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ON TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF CERTAIN TYPES OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper, for a transcendental meromorphic function f and $a \in \mathbb{C}$, we have exhaustively studied the nature and form of solutions of a new type of non-linear differential equation of the following form which has never been investigated earlier:

$$f^n + af^{n-2}f' + P_d(z, f) = \sum_{i=1}^{k} p_i(z)e^{\alpha_i(z)},$$

where $P_d(z, f)$ is a differential polynomial of f, p_i's and α_i's are non-vanishing rational functions and non-constant polynomials, respectively. When $a = 0$, we have pointed out a major lacuna in a recent result of Xue [17] and rectifying the result, presented the corrected form of the same equation at a large extent. In addition, our main result is also an improvement of a recent result of Chen-Lian [2] by rectifying a gap in the proof of the theorem of the same paper. The case $a \neq 0$ has also been manipulated to determine the form of the solutions. We also illustrate a handful number of examples for showing the accuracy of our results.

1. Introduction and definitions

Let $\mathbb{C}$ denote the field of complex numbers and $\mathcal{M}(\mathbb{C})$ be the field of meromorphic functions on $\mathbb{C}$. Throughout this paper we consider $f \in \mathcal{M}(\mathbb{C})$. We assume that the readers are familiar with basic Nevanlinna theory and usual notations such as proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, first and second main theorems, etc (see [4]). Recall that $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic (linear) measure. A meromorphic function α is called a small function of f if and only if $T(r, \alpha) = S(r, f)$. The order of a meromorphic function f is denoted by $\rho(f)$ and defined by $\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$. For

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$\alpha \in \mathbb{C}$, the deficiency $\delta(\alpha, f)$ is defined as $\delta(\alpha, f) = 1 - \limsup\limits_{r\to\infty} \frac{N(r,\alpha;f)}{T(r,f)}$. Nowadays differential equation plays a prominent role in many disciplines and so the study on the different features of differential equation over $\mathbb{C}$ has become an interesting topic. Speculating over the existence of solutions of non-linear differential equation and subsequently finding the exact form of the same equation are really challenging problems. In the present paper we wish to contribute in this perspective. To this end, we denote by $P_d(z, f)$, the non-linear differential polynomial of $f(z)$ of degree d defined by

$$P_d(z,f) = \sum_{\lambda\in\Lambda} a_\lambda \prod_{i=0}^{n_1} \left(f^{(i)}(z)\right)^{\lambda_i}, \tag{1.1}$$

where a_λ's are rational functions, Λ is the index set of non-negative integers with finite cardinality and $\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n_1})$ also $d := \deg(P_d(z, f)) = \max\limits_{\lambda\in\Lambda} \{\sum_{i=0}^{n_1} \lambda_i\}$.

For the last two decades researchers have extensively studied the differential equation of the following form

$$f^n + P_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{1.2}$$

where $P_d(z, f)$ is defined as in (1.1) with some restriction on degree d and $p_1(z)$, $p_2(z)$ are non-zero rational functions and $\alpha_1(z), \alpha_2(z)$ are non-constant polynomials. In this paper, we also like to contribute in this aspect under a more general setting.

2. Backgrounds and main results

In 2006, on the existence of solution of differential equation, Li-Yang [9] obtained the following result.

Theorem A ([9]). *Let $n \geq 4$ be an integer. Consider the differential equation* (1.2)*, where $d \leq n-3$, $p_j(z)$ $(j = 1, 2)$ are two non-vanishing polynomials and $\alpha_j(z) := \alpha_j z$ $(j = 1, 2)$ are two non-zero one degree polynomials such that $\frac{\alpha_1}{\alpha_2}$ is not a rational number. Then the equation* (1.2) *has no transcendental entire solutions.*

In 2011, Li [8] removed the extra supposition "$\frac{\alpha_1}{\alpha_2}$ is not rational" in Theorem A and established the form of the meromorphic solution.

Theorem B ([8]). *Let $n \geq 2$ be an integer. Consider the differential equation* (1.2)*, where $d \leq n-2$, $p_j(z)$ $(j = 1, 2)$ are two non-zero constants and $\alpha_j(z) := \alpha_j z$ $(j = 1, 2)$ are two non-zero one degree polynomials such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the equation* (1.2) *and satisfying $N(r, f) = S(r, f)$, then one of the following holds*:

(i) $f(z) = c_0 + c_1 e^{\alpha_1 z/n}$;

(ii) $f(z) = c_0 + c_2 e^{\alpha_2 z/n}$;

(iii) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$, *and* $\alpha_1 + \alpha_2 = 0$,
where c_0 *is a small function of* $f(z)$ *and* c_1, c_2 *are constants satisfying* $c_1^n = p_1$, $c_2^n = p_2$.

In 2013, considering $p_1(z)$, $p_2(z)$ as non-vanishing rational functions and $\alpha_1(z)$, $\alpha_2(z)$ as non-constant polynomials, Liao-Yang-Zhang [11] generalized Theorem B as follows.

Theorem C ([11])**.** *Let* $n \geq 3$ *be an integer,* $P_d(z, f)$ *be a differential polynomial in* $f(z)$ *of degree* d *with rational functions as its coefficients. Suppose* $p_1(z)$, $p_2(z)$ *are non-vanishing rational functions and* $\alpha_1(z), \alpha_2(z)$ *are non-constant polynomials. If* $d \leq n - 2$ *and the differential equation* (1.2) *admits a meromorphic function solution* $f(z)$ *with finitely many poles, then* $\frac{\alpha_1'}{\alpha_2'}$ *is a rational number. Furthermore, only one of the following four cases holds*:

(i) $f(z) = q(z)e^{P(z)}$, $\frac{\alpha_1'}{\alpha_2'} = 1$, *where* $q(z)$ *is a rational function and* $P(z)$ *is a polynomial with* $nP' = \alpha_1' = \alpha_2'$;

(ii) $f(z) = q(z)e^{P(z)}$, *either* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$ *or* $\frac{k}{n}$, *where* $q(z)$ *is a rational function,* k *is an integer with* $1 \leq k \leq d$ *and* $P(z)$ *is a polynomial with* $nP' = \alpha_1'$ *or* α_2';

(iii) f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1'\right) f + \varphi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1}$ *or* f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) f + \varphi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$, *where* φ *is a rational function*;

(iv) $f(z) = c_1(z)e^{\beta(z)} + c_2(z)e^{-\beta(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = -1$, *where* $c_1(z), c_2(z)$ *are rational functions and* $\beta(z)$ *is a polynomial with* $n\beta' = \alpha_1'$ *or* α_2'.

In 2018, Zhang [19] established the following result in this direction.

Theorem D ([19])**.** *Under the same assumption as in Theorem C if* $n \geq 4$ *is an integer,* $d \leq n - 3$, *and the complex differential equation* (1.2) *admits a transcendental meromorphic function solution* f *with finitely many poles, then* $\frac{\alpha_1'}{\alpha_2'}$ *is a rational number and* $f(z)$ *must be of the following form*:

$$f(z) = q(z)e^{P(z)},$$

where $q(z)$ *is a non-vanishing rational function and* $P(z)$ *is a non-constant polynomial. Moreover, only one of the following two cases holds*:

(i) $\frac{\alpha_1'}{\alpha_2'} = 1$, $P_d(z, f) \equiv 0$ *and* $nP' = \alpha_1' = \alpha_2'$;

(ii) $\frac{\alpha_1'}{\alpha_2'} = \frac{t}{n}$, *where* t *is an integer satisfying* $1 \leq t < d$, $P_d(z, f) \equiv p_1(z)e^{\alpha_1(z)}$ *and* $nP' = \alpha_2'$ *or* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{t}$, *where* t *is an integer satisfying* $1 \leq t < d$, $P_d(z, f) \equiv p_2(z)e^{\alpha_2(z)}$ *and* $nP' = \alpha_1'$.

