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ON ALMOST QUASI-COHERENT RINGS AND ALMOST VON NEUMANN RINGS

Haitham El Alaoui, Mourad El Maalmi, and Hakima Mouanis

Abstract. Let R be a commutative ring with identity. We call the ring R to be an almost quasi-coherent ring if for any finite set of elements $a_1, \ldots, a_p$ and a of R, there exists a positive integer m such that the ideals $\bigcap_{i=1}^{p} Ra_i^m$ and $Ann_R(a^m)$ are finitely generated, and to be almost von Neumann regular rings if for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is generated by an idempotent element. This paper establishes necessary and sufficient conditions for the Nagata's idealization and the amalgamated algebra to inherit these notions. Our results allow us to construct original examples of rings satisfying the above-mentioned properties.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. If A is a ring, $\sqrt{I}$ denotes the radical of an ideal I of A, in the sense of [23]; $Nil(A) := \sqrt{0}$ the set (ideal) of all nilpotent elements of A; $Idem(A)$ the set of all idempotent elements of A and $Ann_A(E) := Ann(E)$ denotes the annihilator of an A-module E.

In 1960, according to Chase [7], R is a coherent domain if and only if the intersection of any two finitely generated ideals is again finitely generated. In 1973, Dobbs [14] introduced the concept of "finite conductor domain" in which every intersection of two principal ideals is a finitely generated ideal. Quasi-coherent a property intermediate between coherence and finite conductor defined by Barucci, Anderson and Dobbs (see [4]). A domain R is a quasi-coherent if each intersection of finitely many principal ideals of R is finitely generated. Coherent domains and Greatest Common Divisor GCD-domains (such that the intersection of any two principal ideals is again principal) are trivial examples of quasi-coherent domains. In 2000, Glaz extended the definition of quasi-coherent domains to rings with zero divisors, that is, the intersection of finitely many principal ideals of R is finitely generated and $Ann_R(a)$ is finitely

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generated for every element a of R [17], and latter further studied by many authors (see for instance [17, 22]). Moreover, the conception of von Neumann regular rings occurred in 1936 when John von Neumann defined a regular ring as a ring R (associative, with 1, not necessarily commutative) with the property that for each $a \in R$ there exists $b \in R$ such that $a = aba$ [29]. In order to distinguish these rings from the regular Noetherian rings of commutative algebra, noncommutative ring theorists have added von Neumann's name as a modifier. Regular rings are homologically characterized as those rings for which all modules (left or right) are flat (see for instance [3, 18]). Accordingly, the Bourbakian school refers to regular rings as absolutely flat rings. Commutative regular rings may be characterized in many ways: (i) rings in which all prime ideals are maximal and their nilradical are zeros; (ii) rings for which all simple modules are injective [28]; (iii) rings for which localization at any maximal ideal yields a field [28]; (iv) the polynomial ring in one variable is semi-hereditary (see [6, 25]). Further, R. S. Pierce's 1967 Memoir [27], amply demonstrated the rich connection between the theory of sheaves and commutative regular rings. We introduce a new concept of an "almost von Neumann regular ring". A ring R is an almost von Neumann regular ring (AVN-ring for short) if, for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is generated by an idempotent element. A von Neumann regular ring is naturally an AVN-ring.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$. Trivial ring extensions have been studied extensively; and considerable work, which is summarized in Glaz's book [16] and Huckaba's book [19], has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [2, 19, 21, 22].

Let A and B be two rings with identity elements, J be an ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the subring of $A \times B$, $A \bowtie^f J := \{(a, f(a) + j) : a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [9, 12, 13]). Moreover, other classical constructions (such as $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([10, Examples 2.5 and 2.6]). Other classical constructions, such as the Nagata idealization, also called trivial ring extension [26, p. 2], and the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it [10, Example 2.7 and Remark 2.8]. Unreferenced material is standard as in [15].

The purpose of this paper is to investigate the possible transfer of the notions of almost quasi-coherent rings and almost von Neumann regular rings to various context of trivial ring extensions and amalgamated. Examples of almost quasi-coherent rings are quasi-coherent rings and almost GCD-domains (i.e., for all $a, b \in R$ there exists an $n \in \mathbb{N}$ such that $a^nR \cap b^nR$ is principal). The latter introduced by Zafrullah in [30] as a generalization of GCD-domains.

