left(1+\frac{|g^{-1}h|}{r}\right)^{N\overline{q}}|g^{-1}h|^{(Q-(4\beta-j))\overline{q}}}\mathrm{d}g\\ &\leq C\sum_{j=1}^{\infty}2^{jN\overline{q}}(2^jr)^{Q-(Q-(4\beta-j))\overline{q}}\frac{1}{(2^jr)^Q}\int_{|g^{-1}h|\leq 2^jr}V(g)^{2\alpha\overline{q}}\mathrm{d}g\\ &\leq C\sum_{j=1}^{\infty}2^{j(N-2\alpha l_0)\overline{q}}(2^jr)^{Q-(Q-(4\beta-j))\overline{q}-4\alpha\overline{q}}\leq C.\end{aligned}$$

Then

$$\left(\int_{\mathbb{H}^n} |V(g)^{2\alpha} u_2(g)|^{\overline{q}} \mathrm{d}g\right)^{1/\overline{q}} \le C\|f\|_{L^1(\mathbb{H}^n)}.$$

This finishes the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

For $j = 1,2,3$, let $K_{\beta,j}$ be the kernel of the operators $\mathcal{W}_{\beta,j} = \nabla^j_{\mathbb{H}^n}\mathcal{L}^{-\beta}$. We first give the kernel estimates.

Lemma 4.1. *For $j = 1,2,3$, let $j/4 < \beta \le 1$. Suppose $V \in RH_s$, $Q/2 \le s < 2Q/(4-j)$. Then, for any positive integer N there exists a constant C_N such that*

$$|K_{\beta,j}(g,h)|$$
$$\le \frac{C_N}{\big(1+|g^{-1}h|m(g,V)\big)^N} \frac{1}{|g^{-1}h|^{n-4\beta}} \left(\int_{B(h,|g^{-1}h|/4)} \frac{V(\xi)^2 \mathrm{d}\xi}{|h^{-1}\xi|^{Q-(4-j)}} + \frac{1}{|g^{-1}h|^j}\right).$$

Moreover, the inequality above also holds with $m(g,V)$ replaced by $m(h,V)$.

Proof. Let $\Gamma_{\mathcal{L}}(g,h,\lambda)$ be the fundamental solution of $\mathcal{L}+\lambda$, where $\lambda \ge 0$. By the functional calculus, for any $j/4 < \beta < 1$,

$$\mathcal{L}^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta} (\mathcal{L}+\lambda)^{-1} \mathrm{d}\lambda.$$

Let $f \in C_0^\infty(\mathbb{H}^n)$. It follows from $(\mathcal{L}+\lambda)^{-1} f(g) = \int_{\mathbb{H}^n} \Gamma_{\mathcal{L}}(g,h,\lambda) f(h) \mathrm{d}h$ that

$$\mathcal{W}_{\beta,j}(f)(g) = \nabla^j_{\mathbb{H}^n} \mathcal{L}^{-\beta}(f)(g) = \int_{\mathbb{H}^n} K_{\beta,j}(g,h) f(h) \mathrm{d}h.$$

Then

$$\mathcal{W}^*_{\beta,j}(f)(g) = \int_{\mathbb{H}^n} K^*_{\beta,j}(g,h) f(h) \mathrm{d}h,$$

where

$$K^*_{\beta,j}(g,h) = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta} \nabla^j_{\mathbb{H}^n,h} \Gamma_{\mathcal{L}}(h,g,\lambda) \mathrm{d}\lambda$$

for $j/4 < \beta < 1$, and

$$K^*_{\beta,j}(g,h) = \nabla^j_{\mathbb{H}^n,h} \Gamma_{\mathcal{L}}(h,g,0)$$

for $\beta = 1$.