For more results related to this area readers can see [1, 5, 6, 12–14, 16]. We note that the right hand side of (1.2) consisting of two exponential terms. So investigations on the case when the right hand side of (1.2) contains three or more exponential terms attract the researchers. In this perspective, in 2020, Xue [17] found the following result:

Theorem E ([17]). *Let $n \geq 2$ and $P_d(z,f)$ be an algebraic differential polynomial in $f(z)$ of degree $d \leq n-1$. Suppose that p_j, α_j are non-zero constants for $j = 1,2,3$ and $|\alpha_1| > |\alpha_2| > |\alpha_3|$. If $f(z)$ is a transcendental entire solution of the differential equation*

$$f^n + P_d(z,f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z},$$

then $f(z) = a_1 e^{\alpha_1 z/n}$, where a_1 is non-zero constant such that $a_1^n = p_1$, and α_j are in one line for $j = 1,2,3$.

Remark 2.1. Although Theorem E is a novel approach in this direction but we find that there is a major drawback in the same theorem. The following example shows that Theorem E does not hold good.

Example 2.1. The function $f(z) = e^z + 1$ is a transcendental entire solution of

$$f^3 + 4f'f + f' - f = e^{3z} + 7e^{2z} + 7e^z.$$

But $e^z + 1$ is not of the form of $f(z)$ given in Theorem E.

Also in 2020, Chen-Lian [2] studied on the differential equation with three exponential terms in the right hand side and established the following:

Theorem F ([2]). *Let $n \geq 5$ be an integer, $P_d(z,f)$ be a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients. Suppose $p_j(z)$ ($j = 1,2,3$) are non-vanishing rational functions and $\alpha_j(z)$ ($j = 1,2,3$) are non-constant polynomials such that $\alpha_j'(z)$ ($j = 1,2,3$) are distinct each other. If $d \leq n-4$ and the differential equation*

$$f^n + P_d(z,f) = \sum_{j=1}^{3} p_j(z) e^{\alpha_j(z)}$$

admits a transcendental meromorphic function solution f with finitely many poles, then $\frac{\alpha_1'}{\alpha_2'}$, $\frac{\alpha_2'}{\alpha_3'}$ are rational numbers and $f(z)$ must be of the following form:

$$f(z) = q(z) e^{P(z)},$$

where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial. Moreover, there must exist positive integers l_0, l_1, l_2 with $\{l_0, l_1, l_2\} = \{1,2,3\}$ and distinct integers k_1, k_2 with $1 \leq k_1, k_2 \leq d$ such that $\alpha_{l_0}' : \alpha_{l_1}' : \alpha_{l_2}' = n : k_1 : k_2$, $nP' = \alpha_{l_0}'$ and $P_d(z,f) \equiv p_{l_1}(z) e^{\alpha_{l_1}(z)} + p_{l_2}(z) e^{\alpha_{l_2}(z)}$.

Remark 2.2. First of all we would like to mention that Chen-Lian [2] did not point out the lacuna of Theorem E, but their result automatically rectifies Theorem E. Next we note that to prove Theorem F, the basic methods used by the authors are well known and is same as that adopted in several papers like [7,11]. The only innovative ideas exhibited by Chen-Lian [2] was to manipulate the notion of Cramer's rule in the proof. Unfortunately, at the time of execution of Cramer's rule, Chen-Lian [2] made a mistake. Consider equation (3.2) (see [2, p. 1067]). In view of Crammer's rule, if $D_0 \neq 0$, then only from equation (3.2), one can write equation (3.3) (see [2, p. 1067]). But in the line before (3.4), the authors used $D_0 = 0$ in (3.3) to deduce $D_1 = 0$, which is not at all acceptable and so to tackle this situation further investigations are needed.

Next let us accumulate the following points:

(i) From Remark 2.2 we see that the proof of Theorem F is incomplete and it is high time to dispel all the confusions cropped up from Theorems E-F.

(ii) Considering all the results so far stated, a natural inquisition would be to investigate the case when the right hand side of the differential equation in Theorem F contains k-terms.

(iii) In 2011, Li [8] proposed an open question that how to find the solutions of (1.2), where p_1, p_2 are constants and degree of the differential term $P_d(z,f)$ is equal to $n-1$. But till now, without any extra supposition, nobody has been able to obtain any fruitful result in the literature. So Li's [8] question is still open.

In this respect, in view of previous points, considering the special case $f^{n-2}f' + P_d(z,f)$, it will be interesting to characterize the solutions of a more general form of differential equation, namely,

$$f^n + af^{n-2}f' + P_d(z,f) = \sum_{i=1}^{k} p_i(z)e^{\alpha_i(z)}, \tag{2.1}$$

where $a \in \mathbb{C}$, $P_d(z,f)$ is defined as in (1.1), $p_i(z)$ $(i = 1, 2, \ldots, k)$ are non-vanishing rational functions and $\alpha_i(z)$ $(i = 1, 2, \ldots, k)$ are distinct non-constant polynomials.

The above three points are the main motivations of writing this paper. Our main result is the following:

Theorem 2.1. *Consider the non-linear differential equation* (2.1) *with* $\deg(\alpha_i - \alpha_j) \geq 1$ $(1 \leq i \neq j \leq k)$.

(I) *Let* $a = 0$ *and* $d \leq n-k-1$. *Then the following two cases hold*:

(IA) *Consider* $k = 1$ *and* $n \geq 2$. *If* (2.1) *admits a meromorphic solution* f *with finitely many poles, then* f *is of the form* $f(z) = q(z)e^{P(z)}$, *where* $q(z)$ *is a non-vanishing rational function and* $P(z)$ *is a non-constant polynomial with* $q^n(z) = p_1(z)$, $nP(z) = \alpha_1(z)$ *and* $P_d(z,f) \equiv 0$.

(IB) *Consider* $k \geq 2$ *and* $n \geq k+2$. *If* (2.1) *has a meromorphic solution* $f(z)$ *with finitely many poles, then* $\frac{\alpha_i'}{\alpha_j'}$ $(1 \leq i \neq j \leq k)$ *are rational numbers and* $f(z)$ *must be of the form*:

$$f(z) = q(z)e^{P(z)},$$

where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial.
Also, if we rearrange $\{1,2,\ldots,k\}$ to $\{\tau_0,\tau_1,\ldots,\tau_{k-1}\}$ such that

$$p_i(z)e^{\alpha_i(z)} = p_{\tau_{i-1}}(z)e^{\alpha_{\tau_{i-1}}(z)} \text{ for } i=1,2,\ldots,k,$$

then $q^n = e^{-\mathcal{A}_0}p_{\tau_0}$, $\mathcal{A}_0$ is any constant with

$$nP'(z)=\alpha'_{\tau_0}(z) \text{ and } P_d(z,f)=\sum_{i=1}^{k-1} p_{\tau_i}(z)e^{\alpha_{\tau_i}(z)}.$$

(II) *Next let $a\neq 0$ and $d\leq n-k-3$. Then the following three cases hold*:

(IIA) *Suppose $k=1$ and $n\geq 5$. Then the equation (2.1) does not admit any meromorphic solution with finitely many poles.*

(IIB) *Suppose $k=2$ and $n\geq 6$. If (2.1) has a meromorphic solution $f(z)$ with finitely many poles, then $\frac{\alpha'_1}{\alpha'_2}$ are rational numbers and $f(z)$ must be of the form*:

$$f(z)=q(z)e^{P(z)},$$

where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial and $P_d(z,f)\equiv 0$. Also,

(i) $q^n = e^{-\mathcal{B}_1}p_1$, $aq^{n-2}(q'+qP') = e^{-\mathcal{B}_2}p_2$, $\frac{\alpha'_1}{\alpha'_2}=\frac{n}{n-1}$, *where $\mathcal{B}_i$ $(i=1,2)$ are constants.*

(ii) *or* $q^n = e^{-\tilde{\mathcal{B}}_1}p_1$, $aq^{n-2}(q'+qP') = e^{-\tilde{\mathcal{B}}_2}p_2$, $\frac{\alpha'_1}{\alpha'_2}=\frac{n-1}{n}$, *where $\tilde{\mathcal{B}}_i$ $(i=1,2)$ are constants.*