2. On AQC-ring property

Definition 2.1. Let R be a ring. R is called an almost quasi-coherent ring (AQC-ring) if, for any finite set of elements $a_1, \dots, a_n$ and a of R, there exists a positive integer m such that the ideals $\bigcap_{i=1}^n Ra_i^m$ and $Ann_R(a^m)$ are finitely generated.

The first result of this section, inverstigates the possible transfer of the almost quasi-coherent property to various trivial extension contexts. Recall that a module over a domain is divisible if each element of the module is divisible by every nonzero element of the domain.

Theorem 2.1. *Let A be a ring, E be a nonzero A-module, and $R := A \ltimes E$.*

(1) *If R is an AQC-ring, then A is an AQC-ring.*
(2) *Suppose that A is a domain and E is a divisible A-module. Then R is an AQC-ring if and only if for all $((a_i)_{1\leq i\leq n}, a) \in A^n \times A \setminus \{0\}$, there exists positive integer m such that $\bigcap_{i=1}^n Aa_i^m$ is a finitely generated ideal of A (i.e., A is an AQC-ring) and $Ann_E(a^m)$ is a finitely generated submodule of E.*
(3) *Let A be a local ring with maximal ideal M, and E be a finitely generated A-module such that $M = \sqrt{Ann_A(E)}$. Then R is an AQC-ring if and only if so is A.*

Proof. (1) Suppose that R is an AQC-ring, and let $((a_i)_{1\leq i\leq n}, a) \in A^n \times A$ for some integer $n \geq 2$. Then there exists a positive integer m such that $\bigcap_{i=1}^n R(a_i, 0)^m = \bigcap_{i=1}^n R(a_i^m, 0)$ is a finitely generated ideal of R. Hence, $\bigcap_{i=1}^n Aa_i^m$ is a finitely generated ideal of A. Further, obviously $Ann_R(a^m, 0) = Ann_A(a^m) \ltimes Ann_E(a^m)$ is finitely generated. Hence, A is an AQC-ring.

(2) Suppose that R is an AQC-ring and A is a domain. Let $((a_i)_{1\leq i\leq n}, a) \in A^n \times A \setminus \{0\}$. With a similar arguments used in the proof of (1) lead to the fact A is an AQC-ring and $Ann_R(a^m, 0) = 0 \propto Ann_E(a^m)$, and therefore $Ann_E(a^m)$ is a finitely generated submodule of E. Conversely, let $((a_i, e_i)_{i=1}^{i=n}, (b, f)) \in R^{n+1}$ for some positive integer n. We will show that it exists a positive integer m such that $\bigcap_{i=1}^n R(a_i, e_i)^m$ and $(0 : (b, f)^m)$ are finitely generated ideals of R. Four cases are possibles:

Case 1. If there exists $i \in \{1, \dots, n\}$ such that $a_i = 0$ and $b = 0$, it suffices to take $m = 2$.

Case 2. If there exists $i \in \{1, \dots, n\}$ such that $a_i = 0$ and $b \neq 0$. First, notice that $(a, e)^n = (a^n, na^{n-1}e)$ for all $n \geq 1$. Since $b^2 \neq 0$, there exists a positive integer p such that $Ann_E(b^{2p})$ is a finitely generated submodule of E (by hypothesis). Moreover, one can easily check $Ann_R((b, f)^{2p}) = 0 \ltimes$

$Ann_E(b^{2p})$. Hence, $\bigcap_{i=1}^n R(a_i,e_i)^{2p} = R(0,0)$ and $Ann_R((b,f)^{2p})$ are finitely generated ideals of R.

Case 3. If $a_i \neq 0$ for all $i \in \{1,\dots,n\}$ and $b = 0$, then $a_i^2 \neq 0$ for all $i \in \{1,\dots,n\}$ (since A is a domain) and $b^2 = 0$. But since A is an AQC-ring, there exists a positive integer p such that $\bigcap_{i=1}^n Aa_i^{2p} = \sum_{j=1}^r Ac_j$, where $c_j \in \bigcap_{i=1}^n Aa_i^{2p}$ for every $j \in \{1,\dots,r\}$. Let $y \in \bigcap_{i=1}^n R(a_i,e_i)^{2p}$. It is easily seen that y is written in the form $y = (\sum_{j=1}^r \alpha_j c_j, z) = (\alpha_1 c_1, z) + (\alpha_2 c_2, 0) + \cdots + (\alpha_r c_r, 0)$, where $\alpha_j \in A$ for every $j \in \{1,\dots,r\}$ and $z \in E$. By divisibility assumption, we obtain $z = c_1\beta$ for some $\beta \in E$. Hence $y = (\alpha_1,\beta)(c_1,0) + (\alpha_2 c_2, 0) + \cdots + (\alpha_r c_r, 0)$. Therefore, $\bigcap_{i=1}^n R(a_i,e_i)^{2p} \subseteq \sum_{j=1}^r R(c_j,0)$. For the reverse inclusion, we have $c_j \in \bigcap_{i=1}^n Aa_i^{2p}$ for all $j \in \{1,\dots,r\}$. So, there exist $\alpha_{j,1}$, $\alpha_{j,2}$,…,$\alpha_{j,n}$ such that $c_j = \alpha_{j,1}a_1^{2p} = \alpha_{j,2}a_2^{2p} = \cdots = \alpha_{j,n}a_n^{2p}$ for all $j \in \{1,\dots,r\}$. Further, by divisibility, we obtain for each $j \in \{1,\dots,r\}$:

$$\begin{cases} \alpha_{j,1}2pa_1^{2p-1}e_1 - k_{j,1}a_1^{2p} = 0, \\ \alpha_{j,2}2pa_2^{2p-1}e_2 - k_{j,2}a_2^{2p} = 0, \\ \qquad \vdots \\ \alpha_{j,n}2pa_n^{2p-1}e_n - k_{j,n}a_n^{2p} = 0 \end{cases}$$

for some $k_{j,1}, k_{j,2}, \dots, k_{j,n} \in E$. Hence

$$\begin{aligned}(c_j,0) &= (\alpha_{j,1}, -k_{j,1})(a_1,e_1)^{2p} \\ &= \cdots = (\alpha_{j,n}, -k_{j,n})(a_n,e_n)^{2p} \in \bigcap_{i=1}^n R(a_i,e_i)^{2p}.\end{aligned}$$

Thus, $\bigcap_{i=1}^n R(a_i,e_i)^{2p} = \sum_{j=1}^r R(c_j,0)$ and $Ann_R((b,f)^{2p}) = R$ are finitely generated ideals of R.

Case 4. Suppose $a_i \neq 0$ for all $i \in \{1,\dots,n\}$ and $b \neq 0$. In this case, there exists a positive integer p such that $\bigcap_{i=1}^n Aa_i^p = \sum_{j=1}^r Ac_j$, where $c_j \in \bigcap_{i=1}^n Aa_i^p$ for every $j \in \{1,\dots,r\}$ and $Ann_E(b^p)$ is a finitely generated submodule of E. With a similar argument of the proof of Case 2 and Case 3, we get that $\bigcap_{i=1}^n R(a_i,e_i)^p = \sum_{j=1}^r R(c_j,0)$ and $Ann_R((b,f)^p) = 0 \ltimes Ann_E(b^p)$. Hence, $\bigcap_{i=1}^n R(a_i,e_i)^p$ and $Ann_R((b,f)^p)$ are finitely generated ideals of R. It follows that R is an AQC-ring.

(3) Suppose that (A,M) is a local ring, and assume that E is a finitely generated A-module. If R is an AQC-ring, then so is A by (1). Conversely, let $((a_i,e_i)_{i=1}^{i=n}, (b,f)) \in R^{n+1}$ for some $n \geq 2$. If there exists $i \in \{1,\dots,n\}$ such that a_i is a unit of A, then (a_i,e_i) is a unit of R by [19, Theorem 25.1]. So, $R(a_i,e_i)^s = (R(a_i,e_i))^s = R^s$ for all integer $s \geq 1$. Thus, we may assume without loss of generality, that the a_i are in M for each $i \in \{1,\dots,n\}$. As $M = \sqrt{Ann_A(E)}$, there exist positive integers n_i such that $a_i^{n_i} \in Ann(E)$.

Then for each $i \in \{1, \ldots, n\}$, we get:

$$(a_i, e_i)^{\prod_{j=1}^{n} n_j+1} = (a_i^{\prod_{j=1}^{n} n_j+1}, (\prod_{j=1}^{n} n_j + 1)a_i^{\prod_{j=1}^{n} n_j} e_i) = (a_i^{\prod_{j=1}^{n} n_j+1}, 0) \text{ and}$$

$$(b, f)^{\prod_{j=1}^{n} n_j+1} = (b^{\prod_{j=1}^{n} n_j+1}, (\prod_{j=1}^{n} n_j + 1)b^{\prod_{j=1}^{n} n_j} f),$$