Fix $g_0, h_0 \in \mathbb{H}^n$. Let $u(h) = \Gamma_{\mathcal{L}}(h,g_0,\lambda)$ and $R = \frac{|g_0^{-1}h_0|}{4}$. It follows from the proof of Lemma 13 in [15] and Lemma 2.5 that

$$|\nabla^j_{\mathbb{H}^n} u(h_0)|$$
$$\le C \frac{1}{(1+\lambda^{\frac{1}{2}} R^2)^N \big(1+Rm(g_0,V)\big)^N} \left\{\frac{1}{R^{n-4}} \int_{B(h_0,R)} \frac{V(\xi)^2 \mathrm{d}\xi}{|h_0^{-1}\xi|^{n-4+j}} + \frac{1}{R^{n-4+j}}\right\}.$$

Then, for $\beta = 1$ we have

$$|K^*_{\beta,j}(g_0,h_0)| = |\nabla^j_{\mathbb{H}^n,h}\Gamma_{\mathcal{L}}(h_0,g_0,0)| \\ \leq C\frac{1}{\big(1+Rm(g_0,V)\big)^N}\frac{1}{R^{n-4}}\bigg\{\int_{B(h_0,R)}\frac{V(\xi)^2 d\xi}{|h_0^{-1}\xi|^{n-4+j}}+\frac{1}{R^j}\bigg\}.$$

Note that

$$\int_0^\infty \frac{\lambda^{-\beta}}{\big(1+\lambda^{\frac{1}{2}}R^2\big)^N}d\lambda \leq CR^{4\beta-4}.$$

So, for $j/4 < \beta < 1$, we get

$$|K^*_{\beta,j}(g_0,h_0)| \leq C\frac{C_N}{\big(1+Rm(g_0,V)\big)^N}\frac{1}{R^{n-4\beta}}\bigg\{\int_{B(h_0,R)}\frac{V(\xi)^2 d\xi}{|h_0^{-1}\xi|^{n-4+j}}+\frac{1}{R^j}\bigg\}.$$

□

Next, we give the maximal function estimates for $T^*_{\alpha,\beta,j},\ j=1,2,3.$

Lemma 4.2. *Suppose $V \in RH_s$ with $s \geq Q/2$. Let $0 < \alpha \leq 1 - j/4$, $j/4 < \beta \leq 1$, $\beta - \alpha \geq j/4$. Then for any $f \in C_0^\infty(\mathbb{H}^n)$,*

$$|T^*_{\alpha,\beta,j}(f)(g)| \leq C\{M_{\gamma q}(|f|^q)(g)\}^{\frac{1}{q}}$$

holds for some $1 < q < q_1$, where $\frac{1}{q_1} = 1 - \frac{1}{p_\alpha}$, $\frac{1}{p_\alpha} = \frac{2\alpha+2}{s} - \frac{4-j}{Q}$, and $\gamma = 4(\beta-\alpha) - j$.

Proof. Let $r = \rho(g)$, $C_k = \{h : 2^{k-1}r < |g^{-1}h| \leq 2^k r\}$. We choose t such that $1/t = 2/s - (4-j)/Q$. Then $1/t + 1/q_1 + (2\alpha)/s = 1$. By the Hölder inequality,

$$\begin{aligned}
&|T^*_{\alpha,\beta,j}(f)(g)| \\
&\leq \sum_{k=-\infty}^{+\infty}\int_{C_k}|K^*_{\beta,j}(g,h)|V(h)^{2\alpha}|f(h)|\mathrm{d}h \\
&\leq C\sum_{k=-\infty}^{+\infty}(2^k r)^Q\left(\frac{1}{(2^k r)^Q}\int_{C_k}|K^*_{\beta,j}(g,h)|^t\mathrm{d}h\right)^{1/t} \\
&\quad\times\left(\frac{1}{(2^k r)^Q}\int_{B(g,2^k r)}V(h)^s\mathrm{d}h\right)^{2\alpha/s}\left(\frac{1}{(2^k r)^Q}\int_{B(g,2^k r)}|f(h)|^{q_1}\mathrm{d}h\right)^{1/q_1}.
\end{aligned}$$

Due to $V \in RH_s$, we have

$$\left(\frac{1}{(2^k r)^Q}\int_{B(g,2^k r)}V(h)^s\mathrm{d}h\right)^{2\alpha/s} \leq C(2^k r)^{-4\alpha}\left(\frac{(2^k r)^2}{(2^k r)^Q}\int_{B(g,2^k r)}V(h)\mathrm{d}h\right)^{2\alpha}.$$