(IIC) *Suppose $k\geq 3$ and $n\geq k+4$. If (2.1) has a meromorphic solution $f(z)$ with finitely many poles, then $\frac{\alpha'_i}{\alpha'_j}$ $(1\leq i\neq j\leq k)$ are rational numbers and $f(z)$ must be of the form*:

$$f(z)=q(z)e^{P(z)},$$

where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial.
Also, if we rearrange $\{1,2,\ldots,k\}$ to $\{\mu,\nu,\kappa_1,\kappa_2,\ldots,\kappa_{k-2}\}$ such that $p_1(z)e^{\alpha_1(z)} = p_\mu(z)e^{\alpha_\mu(z)}$, $p_2(z)e^{\alpha_2(z)} = p_\nu(z)e^{\alpha_\nu(z)}$, $p_i(z)e^{\alpha_i(z)} = p_{\kappa_{i-2}}(z)e^{\alpha_{\kappa_{i-2}}(z)}$ for $i=3,4,\ldots,k$, then $q^n=e^{-\mathcal{C}_\mu}p_\mu$ and $aq^{n-2}(q'+qP')=e^{-\mathcal{C}_\nu}p_\nu$, where $\mathcal{C}_\mu$, $\mathcal{C}_\nu$ are any constants with

$$nP'(z)=\alpha'_\mu(z),\ (n-1)P'(z)=\alpha'_\nu(z) \text{ and}$$

$$P_d(z,f)=\sum_{i=1}^{k-2} p_{\kappa_i}(z)e^{\alpha_{\kappa_i}(z)}.$$

Corollary 2.1. *Putting $k=3$, in* (IB) *of Theorem 2.1, we have Theorem F. Therefore, our result is a huge improvement of Theorem F.*

The following examples show that all conclusions of Theorem 2.1 actually occurs for the cases $a = 0$ and $a \neq 0$, respectively:

Example 2.2. Let $k = 2$ and $n = 4$. Then $f(z) = \frac{z}{z+1}e^{z^2+2}$ satisfies the differential equation

$$f^4 + P_d(z, f) = e^3 \left(\frac{z}{z+1}\right)^4 e^{\alpha_1} + \frac{2z^2(z+1)+1}{(z+1)^2} e^{\alpha_2},$$

where $P_d(z, f) = f'$, $\alpha_1 = 4z^2 + 5$ and $\alpha_2 = z^2 + 2$. Clearly, $\frac{\alpha_1'}{\alpha_2'}$ is rational.

Example 2.3. Let $k = 2$ and $n = 6$. Then it is easy to verify that $f(z) = e^{\frac{2z}{3}}$ satisfies the differential equation

$$f^6 + af^4 f' = e^{4z} + \frac{2a}{3} e^{\frac{10z}{3}}.$$

Note that here $P_d(z, f) \equiv 0$.

Example 2.4. Let $k = 3$ and $n = 7$. Then choosing $P_d(z, f) = f''$, one can show that $f(z) = e^{\frac{2z}{7}}$ satisfies the differential equation

$$f^7 + af^5 f' + P_d(z, f) = e^{2z} + \frac{2a}{7} e^{\frac{12z}{7}} + \frac{4}{49} e^{\frac{2z}{7}}.$$

The next two examples show that in (I) of Theorem 2.1, the bound $d \leq n-k-1$ can not be extended to $d \leq n-k$ and it is the best possible estimation.

Example 2.5. Let $k = 2$, $n = 4$ and $d = 2$. Then it is easy to verify that $f(z) = e^z + z + 1$ is a solution of the differential equation

$$f^4 + P_d(z, f) = e^{4z} + 4(z+1)e^{3z},$$

where $P_d(z, f) = -6(z+1)^2(f'')^2 - 3(z+1)^3 f'' - (z+1)^3 f$, but $e^z + z + 1$ can not be expressed as $q(z)e^{P(z)}$.

Example 2.6. Let $k = 3$, $n = 5$ and $d = 2$. Then it is easy to verify that $f(z) = e^z - 1$ is a solution of the differential equation

$$f^5 + P_d(z, f) = e^{5z} - 5e^{4z} + 10e^{3z},$$

where $P_d(z, f) = 10ff' + 5f' + 1$, but $e^z - 1$ can not be written as $q(z)e^{P(z)}$.

Notice that, if $d \leq n - k - 1$, then the bound $n \geq k + 2$ can not be replaced by $n \geq k + 1$, because in that case the equation (2.1) is not at all a differential equation. Now if we consider $n = k + 1$ and $d = n - k$, then the next example shows that the conclusion (IB) of Theorem 2.1 cease to be hold. So we can say the bound for n is the best possible.

Example 2.7. Let $k = 4$, $n = 5$ and $d = 1$. Here $f(z) = e^z + z - 1$ is a solution of the differential equation

$$f^5 + P_d(z, f) = e^{5z} + 5(z-1)e^{4z} + 10(z-1)^2 e^{3z} + 10(z-1)^3 e^{2z},$$

where $P_d(z, f) = -4(z-1)^4 f'' - (z-1)^4 f$, but $e^z + z - 1$ can not be expressed as $q(z)e^{P(z)}$.

The following examples show that in (II) of Theorem 2.1, the bound $d \leq n-k-3$ is sharp.

Example 2.8. Let $k=3$, $n=7$ and $d=2$. Then it is easy to verify that $f(z)=e^z+1$ is a solution of the differential equation

$$f^7 - \frac{7}{2} f^5 f' + P_d(z,f) = e^{7z} + \frac{7}{2} e^{6z} + \frac{7}{2} e^{5z},$$

where $P_d(z,f) = -\frac{7}{2} f f' - 1$, but e^z+1 is not of the form $q(z)e^{P(z)}$. Here we note that $2 = d > n-k-3 = 1$.

3. Lemmas

The following lemma can be easily derived from the proof of the Clunie lemma (see [3, 6]).

Lemma 3.1. *Let $f(z)$ be a transcendental meromorphic solution of the differential equation*

$$f^n(z) P(z,f) = Q(z,f),$$

where $P(z,f), Q(z,f)$ are polynomials in f and its derivatives such that the coefficients are small meromorphic functions of f. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(z,f)) = S(r,f)$$

for all r out of a possible exceptional set of finite logarithmic measure. In particular, if f is finite order, then

$$m(r, P(z,f)) = O(\log\ r) \text{ as } r \to \infty.$$

Lemma 3.2 ([15, Cramer's rule])**.** *Consider the system of linear equation $AX = B$, where*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ \text{ and } \ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $\det(A) \neq 0$, then the system has unique solution

$$(x_1, x_2, \ldots, x_n) = \left(\frac{\det(A_1)}{\det(A)}, \frac{\det(A_2)}{\det(A)}, \ldots, \frac{\det(A_n)}{\det(A)} \right),$$

where

$$A_i = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{pmatrix} \ \text{ for } \ i = 1, 2, \ldots, n.$$

Lemma 3.3 ([18])**.** *Let* $a_j(z)$ *be an entire function of finite order* $\leq \rho$*. Let* $g_j(z)$ *be entire and* $g_k(z) - g_j(z)$*,* $(k \neq j)$ *be a transcendental entire function or a polynomial of degree greater than* ρ*. Then*

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)} = a_0(z),$$

holds only when

$$a_0(z) = a_1(z) = \cdots = a_n(z) \equiv 0.$$

Lemma 3.4 ([18, Hadamard's factorization theorem])**.** *Let* $f(z)$ *be a meromorphic function of finite order* ρ *and*

$$f(z) = a_k z^k + a_{k+1}z^{k+1} + \cdots \quad (\textit{where } a_k \neq 0)$$

in the neighborhood of $z = 0$*. Suppose that* $b_1, b_2, \ldots$ *are non-zero zeros of* $f(z)$ *and* $c_1, c_2, \ldots$ *are non-zero poles of* $f(z)$*. Then*

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ *is a canonical product of non-zero zeros of* $f(z)$*,* $P_2(z)$ *is a canonical product of nonzero poles of* $f(z)$*, and* $Q(z)$ *is a polynomial of degree at most* ρ*.*

4. Proof of theorems

Proof of Theorem 2.1. In view of Remark 2.2, a detail proof is required for the sake of general reader.