where $j \in \{1, \ldots, n\}$. Since A is an AQC-ring, there exists a positive integer p such that $\bigcap_{i=1}^{n} Aa_i^{(\prod_{j=1}^{n} n_j+1)p}$ and $(0 : b^{(\prod_{j=1}^{n} n_j+1)p})$ are finitely generated ideals of A. If $b^{(\prod_{j=1}^{n} n_j+1)p}$ is a unit of A, then $\bigcap_{i=1}^{n} R(a_i, e_i)^{(\prod_{j=1}^{n} n_j+1)p}$ and $(0 : (b, f)^{(\prod_{j=1}^{n} n_j+1)p}) = (0 : (b^{(\prod_{j=1}^{n} n_j+1)P}, (\prod_{j=1}^{n} n_j + 1)pb^{((\prod_{j=1}^{n} n_j+1)p)-1} f))$ are finitely generated ideals of R; and so we are done. Suppose on the contrary, $b^{(\prod_{j=1}^{n} n_j+1)p} \in M$. So, there exists a positive integer m such that $b^{(\prod_{j=1}^{n} n_j+1)pm} \in Ann(E)$. Then for each $i \in \{1, \ldots, n\}$, we get:

$$\begin{aligned}(a_i, e_i)^{(\prod_{j=1}^{n} n_j+1)pm} &= (a_i^{(\prod_{j=1}^{n} n_j+1)pm}, ((\prod_{j=1}^{n} n_j + 1)pm)a_i^{((\prod_{j=1}^{n} n_j+1)pm)-1} e_i) \\ &= (a_i^{(\prod_{j=1}^{n} n_j+1)pm}, 0)\end{aligned}$$

and

$$\begin{aligned}(b, f)^{(\prod_{j=1}^{n} n_j+1)pm} &= (b^{(\prod_{j=1}^{n} n_j+1)pm}, ((\prod_{j=1}^{n} n_j + 1)pm)b^{((\prod_{j=1}^{n} n_j)pm)-1} f) \\ &= (b^{(\prod_{j=1}^{n} n_j+1)pm}, 0).\end{aligned}$$

Since A is an AQC-ring, there exists a positive integer r such that $\bigcap_{i=1}^{n} Aa_i^{(\prod_{j=1}^{n} n_j+1)mpr}$ and $(0 : b^{(\prod_{j=1}^{n} n_j+1)mpr})$ are finitely generated ideals of A. Therefore,

$$\bigcap_{i=1}^{n} R(a_i, e_i)^{(\prod_{j=1}^{n} n_j+1)mpr} \text{ and } (0 : (b, f)^{(\prod_{j=1}^{n} n_j+1)mpr}) = (0 : b^{(\prod_{j=1}^{n} n_j+1)mpr}) \propto E$$

are finitely generated ideals of R. □

Remark 1. The condition that E is a finitely generated A-module is not necessary in Theorem 2.1(3) (see Example 1).

As immediate corollaries of this theorem, we have the following results.

Corollary 2.1. *Let A be a domain, E a divisible Noetherian A-module, and $R := A \ltimes E$ the trivial ring extension of A by E. Then R is an AQC-ring if and only if so is A.*

Corollary 2.2. *Let A be a local ring with maximal ideal M, E a finitely generated nonzero A-module such that $ME = 0$, and $R := A \ltimes E$ the trivial ring extension of A by E. Then R is an AQC-ring if and only if so is A.*

The theorem above enable us to construct examples of AQC-rings which are not quasi-coherent rings. Recall that a ring R is a quasi-coherent ring if $\bigcap_{i=1}^{n} Ra_i$ and $(0 : a)$ are finitely generated of R for any finite set of elements $a_1, \dots, a_n$ and a of R [17].

Example 1. Let $\mathbb{R}$ denote the field of real numbers and let $R := \mathbb{R} \ltimes \mathbb{R}[X]$ be the trivial extension ring of $\mathbb{R}$ by the polynomial ring $\mathbb{R}[X]$. Then:

(1) R is an AQC-ring by Theorem 2.1(2), since $Ann_{\mathbb{R}[X]}(r) = 0$ for each $r \in \mathbb{R} \setminus \{0\}$.
(2) R is not a quasi-coherent ring. Indeed, let $c := (0, 1) \in R$. It can easily be seen that $(0 : c) = 0 \propto \mathbb{R}[X]$ which is not finitely generated.

Example 2. Let (A, M) be a nondiscrete valuation domain. Then $R := A \ltimes A/M$ satisfies the following statements:

(1) R is an AQC-ring by Corollary 2.2.
(2) R is not a quasi-coherent ring by [22, Theorem 2.6] since M is not a finitely generated ideal of A.