Let $\mathcal{I}_{4-j}(f)(g) = \int_{\mathbb{H}^n}\frac{f(h)\mathrm{d}h}{|g^{-1}h|^{n-(4-j)}}$. By Lemma 4.1, the Minkowski inequality and theorem of the fractional integral on Heisenberg group, we obtain

$$(2^k r)^Q\left(\frac{1}{(2^k r)^Q}\int_{C_k}|K^*_{\beta,j}(g,h)|^t\mathrm{d}h\right)^{1/t}$$

$$\leq C \frac{1}{(1+2^k)^N} \frac{1}{(2^k r)^{\frac{Q}{t}-4\beta}} \left(\left(\int_{C_k} \left(\mathcal{I}_{4-j}(V^2 \chi_{B(g,2^{k+1}r)})(h) \right)^t \mathrm{d}h \right)^{1/t} + (2^k r)^{\frac{Q}{t}-j} \right)$$

$$\leq C \frac{1}{(1+2^k)^N} \frac{1}{(2^k r)^{\frac{Q}{t}-4\beta}} \left((2^k r)^{\frac{2Q}{s}} \left(\frac{1}{(2^k r)^Q} \int_{B(g,2^{k+1}r)} V(h)^s \mathrm{d}h \right)^{2/s} + (2^k r)^{\frac{Q}{t}-j} \right)$$

$$\leq C \frac{1}{(1+2^k)^N} \frac{1}{(2^k r)^{\frac{Q}{t}-4\beta}} \left((2^k r)^{\frac{2Q}{s}-4} \left(\frac{(2^k r)^2}{(2^k r)^Q} \int_{B(g,2^k r)} V(h) \mathrm{d}h \right)^2 + (2^k r)^{\frac{Q}{t}-j} \right)$$

$$\leq C \frac{1}{(1+2^k)^N} \frac{1}{(2^k r)^{j-4\beta}} \left(\left(\frac{(2^k r)^2}{(2^k r)^Q} \int_{B(g,2^k r)} V(h) \mathrm{d}h \right)^2 + 1 \right).$$

For $k \geq 1$, by Lemma 2.2 we have

$$\frac{(2^k r)^2}{(2^j r)^Q} \int_{B(g,2^k r)} V(h) \mathrm{d}h \leq C 2^{k l_0}.$$

For the case $k \leq 0$, by Lemma 2.4 we get

$$\frac{(2^k r)^2}{(2^k r)^Q} \int_{B(g,2^k r)} V(h) \mathrm{d}h \leq C 2^{k(2-Q/s)} \frac{1}{r^{Q-2}} \int_{B(g,r)} V(h) \mathrm{d}h \leq C 2^{k(2-Q/s)}.$$

Then, taking $N \geq 2 l_0(\alpha+1)$ we get

$$|T^*_{\alpha,\beta,j}(f)(g)| \leq C \left(\sum_{k=1}^{\infty} \frac{1}{(2^k)^{N-2l_0(\alpha+1)}} + \sum_{k=-\infty}^{0} (2^k)^{2\alpha(2-\frac{Q}{s})} \right)$$

$$\times \frac{1}{(2^k r)^{j-4(\beta-\alpha)}} \left(\frac{1}{(2^k r)^Q} \int_{B(g,2^k r)} |f(h)|^{q_1} dz \right)^{1/q_1}$$

$$\leq C \left(\frac{1}{(2^k r)^{Q-(4(\beta-\alpha)-j)q_1)}} \int_{B(g,2^k r)} |f(h)|^{q_1} \mathrm{d}h \right)^{1/q_1}$$

$$\leq C \{ M_{\gamma q_1}(|f|^{q_1})(g) \}^{1/q_1},$$

where $\gamma = 4(\beta-\alpha) - j$. By the self-improvement of class RH_s we know that there exists some $1 < q < q_1$ such that

$$|T^*_{\alpha,\beta,j}(f)(g)| \leq C \{ M_{\gamma q}(|f|^q)(g) \}^{1/q}.$$

□

Proof of Theorem 1.2. By Lemma 4.2 we know

$$\|T^*_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \leq C \|f\|_{L^p(\mathbb{H}^n)}$$