To prove this theorem, we distinguish the following cases:

Case I: Let $a = 0$. Then we denote

$$h(z) = f^n + P_d(z, f). \tag{4.1}$$

Let f be a rational solution of the non-linear differential equation (2.1). Then it is easy to see that $h(z)$ is a small function of $p_i(z)e^{\alpha_i(z)}$ $(i = 1, 2, \ldots, k)$. As $\deg\{\alpha_i(z) - \alpha_j(z)\} \geq 1$, so by Lemma 3.3, we get $p_i(z) \equiv 0$ $(i = 1, 2, \ldots, k)$, which contradicts $p_i(z)$ $(i = 1, 2, \ldots, k)$ are non-vanishing rational functions. Hence f must be a transcendental.

Now we will show that order of f is finite. As f has finitely many poles, so

$$\begin{aligned} nT(r, f) &= m(r, f^n) + S(r, f) \\ &\leq T\left(r, \sum_{i=1}^{k} p_i(z)e^{\alpha_i(z)}\right) + m(r, P_d(z, f)) + S(r, f) \\ &\leq Ar^{\eta} + dT(r, f) + S(r, f), \end{aligned}$$

where $\eta = \max\{\deg \alpha_1, \deg \alpha_2, \ldots, \deg \alpha_k\}$. Hence $(n - d)T(r, f) \leq Ar^{\eta} + S(r, f)$ and f is of finite order.

Sub-case IA: First suppose that $k = 1$. Then the result follows from Theorem 1.7 of [10].

Sub-case IB: Next suppose that $k \geq 2$. For $k = 2$, we refer Theorem D. So we assume $k \geq 3$. Differentiating (2.1), $k-1$ times, we get the following system of equations

$$(4.2)\quad \begin{cases} h = & \sum_{i=1}^{k} p_i e^{\alpha_i}, \\ h' = & \sum_{i=1}^{k} \left[p_i' + p_i\alpha_i'\right] e^{\alpha_i}, \\ h'' = & \sum_{i=1}^{k} \left[p_i'' + 2p_i'\alpha_i' + p_i\alpha_i'' + p_i(\alpha_i')^2\right] e^{\alpha_i}, \\ \vdots & \\ h^{(k-1)} = & \sum_{i=1}^{k} \Big[p_i^{(k-1)} + p_i^{(k-2)}Q_1(\alpha_i') + p_i^{(k-3)}Q_2(\alpha_i', \alpha_i'') \\ & \qquad + \cdots + p_i Q_{k-1}(\alpha_i', \alpha_i'', \ldots, \alpha_i^{(k-1)})\Big] e^{\alpha_i}, \end{cases}$$

where $Q_j(\alpha_i', \alpha_i'', \ldots, \alpha_i^{(j)})$ are differential polynomials of α_i' with degree j ($j = 1, 2, \ldots, k-1$; $i = 1, 2, \ldots, k$). From the system (4.2), the determinant of the coefficient matrix is

$$D_0 = \begin{vmatrix} p_1 & \cdots & p_k \\ p_1' + p_1\alpha_1' & \cdots & p_k' + p_k\alpha_k' \\ \vdots & \ddots & \vdots \\ p_1^{(k-1)} + p_1^{(k-2)}Q_1(\alpha_1') + \cdots & \cdots & p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots \\ +p_1Q_{k-1}(\alpha_1', \alpha_1'', \ldots, \alpha_1^{(k-1)}) & & +p_kQ_{k-1,k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) \end{vmatrix}.$$

It is clear that D_0 is a rational function. Let us consider

$$D_1 = \begin{vmatrix} h & p_2 & \cdots & p_k \\ h' & p_2' + p_2\alpha_2' & \cdots & p_k' + p_k\alpha_k' \\ \vdots & \vdots & \ddots & \vdots \\ h^{(k-1)} & p_2^{(k-1)} + p_2^{(k-2)}Q_1(\alpha_2') + \cdots + & \cdots & p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots + \\ & p_2Q_{k-1}(\alpha_2', \alpha_2'', \ldots, \alpha_2^{(k-1)}) & & p_kQ_{k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) \end{vmatrix}.$$

Now we distinguish the following two cases:

Sub-case IB.1: If $D_0 \equiv 0$, then the rank of the coefficient matrix of the system (4.2) is equal to $m \leq k-1$. As the system of equation has a solution, so the rank of the augmented matrix

$$\widetilde{D_0} = \left(\begin{array}{ccc|c} p_1 & \cdots & p_k & h \\ p_1' + p_1\alpha_1' & \cdots & p_k' + p_k\alpha_k' & h' \\ \vdots & \ddots & \vdots & \vdots \\ p_1^{(k-1)} + p_1^{(k-2)}Q_1(\alpha_1') + \cdots + & \cdots & p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots + & h^{(k-1)} \\ p_1Q_{k-1}(\alpha_1', \alpha_1'', \ldots, \alpha_1^{(k-1)}) & & p_kQ_{k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) & \end{array}\right)$$

must be equal to $m \leq k-1$. So, all $k \times k$ minors of $\widetilde{D_0}$ are zero, which implies

$$\begin{vmatrix} p_2 & \cdots & p_k & h \\ p_2' + p_2\alpha_2' & \cdots & p_k' + p_k\alpha_k' & h' \\ \vdots & \ddots & \vdots & \vdots \\ \begin{matrix} p_2^{(k-1)} + p_2^{(k-2)}Q_1(\alpha_2') + \cdots \\ +p_2Q_{k-1}(\alpha_2', \alpha_2'', \ldots, \alpha_2^{(k-1)}) \end{matrix} & \cdots & \begin{matrix} p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots \\ +p_kQ_{k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) \end{matrix} & h^{(k-1)} \end{vmatrix} = 0,$$

which implies, $D_1 \equiv 0$.
So,

$$M_{11}h - M_{21}h' + \cdots + (-1)^{k-1}M_{k1}h^{(k-1)} = 0, \tag{4.3}$$

where

$$M_{11} = \begin{vmatrix} p_2' + p_2\alpha_2' & \cdots & p_k' + p_k\alpha_k' \\ \vdots & \ddots & \vdots \\ \begin{matrix} p_2^{(k-1)} + p_2^{(k-2)}Q_1(\alpha_2') + \cdots \\ +p_2Q_{k-1}(\alpha_2', \alpha_2'', \ldots, \alpha_2^{(k-1)}) \end{matrix} & \cdots & \begin{matrix} p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots \\ +p_kQ_{k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) \end{matrix} \end{vmatrix},$$

$$M_{21} = \begin{vmatrix} p_2 & \cdots & p_k \\ p_2'' + 2p_2'\alpha_2' + p_2\alpha_2'' + p_2(\alpha_2')^2 & \cdots & p_k'' + 2p_k'\alpha_k' + p_k\alpha_k'' + p_k(\alpha_k')^2 \\ \vdots & \ddots & \vdots \\ \begin{matrix} p_2^{(k-1)} + p_2^{(k-2)}Q_1(\alpha_2') + \cdots \\ +p_2Q_{k-1}(\alpha_2', \alpha_2'', \ldots, \alpha_2^{(k-1)}) \end{matrix} & \cdots & \begin{matrix} p_k^{(k-1)} + p_k^{(k-2)}Q_1(\alpha_k') + \cdots \\ +p_kQ_{k-1}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-1)}) \end{matrix} \end{vmatrix}$$

and so

$$M_{k1} = \begin{vmatrix} p_2 & \cdots & p_k \\ p_2' + p_2\alpha_2' & \cdots & p_k' + p_k\alpha_k' \\ \vdots & \ddots & \vdots \\ \begin{matrix} p_2^{(k-2)} + p_2^{(k-3)}Q_1(\alpha_2') + \cdots \\ +p_2Q_{k-2}(\alpha_2', \alpha_2'', \ldots, \alpha_2^{(k-2)}) \end{matrix} & \cdots & \begin{matrix} p_k^{(k-2)} + p_k^{(k-3)}Q_1(\alpha_k') + \cdots \\ +p_kQ_{k-2}(\alpha_k', \alpha_k'', \ldots, \alpha_k^{(k-2)}) \end{matrix} \end{vmatrix},$$

are rational functions.