The next result establishes the transfer of almost quasi-coherent property to amalgamation of rings.

Proposition 2.1. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and $R := A \bowtie^f J$.*

(1) *If R is an AQC-ring, then A is an AQC-ring.*
(2) *Suppose that A is a local ring with maximal ideal M such that $f(M)J = 0$ and $J \subseteq Nil(B)$. Then, R is an AQC-ring if and only if so is A and J is a finitely generated A-module.*

Proof. (1) Suppose that R is an AQC-ring, and let $((a_i)_{1\le i\le n}, b) \in A^{n+1}$ for some integer $n \ge 2$. Then there exists a positive integer m such that the ideals $\bigcap_{i=1}^{n} R(a_i, f(a_i))^m = \bigcap_{i=1}^{n} R(a_i^m, f(a_i^m))$ and $Ann_R(b^m, f(b^m)) = Ann_A(b^m) \bowtie^f Ann_B(f(b^m)) \cap J$ are finitely generated ideals of R. Therefore, $\bigcap_{i=1}^{n} Aa_i^m$ and $Ann_A(b^m)$ are finitely generated ideals of A, and hence A is an AQC-ring.

(2) Assume that A is a local ring with maximal ideal M such that $f(M)J = 0$. Let $((a_i)_{1\le i\le n}, b) \in A^n \times M \setminus \{0\}$. With a similar argument as in the statement (1), we get $Ann_R(b^m, f(b^m)) = Ann_A(b^m) \bowtie^f J$, and therefore J is a finitely generated A-module. Conversely, let $((a_i, f(a_i) + e_i)_{1\le i\le n}, (b, f(b) + k)) \in R^{n+1}$. We will show that it exists a positive integer m such that

$\bigcap_{i=1}^n R(a_i, f(a_i)+e_i)^m$ and $Ann_R((b, f(b)+k))^m$ are finitely generated. First, not that if there exists $i \in \{1, \dots, n\}$ such that a_i is an invertible element in A, then $(a_i, f(a_i)+e_i)$ so is in R by [20, Lemma 3.3]. So, $R(a_i, f(a_i)+e_i)^s = (R(a_i, f(a_i)+e_i))^s = R^s$ for all integer $s \geq 1$. Moreover, if b is a unit of A, then $Ann_R((b, f(b)+k)^s) = 0$ for all integer $s \geq 1$, since $(b, f(b)+k)$ is a regular element of R. Thus, we may assume without loss of generality, that both a_i and b are in M for each $i \in \{1, \dots, n\}$. But since e_i, $k \in J$ for each $i \in \{1, \dots, n\}$, there exist a positives integers n_i and m such that $e_i^{n_i} = 0$ and $k^m = 0$. As A is an AQC-ring, there is a positive integer p such that $\bigcap_{i=1}^n Aa_i^{pm\prod_{j=1}^n n_j} = \sum_{l=1}^q Ac_l$ where $c_l \in A$ for each $l \in \{1, \dots, q\}$ and $Ann_A(b^{pm\prod_{j=1}^n n_j})$ is finitely generated in A. By applying binomial theorem (which is valid in any commutative ring), we get that $\bigcap_{i=1}^n R(a_i, f(a_i)+e_i)^{pm\prod_{j=1}^n n_j} = \bigcap_{i=1}^n R(a_i, f(a_i))^{pm\prod_{j=1}^n n_j}$. Hence $\bigcap_{i=1}^n R(a_i, f(a_i)+e_i)^{pm\prod_{j=1}^n n_j} = \sum_{l=1}^q R(c_l, f(c_l))$, it remains to show that $Ann_R((b, f(b)+k)^{pm\prod_{j=1}^n n_j})$ is finitely generated in R. Indeed, if $b = 0$, then $Ann_R((b, f(b)+k)^{pm\prod_{j=1}^n n_j}) = R$. Next, assume $b \neq 0$. In this case,

$$\begin{aligned} Ann_R((b, f(b)+k)^{pm\prod_{j=1}^n n_j}) &= Ann_R(b^{pm\prod_{j=1}^n n_j}, f(b^{pm\prod_{j=1}^n n_j})) \\ &= Ann_A(b^{pm\prod_{j=1}^n n_j}) \bowtie^f J \end{aligned}$$

which is finitely generated (since by hypothesis J is a finitely generated A-module). □

Proposition 2.1 enriches the literature with original examples of AQC-rings which are not quasi-coherent rings.