holds for $q_1 \leq q < \frac{Q}{4(\beta-\alpha)-j}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$. By duality, we get

$$\|T_{\alpha,\beta,j}(f)\|_{L^q(\mathbb{H}^n)} \leq C \|f\|_{L^p(\mathbb{H}^n)}$$

for $\frac{Q}{Q-(4(\beta-\alpha)-j)} < q \leq p_\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}$. These conditions are equivalent to

$$1 < p \leq \frac{1}{\frac{1}{p_\alpha} + \frac{4(\beta-\alpha)-j}{Q}} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{Q}.$$

□

References

[1] Y. Hu and Y. Wang, *Estimates for the commutators of operator* $V^\alpha\nabla(-\Delta+V)^{-\beta}$, J. Inequal. Appl. **2019** (2019), Paper No. 126, 18 pp. https://doi.org/10.1186/s13660-019-2081-z

[2] H.-Q. Li, *Estimations* L^p *des opérateurs de Schrödinger sur les groupes nilpotents*, J. Funct. Anal. **161** (1999), no. 1, 152–218. https://doi.org/10.1006/jfan.1998.3347

[3] C.-C. Lin and H. Liu, $\mathrm{BMO}_L(\mathbb{H}^n)$ *spaces and Carleson measures for Schrödinger operators*, Adv. Math. **228** (2011), no. 3, 1631–1688. https://doi.org/10.1016/j.aim.2011.06.024

[4] Y. Liu, J. Huang, and D. Xie, *Some estimates of Schrödinger type operators on the Heisenberg group*, Arch. Math. (Basel) **94** (2010), no. 3, 255–264. https://doi.org/10.1007/s00013-009-0098-0

[5] Y. Liu and W. Xie, *Riesz transform related to a Schrödinger type operator on the Heisenberg group*, Georgian Math. J. **22** (2015), no. 3, 397–407. https://doi.org/10.1515/gmj-2015-0024

[6] Z. W. Shen, L^p *estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546.

[7] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.

[8] S. Sugano, *Estimates for the operators* $V^\alpha(-\Delta+V)^{-\beta}$ *and* $V^\alpha\nabla(-\Delta+V)^{-\beta}$ *with certain nonnegative potentials* V, Tokyo J. Math. **21** (1998), no. 2, 441–452. https://doi.org/10.3836/tjm/1270041824

[9] S. Sugano, L^p *estimates for some Schrödinger type operators and a Calderón-Zygmund operator of Schrödinger type*, Tokyo J. Math. **30** (2007), no. 1, 179–197. https://doi.org/10.3836/tjm/1184963655

[10] H. Wang, *Semigroup maximal functions, Riesz transforms, and Morrey spaces associated with Schrödinger operators on the Heisenberg groups*, J. Funct. Spaces **2020** (2020), Art. ID 8839785, 22 pp. https://doi.org/10.1155/2020/8839785

[11] Y. Wang, *Estimates for Riesz transforms associated with Schrödinger type operators*, Bull. Korean Math. Soc. **56** (2019), no. 5, 1117–1127. https://doi.org/10.4134/BKMS.b180836

[12] Y. Wang, *Estimates for the higher order Riesz transforms related to Schrödinger type operators*, Bull. Korean Math. Soc. **58** (2021), no. 1, 235–251. https://doi.org/10.4134/BKMS.b200240

[13] Y. Wang, *Some estimates of Riesz transforms related to Schrödinger-type operators*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 5, 3161–3174. https://doi.org/10.1007/s40840-021-01104-z

[14] Z. Wang, C. Sun, and P. Li, *Regularities of time-fractional derivatives of semigroups related to Schrodinger operators with application to Hardy-Sobolev spaces on Heisenberg groups*, J. Funct. Spaces **2020** (2020), Art. ID 8851287, 21 pp. https://doi.org/10.1155/2020/8851287

[15] L. Yu, L^p *estimates for Schrödinger type operators on the Heisenberg group*, J. Korean Math. Soc. **47** (2010), no. 2, 425–443. https://doi.org/10.4134/JKMS.2010.47.2.425

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