Substituting the expression (4.1) of $h(z)$ into (4.3), we get

$$M_{11}f^n - M_{21}(f^n)' + \cdots + (-1)^{k-1}M_{k1}(f^n)^{(k-1)} = Q_1, \tag{4.4}$$

where

$$Q_1 = -\left[M_{11}P_d(z,f) - M_{21}P_d'(z,f) + \cdots + (-1)^{k-1}M_{k1}P_d^{(k-1)}(z,f)\right]$$

is a differential polynomial in f with rational functions as its coefficients and degree of $Q_1 \leq d$.

One can easily check that

$$(f^n)^{(t)} = (nf^{n-1}f')^{(t-1)} = n\sum_{i=0}^{t-1}\binom{t-1}{i}(f^{n-1})^{(i)}f^{(t-i)} \tag{4.5}$$

$$= n\Big[f^{n-1}f^{(t)} + (t-1)(n-1)f^{n-2}f'f^{(t-1)}$$
$$+\sum_{i=2}^{t-1}\binom{t-1}{i}f^{(t-i)}\Bigg\{(n-1)f^{n-2}f^{(i)} + \sum_{j=2}^{i-1}\sum_{\lambda}\gamma_{j\lambda}f^{n-j-1}(f')^{\lambda_{j,1}}$$
$$(f'')^{\lambda_{j,2}}\cdots(f^{(i-1)})^{\lambda_{j,i-1}} + (n-1)(n-2)\cdots(n-i)f^{n-i-1}(f')^{i}\Bigg\}\Big],$$

where $\gamma_{j\lambda}$ are positive integers, $\lambda_{j,1},\ldots,\lambda_{j,i-1}$ are non-negative integers and sum $\sum\limits_{\lambda}$ is carried out such that

$$\lambda_{j,1}+\lambda_{j,2}+\cdots+\lambda_{j,i-1} = j \text{ and } \lambda_{j,1}+2\lambda_{j,2}+\cdots+(i-1)\lambda_{j,i-1} = i. \tag{4.6}$$

Now we define

$$\xi_t = \frac{(f^n)^{(t)}}{f^{n-k+1}} \tag{4.7}$$

for $t = 1,2,\ldots,k-1$; $k \geq 3$. Using (4.7) in (4.4), we have

$$f^{n-k+1}R_1 = Q_1, \tag{4.8}$$

where

$$R_1 = M_{11}f^{k-1} - M_{21}\xi_1 + \cdots + (-1)^{k-1}M_{k1}\xi_{k-1}. \tag{4.9}$$

As $d \leq n-k-1$ and f is finite order, so combining (4.8) with Lemma 3.1 we get

$$m(r,R_1) = O(\log r).$$

On the other hand, as we assume, f has finitely many poles, so we have

$$T(r,R_1) = m(r,R_1) + N(r,R_1) = O(\log r),$$

i.e., R_1 is a rational function. Next we study the following two subcases:

Sub-case IB.1.1: If $R_1(z) \equiv 0$, then from (4.9) we have

$$M_{11}f^{k-1} = -\left[-M_{21}\xi_1 + \cdots + (-1)^{k-1}M_{k1}\xi_{k-1}\right]. \tag{4.10}$$

We will show that f has at most finitely many zeros. On the contrary, suppose that f has infinitely many zeros. So we can consider a point z_1 such that $f(z_1) = 0$ but z_1 is neither a zero nor a pole of M_{j1} $(j = 1,2,\ldots,k)$.

Now let, $M_{k1} \not\equiv 0$. From the construction of ξ_{k-1} and (4.5) we know ξ_{k-1} contains one term, corresponding to $i = k-2$ in which, power of f is 0. Considering this term in ξ_{k-1}, we see that $f'(z_1) = 0$, which implies z_1 is a multiple zero of f of multiplicity $p_1 \geq 2$. It follows that z_1 is a zero of the left hand side of (4.10) with multiplicity $(k-1)p_1$, where as it is a zero of the right hand side of (4.10) with multiplicity $(k-1)(p_1-1)$, a contradiction.

Next, assume $M_{k1} \equiv 0$. If z_1 is a simple zero, then z_1 is a zero with multiplicity $k-1$ of left hand side of (4.10) and a zero with multiplicity 1 of right hand side of (4.10), which is a contradiction. If z_1 is a multiple zero with multiplicity $q_1 \geq 2$, then z_1 is a zero with multiplicity $(k-1)q_1$ of left hand side of (4.10) and a zero with multiplicity $(k-1)q_1-(k-2)$ of right hand side of (4.10), which is a contradiction. Thus, f has at most finitely many zeros.

Sub-case IB.1.2: If $R_1(z)\not\equiv 0$, then (4.8) becomes

$$f^{n-k}(fR_1)=Q_1. \tag{4.11}$$

By Lemma 3.1, we have

$$m(r,fR_1)=O(\log r).$$

From our assumption, since fR_1 has finitely many poles. Then

$$T(r,fR_1)=m(r,fR_1)+N(r,fR_1)=O(\log r).$$

Therefore, fR_1 is a rational function, which contradicts that f is transcendental.

Sub-case IB.2: If $D_0\not\equiv 0$, then by Lemma 3.2 we get

$$D_0e^{\alpha_1}=D_1. \tag{4.12}$$

Differentiating (4.12), we have

$$(D_0'+D_0\alpha_1')e^{\alpha_1}=D_1'. \tag{4.13}$$

Eliminating e^{α_1} from (4.12) and (4.13), we get

$$D_1'D_0-D_1D_0'=\alpha_1'D_1D_0. \tag{4.14}$$

Substituting,
$D_1=M_{11}h-M_{21}h'+\cdots+(-1)^{k-1}M_{k1}h^{(k-1)}$ and $D_1'=M_{11}'h+(M_{11}-M_{21}')h'-(M_{21}-M_{31}')h''+\cdots+(-1)^{k-2}(M_{k-1\,1}-M_{k1}')h^{(k-1)}+(-1)^{k-1}M_{k1}h^{(k)}$ in (4.14), we have

$$A_1h+A_2h'+\cdots+A_{k+1}h^{(k)}=0, \tag{4.15}$$

where

$$\begin{aligned}
A_1&=M_{11}'D_0-M_{11}(D_0'+\alpha_1'D_0),\\
A_2&=(M_{11}-M_{21}')D_0+M_{21}(D_0'+\alpha_1'D_0),\\
&\ \ \vdots\\
A_k&=(-1)^{k-2}(M_{k-1\,1}-M_{k\,1}')D_0-(-1)^{k-1}M_{k1}(D_0'+\alpha_1'D_0),\\
A_{k+1}&=(-1)^{k-1}M_{k1}D_0,
\end{aligned}$$

are rational functions.

Substituting the expressions of $h',h'',\dots,h^{(k)}$ into (4.15), we get

$$A_1f^n+A_2(f^n)'+\cdots+A_k(f^n)^{(k-1)}+A_{k+1}(f^n)^{(k)}=Q_2, \tag{4.16}$$

where $Q_2=-[A_1P_d(z,f)+A_2P_d'(z,f)+\cdots+A_kP_d^{(k-1)}(z,f)+A_{k+1}P_d^{(k)}(z,f)]$ is a differential polynomial in f with rational functions as its coefficients and degree of $Q_2\le d$.

Using (4.5) we define

$$\psi_t = \frac{(f^n)^{(t)}}{f^{n-k}}, \tag{4.17}$$

$t = 1, 2, \ldots, k$; $k \geq 3$. Now applying (4.5) and (4.17) in (4.16), we have

$$f^{n-k} R_2 = Q_2, \tag{4.18}$$

where

$$R_2 = f^k A_1 + A_2 \psi_1 + \cdots + A_{k+1} \psi_k. \tag{4.19}$$

Noting the fact $d \leq n-k-1$ and combining (4.18) with Lemma 3.1 we obtain

$$m(r, R_2) = O(\log r).$$

By the assumption, f has finitely many poles, then

$$T(r, R_2) = m(r, R_2) + N(r, R_2) = O(\log r),$$

i.e., R_2 is a rational function.

Next we discuss two sub cases as follows:

Sub-case IB.2.1: If $R_2(z) \equiv 0$, then by (4.19) we have

$$f^k A_1 = -(A_2 \psi_1 + \cdots + A_{k+1} \psi_k). \tag{4.20}$$

Now proceeding in the same way as done in Sub-case IB.1.1 and replacing $k-1$ by k we can show that f has finitely many zeros. So we omit the details.

Sub-case IB.2.2: If $R_2(z) \not\equiv 0$, then (4.18) becomes

$$f^{n-k-1}(f R_2) = Q_2. \tag{4.21}$$

Next similar to subcase Sub-case IB.1.2 we can get a contradiction.

So from the above discussion we conclude that f is a transcendental meromorphic function with finite zeros and poles. Now by Lemma 3.4, we can say that

$$f(z) = q(z) e^{P(z)}, \tag{4.22}$$

where $q(z)$ is a non-vanishing rational function, and $P(z)$ is a non-constant polynomial. Substituting (4.22) into (2.1) yields

$$q^n(z) e^{nP(z)} + \sum_{j=0}^{d} \beta_j(z) e^{jP(z)} = \sum_{i=1}^{k} p_i(z) e^{\alpha_i(z)}, \tag{4.23}$$

where $\beta_j(z)$ are rational functions.

Now we can rearrange $\{1, 2, \ldots, d\}$ to $\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ such that $\beta_j(z) e^{jP(z)} = \beta_{\sigma_j}(z) e^{\sigma_j P(z)}$ for $j = 1, \ldots, d$ and $\{1, 2, \ldots, k\}$ to $\{\tau_0, \tau_1, \ldots, \tau_{k-1}\}$ such that $p_i(z) e^{\alpha_i(z)} = p_{\tau_{i-1}}(z) e^{\alpha_{\tau_{i-1}}(z)}$ for $i = 1, 2, \ldots, k$. Then (4.23) can be written as

$$q^n(z) e^{nP(z)} + \sum_{j=0}^{d} \beta_{\sigma_j}(z) e^{\sigma_j P(z)} = \sum_{i=1}^{k} p_{\tau_{i-1}}(z) e^{\alpha_{\tau_{i-1}}(z)}. \tag{4.24}$$

Since $n > d$, $\deg(\alpha_{\tau_i} - \alpha_{\tau_j}) \geq 1$ $(1 \leq \tau_i \neq \tau_j \leq k)$ and q, p_i $(i = 1, 2, \ldots, k)$ are all non-zero rational functions, then using Lemma 3.3 on (4.24) we have

$$nP = \alpha_{\tau_0} + \mathcal{A}_0, \sigma_1 P = \alpha_{\tau_1} + \mathcal{A}_1, \sigma_2 P = \alpha_{\tau_2} + \mathcal{A}_2, \ldots, \sigma_{k-1} P = \alpha_{\tau_{k-1}} + \mathcal{A}_{k-1},$$

where $\mathcal{A}_i$ $(i = 0, 1, \ldots, k-1)$ are constants such that

$$q^n = e^{-\mathcal{A}_0} p_{\tau_0}, \beta_{\sigma_1} = e^{-\mathcal{A}_1} p_{\tau_1}, \beta_{\sigma_2} = e^{-\mathcal{A}_2} p_{\tau_2}, \ldots, \beta_{\sigma_{k-1}} = e^{-\mathcal{A}_{k-1}} p_{\tau_{k-1}}.$$

Also, $\beta_0(z) = 0$ and $\beta_{\sigma_j}(z) \equiv 0$ for all $\sigma_j \neq \sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ with $0 \leq \sigma_j \leq d$.

Therefore,

$$\alpha'_{\tau_0} : \alpha'_{\tau_1} : \cdots : \alpha'_{\tau_{k-1}} = n : \sigma_1 : \cdots : \sigma_{k-1},$$

$$nP' = \alpha'_{\tau_0} \text{ and } P_d(z, f) = \sum_{i=1}^{k-1} p_{\tau_i}(z) e^{\alpha_{\tau_i}(z)}.$$

Case II: Let $a \neq 0$. In this case we consider

$$\tilde{h}(z) = f^n + af^{n-2}f' + P_d(z, f) \tag{4.25}$$

and similar as Case I, we can prove that f is a finite order transcendental meromorphic function.

Sub-case IIA: Suppose that $k = 1$. Then (2.1) becomes

$$f^n + af^{n-2}f' + P_d = p_1 e^{\alpha_1}. \tag{4.26}$$

Differentiating (4.26) we have,

$$nf^{n-1}f' + a(n-2)f^{n-3}(f')^2 + af^{n-2}f'' + P'_d = p_1 \left(\frac{p'_1}{p_1} + \alpha'_1\right) e^{\alpha_1}. \tag{4.27}$$

Eliminating e^{α_1} from (4.26) and (4.27), we have

$$f^{n-3} R_3 = Q_3, \tag{4.28}$$

where

$$R_3 = nf^2 f' + a(n-2)(f')^2 + aff'' - \left(\alpha'_1 + \frac{p'_1}{p_1}\right)(f^3 + aff') \tag{4.29}$$

and $Q_3 = \left(\alpha'_1 + \frac{p'_1}{p_1}\right) P_d - P'_d$ with degree d. Since $d \leq n - k - 3 = n - 4$, so from (4.29) and Lemma 3.1, we have

$$m(r, R_3) = O(\log r).$$

By the hypothesis, f has finitely many poles, thus

$$T(r, R_3) = m(r, R_3) + N(r, R_3) = O(\log r),$$

i.e., R_3 is a rational function.

Sub-case IIA.1: Let $R_3 \equiv 0$. Then from (4.29) we have

$$\left(\alpha'_1 + \frac{p'_1}{p_1}\right) f^3 = nf^2 f' + a(n-2)(f')^2 + aff'' - a\left(\alpha'_1 + \frac{p'_1}{p_1}\right) ff'. \tag{4.30}$$

Now we will prove that f has only finitely many zeros. On the contrary, suppose that f has infinitely many zeros. It is clear from (4.30) that all zeros of f are multiple zeros. Assume z_3 is a zero of f but not the zeros or poles of the coefficients of (4.30), with multiplicity $p_3 \geq 2$. Then comparing the multiplicity of f on both side of (4.30) we have, z_3 is a zero of the left hand side of (4.30) with multiplicity $3p_3$ and a zero of the right hand side of (4.30) with multiplicity $2(p_3-1)$, which is a contradiction. So, f has at most finitely many zeros.

Hence f is a transcendental meromorphic function with finite zeros and poles. Now by Lemma 3.4, we can say that $f(z) = q(z)e^{P(z)}$, where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial. Substituting the form of f into (4.26) yields

$$q^n(z)e^{nP(z)} + aq^{n-2}(q'+qP')e^{(n-1)P(z)} + \sum_{j=0}^{d} \eta_j(z)e^{jP(z)} = p_1(z)e^{\alpha_1(z)}, \tag{4.31}$$

where $\eta_j(z)$ are rational functions. As $q \neq 0$, so by applying Lemma 3.3 on (4.31) we get $q' + qP' = 0$, i.e., $q(z) = D_1/e^{P(z)}$ for a constant D_1, which is a contradiction that q is a rational function.

Sub-case IIA.2: Let $R_3 \not\equiv 0$. Then from (4.28) we have,

$$f^{n-4}(fR_3) = Q_3. \tag{4.32}$$

Since $d \leq n-4$, combining (4.32) with Lemma 3.1, we have

$$m(r, fR_3) = O(\log r).$$

On the other hand, as we assume f has finitely many poles, thus

$$T(r, fR_3) = m(r, fR_3) + N(r, fR_3) = O(\log r),$$

i.e., fR_3 is a rational function and we have R_3 is rational, but f is transcendental, a contradiction. Therefore, in this case, there does not exist any transcendental meromorphic solution.

Sub-case IIB: In this case $k = 2$. Then (2.1) becomes

$$f^n + af^{n-2}f' + P_d = p_1e^{\alpha_1} + p_2e^{\alpha_2}. \tag{4.33}$$

Differentiating (4.33) and eliminating e^{α_2} we get

$$\begin{aligned} &\left(\frac{p_2'}{p_2} + \alpha_2'\right) f^n + a\left(\frac{p_2'}{p_2} + \alpha_2'\right) f^{n-2}f' - nf^{n-1}f' \\ &-a(n-2)f^{n-3}(f')^2 - af^{n-2}f'' - P_d' + \left(\frac{p_2'}{p_2} + \alpha_2'\right) P_d = p_1Ae^{\alpha_1}, \end{aligned} \tag{4.34}$$

where $A \equiv \left(\frac{p_2'}{p_2} - \frac{p_1'}{p_1} + \alpha_2' - \alpha_1'\right) \not\equiv 0$, otherwise it contradicts our assumption $\deg(\alpha_i - \alpha_j) \geq 1$ $(1 \leq i \neq j \leq k)$.

Now differentiating (4.34) and eliminating e^{α_1}, we have

$$f^{n-4}R_4 = Q_4, \tag{4.35}$$

where

$$R_4 = h_1(z)(f^4 + af^2 f') + h_2(z)(nf^3 f' + af^2 f'' + a(n-2)f(f')^2) + af^2 f^{(3)} + 3a(n-2)ff'f'' + n(n-1)f^2(f')^2 + nf^3 f'' + a(n-2)(n-3)(f')^3 \tag{4.36}$$

and

$$Q_4 = -P_d'' - h_2(z)P_d' - h_1(z)P_d,$$

such that

$$h_1(z) = \left(\frac{p_2'}{p_2} + \alpha_2'\right)\left(\frac{p_1'}{p_1} + \frac{A'}{A} + \alpha_1' - \frac{\left(\frac{p_2'}{p_2} + \alpha_2'\right)'}{\left(\frac{p_2'}{p_2} + \alpha_2'\right)}\right),$$

$$h_2(z) = -\left(\frac{p_1'}{p_1} + \frac{A'}{A} + \alpha_1' + \frac{p_2'}{p_2} + \alpha_2'\right).$$

Then it follows from Lemma 3.1 that R_4 is a rational function. Now we consider the following two cases:

Sub-case IIB.1: Let $R_4 = 0$.

Then (4.36) can be written as

$$h_1(z)(f^4 + af^2 f') = -[h_2(z)(nf^3 f' + af^2 f'' + a(n-2)f(f')^2) + af^2 f^{(3)} + 3a(n-2)ff'f'' + n(n-1)f^2(f')^2 + nf^3 f'' + a(n-2)(n-3)(f')^3]. \tag{4.37}$$

If f has infinitely many zeros, it follows from (4.37) that zeros of f are of multiplicity $p_4 \geq 2$. Let z_4 be a zero of f but not zeros or poles of the coefficients. Then comparing the multiplicity of f on both side of (4.37) we have, z_4 is a zero of the left hand side of (4.37) with multiplicity $3p_4 - 1$ and a zero of the right hand side of (4.37) with multiplicity $3(p_4 - 1)$, which is a contradiction. So, f has at most finitely many zeros.

So, f is a transcendental meromorphic function with finite zeros and poles. Now by Lemma 3.4, we can say that $f(z) = q(z)e^{P(z)}$, where $q(z)$ is a non-vanishing rational function and $P(z)$ is a non-constant polynomial. Substituting the form of f into (4.33) yields

$$q^n(z)e^{nP(z)} + aq^{n-2}(q' + qP')e^{(n-1)P(z)} + \sum_{j=0}^{d} \zeta_j(z)e^{jP(z)} = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{4.38}$$

where $\zeta_j(z)$ are rational functions. Applying Lemma 3.3 on (4.38), we get

$$nP = \alpha_1 + \mathcal{B}_1, \ \ (n-1)P = \alpha_2 + \mathcal{B}_2,$$

where $\mathcal{B}_i$ $(i = 1, 2)$ are constants such that

$$q^n = e^{-\mathcal{B}_1}p_1, \ \ aq^{n-2}(q' + qP') = e^{-\mathcal{B}_2}p_2$$

or

$$nP = \alpha_2 + \tilde{\mathcal{B}}_1, \ \ (n-1)P = \alpha_1 + \tilde{\mathcal{B}}_2,$$

where $\tilde{\mathcal{B}}_i$ $(i = 1, 2)$ are constants such that

$$q^n = e^{-\tilde{\mathcal{B}}_1} p_1, \ \ aq^{n-2}(q' + qP') = e^{-\tilde{\mathcal{B}}_2} p_2.$$

Also, $\zeta_j(z) = 0$ for all $j = 0, 1, \dots, d$, i.e., $P_d(z, f) \equiv 0$. Clearly,

$$\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1} \ \text{ and } \ \frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}.$$

Sub-case IIB.2: Let $R_4 \neq 0$. Then from (4.35) we have,

$$f^{n-5}(fR_4) = Q_4. \tag{4.39}$$

Since $d \leq n-5$, combining (4.39) with Lemma 3.1, we have

$$m(r, fR_4) = O(\log r).$$

By the hypothesis, f has finitely many poles, thus

$$T(r, fR_4) = m(r, fR_4) + N(r, fR_4) = O(\log r),$$

i.e., fR_4 is a rational function and we have R_4 is rational, which is a contradiction that f is transcendental.

Sub-case IIC: In this case, we suppose that $k \geq 3$. We consider $\tilde{h}(z)$ in (4.25) instead of $h(z)$ in (4.1) and proceed similarly as Sub-case IB with necessary changes.

Sub-case IIC.1: Now similar as Sub-case IB.1 we proceed upto (4.4). In this case, (4.4) becomes

$$M_{11}(f^n + af^{n-2}f') + \cdots + (-1)^{k-1} M_{k1}(f^n + af^{n-2}f')^{(k-1)} = Q_1 \tag{4.40}$$

and similar as (4.5), we can check that

$$\begin{aligned} (f^{n-2}f')^{(t)} &= f^{n-2}f^{(t+1)} + t(n-2)f^{n-3}f'f^{(t)} \\ &\quad + \sum_{i=2}^{t} \binom{t}{i} f^{(t-i+1)} \Bigg\{ (n-2)f^{n-3}f^{(i)} + \sum_{j=2}^{i-1} \sum_{\lambda} \gamma_{j\lambda} f^{n-j-2}(f')^{\lambda_{j,1}} \\ &\quad (f'')^{\lambda_{j,2}} \cdots (f^{(i-1)})^{\lambda_{j,i-1}} + (n-2)(n-3)\cdots(n-i-1)f^{n-i-2}(f')^{i} \Bigg\}. \end{aligned} \tag{4.41}$$

In this case, using (4.5) and (4.41), we define

$$\xi_t = \frac{(f^n + af^{n-2}f')^{(t)}}{f^{n-k-1}} \tag{4.42}$$

for $t = 1, 2, \dots, k-1$; $k \geq 3$. Then using (4.42), (4.8) changes to

$$f^{n-k-1} R_5 = Q_5. \tag{4.43}$$

After that proceeding similarly, we can prove that R_5 is rational and can consider the following two subcases.

Sub-case IIC.1.1: If $R_5(z) \equiv 0$, then (4.9) changes to

$$M_{11}(f^{k+1} + af^{k-1}f') = -\left[-M_{21}\xi_1 + \cdots + (-1)^{k-1}M_{k1}\xi_{k-1}\right]. \tag{4.44}$$

We will show that f has at most finitely many zeros. On the contrary, suppose that f has infinitely many zeros. So we can consider such a point z_5 such that $f(z_5) = 0$ but z_5 is neither a zero nor a pole of M_{j1} $(j = 1, 2, \ldots, k)$.

Now let, $M_{k1} \not\equiv 0$. Notice that, ξ_{k-1} contain one term, in which, power of f is 0. Thus we can deduce that $f'(z_5) = 0$, which implies that z_5 is a multiple zero of f with multiplicity say $p_5 \geq 2$. Then z_5 will be a zero of the left and right hand side of (4.44) with multiplicity $kp_5 - 1$ and $k(p_5 - 1)$, respectively, which is a contradiction.

Next, assume $M_{k1} \equiv 0$. If z_5 is a simple zero, then z_5 is a zero with multiplicity $k-1$ of left hand side of (4.44) and a zero with multiplicity 1 of right hand side of (4.44), which is a contradiction. If z_5 is a multiple zero with multiplicity $q_5 \geq 2$, then z_5 will respectively be a zero of multiplicity $kq_5 - 1$ and $k(q_5 - 1) + 1$ of the left and right hand side of (4.44) respectively to yield a contradiction.

Thus, f has at most finitely many zeros.

Sub-case IIC.1.2: Let $R_5(z) \not\equiv 0$. Then (4.11) changes to $f^{n-k-2}(fR_5) = Q_5$ and similarly we get a contradiction.

Sub-case IIC.2: When $D_0 \not\equiv 0$. Proceed similar as Sub-case IB.2 and in this case (4.16), (4.17) and (4.18) changes respectively to

$$A_1(f^n + af^{n-2}f') + \cdots + A_{k+1}(f^n + af^{n-2}f')^{(k)} = Q_2, \tag{4.45}$$

$$\psi_t = \frac{(f^n + af^{n-2}f')^{(t)}}{f^{n-k-2}}, \tag{4.46}$$

$t = 1, 2, \ldots, k$; $k \geq 3$ and

$$f^{n-k-2}R_6 = Q_6. \tag{4.47}$$

Here also R_6 is rational. Now, we distinguish following two cases:

Sub-case IIC.2.1: Let $R_6(z) \equiv 0$, here (4.20) changes to

$$(f^{k+2} + f^k f')A_1 = -(A_2\psi_1 + \cdots + A_{k+1}\psi_k). \tag{4.48}$$

We have f has only finitely many zeros. If not, suppose that f has infinitely many zeros. Consider a point z_6 such that $f(z_6) = 0$ but z_6 is not a zero or a pole of A_j $(j = 1, 2, \ldots, k+1)$.

Now let, $A_{k+1} \not\equiv 0$. Noticing the fact that ψ_k contains a term independent of f, we can deduce $f'(z_6) = 0$, which implies that z_6 is a multiple zero of f with multiplicity say $p_6 \geq 2$. Clearly z_6 is a zero of multiplicity $(k+1)p_6 - 1$ and $(k+1)(p_6 - 1)$ respectively of the left hand side and right hand side of (4.48), a contradiction.

Next, assume $A_{k+1} \equiv 0$. If z_6 is a simple zero, then z_6 is a zero with multiplicity k of left hand side of (4.48) and a zero with multiplicity 1 of right hand side of (4.48), which is a contradiction. If z_6 is a multiple zero with multiplicity $q_6 \geq 2$, then z_6 is a zero of left hand side of (4.48) with multiplicity $(k+1)q_6-1$ and a zero of right hand side of (4.48) with multiplicity $(k+1)q_6-k$, again gives a contradiction. It follows that f has at most finitely many zeros.

Sub-case IIC.2.2: If $R_6(z) \not\equiv 0$, then (4.21) changes to

$$f^{n-k-3}(fR_6) = Q_6$$

and adopting similar procedure we can get a contradiction.

As usual from the above two cases we conclude that f is a transcendental meromorphic function with finite zeros and poles. Now by Lemma 3.4, we can say that

$$f(z) = q(z)e^{P(z)}, \tag{4.49}$$

where $q(z)$ is a non-vanishing rational function, and $P(z)$ is a non-constant polynomial. Substituting (4.49) into (2.1) yields

$$q^n(z)e^{nP(z)} + aq^{n-2}(q' + qP')e^{(n-1)P(z)} + \sum_{j=0}^{d} \varphi_j(z)e^{jP(z)} = \sum_{i=1}^{k} p_i(z)e^{\alpha_i(z)}, \tag{4.50}$$

where $\varphi_j(z)$ are rational functions.

Now we can rearrange $\{1, 2, \ldots, d\}$ to $\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ such that $\varphi_j(z)e^{jP(z)} = \varphi_{\sigma_j}(z)e^{\sigma_j P(z)}$ for $j = 1, \ldots, d$ and $\{1, 2, \ldots, k\}$ to $\{\mu, \nu, \kappa_1, \kappa_2, \ldots, \kappa_{k-2}\}$ such that $p_1(z)e^{\alpha_1(z)} = p_\mu(z)e^{\alpha_\mu(z)}$, $p_2(z)e^{\alpha_2(z)} = p_\nu(z)e^{\alpha_\nu(z)}$ and $p_i(z)e^{\alpha_i(z)} = p_{\kappa_{i-2}}(z)e^{\alpha_{\kappa_{i-2}}(z)}$ for $i = 3, 4, \ldots, k$. Then (4.50) can be written as

$$q^n(z)e^{nP(z)} + aq^{n-2}(q' + qP')e^{(n-1)P(z)} + \sum_{j=0}^{d} \beta_{\sigma_j}(z)e^{\sigma_j P(z)} = p_\mu(z)e^{\alpha_\mu(z)} + p_\nu(z)e^{\alpha_\nu(z)} + \sum_{i=3}^{k} p_{\kappa_{i-2}}(z)e^{\alpha_{\kappa_{i-2}}(z)}. \tag{4.51}$$

Since $n > n-1 > d$, $\deg(\alpha_{\kappa_i} - \alpha_{\kappa_j}) \geq 1$ $(1 \leq \kappa_i \neq \kappa_j \leq k)$ and q, $p_{\kappa_{i-2}}$ $(i = 3, 4, \ldots, k)$ are all non-zero rational functions and $q' + qP' \not\equiv 0$, using Lemma 3.3 on (4.51) we have

$$nP = \alpha_\mu + \mathcal{C}_\mu,\ (n-1)P = \alpha_\nu + \mathcal{C}_\nu,\ \sigma_1 P = \alpha_{\kappa_1} + \mathcal{C}_1,$$
$$\sigma_2 P = \alpha_{\kappa_2} + \mathcal{C}_2, \ldots, \sigma_{k-2}P = \alpha_{\kappa_{k-2}} + \mathcal{C}_{k-2},$$

where $\mathcal{C}_i$ $(i = 1, 2, \ldots, k-2)$ are constants such that

$$q^n = e^{-\mathcal{C}_\mu}p_\mu,\ aq^{n-2}(q' + qP') = e^{-\mathcal{C}_\nu}p_\nu,\ \beta_{\sigma_1} = e^{-\mathcal{C}_1}p_{\kappa_1},$$
$$\beta_{\sigma_2} = e^{-\mathcal{C}_2}p_{\kappa_2}, \ldots, \beta_{\sigma_{k-2}} = e^{-\mathcal{C}_{k-2}}p_{\kappa_{k-2}}.$$

Also, $\beta_0(z) = 0$ and $\beta_{\sigma_j}(z) \equiv 0$ for all $\sigma_j \neq \sigma_1, \sigma_2, \ldots, \sigma_{k-2}$ with $0 \leq \sigma_j \leq d$.

Therefore,

$$\alpha'_\mu : \alpha'_\nu : \alpha'_{\kappa_1} : \cdots : \alpha'_{\kappa_{k-2}} = n : n-1 : \sigma_1 : \cdots : \sigma_{k-2},$$

$$nP' = \alpha'_\mu,\ (n-1)P' = \alpha'_\nu \ \text{ and } P_d(z,f) = \sum_{i=1}^{k-2} p_{\kappa_i}(z) e^{\alpha_{\kappa_i}(z)}.$$

This completes the proof of the theorem. □

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