Example 3. Let $A := \mathbb{R} \ltimes \mathbb{R}[X]$ and $M := 0 \ltimes \mathbb{R}[X]$. Set $B := A \ltimes A/M$ and $J := 0 \ltimes A/M$. Consider the homomorphism $f : A \hookrightarrow B$, $(f(a) = (a, 0))$. Then:

(1) $A \bowtie^f J$ is an AQC-ring.
(2) $A \bowtie^f J$ is not a quasi-coherent ring.

Proof. (1) It follows from Proposition 2.1(2).

(2) $A \bowtie^f J$ is not a finite conductor ring by [20, Theorem 2.1(2)] since $M := 0 \ltimes \mathbb{R}[X]$ is not a finitely generated ideal of A, so that $A \bowtie^f J$ is not a quasi-coherent ring. □

3. On AVN-ring property

Definition 3.1. A ring R is an almost von Neumann regular (AVN-ring for short) if for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is generated by an idempotent element.

Recall that a ring R is called von Neumann regular if every finitely generated ideal is generated by an idempotent element.

Clearly, a von Neumann regular ring is an AVN-ring, while the converse fails; e.g., $\mathbb{Z}/4\mathbb{Z}$.

Our first result investigates the transfer of AVN-ring to trivial ring extension $A \ltimes E$ in case (A, M) is a local ring (with maximal ideal M) and E is an A-module such that $M = \sqrt{Ann(E)}$.

Theorem 3.1. *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Then, the following statements hold*:

1. *If R is an AVN-ring, then so is A.*
2. *Let (A, M) be a local ring (with maximal ideal M) and let E be an A-module such that $M = \sqrt{Ann(E)}$. Then R is an AVN-ring if and only if A is an AVN-ring.*

Proof. (1) Assume that R is an AVN-ring, and let a, $b \in A$. Then, $(a, 0)$, $(b, 0) \in A \ltimes E$ and so there exist $n \in \mathbf{N}$ and $e \in Idem(A)$ such that the ideal $R((a, 0)^n, (b, 0)^n) = R(a, 0)^n + R(b, 0)^n = R(a^n, 0) + R(b^n, 0) = R(e, 0)$. Therefore, $A(a^n, b^n) = Aa^n + Ab^n = Ae$, and hence A is an AVN-ring.

(2) By (1), we need only prove that if A is an AVN-ring, along with the hypothesis that $M = \sqrt{Ann(E)}$, then R is an AVN-ring. Let (a, e), $(b, f) \in R$. Two cases are then possible:

Case 1: a or $b \notin M$. Then a or b is invertible in A. Assume without loss of generality that a is invertible. Then (a, e) is a unit in R by [19, Theorem 25.1], so that $R(a, e) + R(b, f) = R(a, e) = R(1, 0)$.

Case 2: a and $b \in M$. Using the fact that $M = \sqrt{Ann(E)}$, then there exist positive integers n and m such that $a^n \in Ann(E)$ and $b^m \in Ann(E)$. Therefore, $(a, e)^{nm+1} = (a^{nm+1}, (nm+1)a^{nm}e) = (a^{nm+1}, 0)$ and $(b, f)^{nm+1} = (b^{nm+1}, (nm+1)b^{nm}f) = (b^{nm+1}, 0)$. As A is an AVN-ring, there exists a positive integer s such that $Aa^{(nm+1)s} + Ab^{(nm+1)s} = Ae$ for some $e \in Idem(A)$. Thus, $R(a^{nm+1}, 0)^s + R(b^{nm+1}, 0)^s = R(e, 0)$ since $a^n \in Ann(E)$ and $b^m \in Ann(E)$, and this completes the proof. $\square$

The next result shows that the characterization for $A \ltimes E$ to be AVN-ring can be reconducted to the case where (A, M) is a local ring and E is an A-module such that $ME = 0$.

Proposition 3.1. *Let (A, M) be a local ring, E an A-module such that $ME = 0$, and $R := A \ltimes E$. Then R is an AVN-ring if and only if A is an AVN-ring.*

Proof. We may assume without loss of generality, that $E \neq 0$. Then $Ann(E) \neq A$, hence $M = \sqrt{Ann(E)}$, and so an application of Theorem 3.1(2) completes the proof. $\square$

Next, as an illustrative example for Theorem 3.1 and Proposition 3.1, we provide a new example of AVN-ring which arises as a trivial ring extension.

Example 4. Let A be a von Neumann local ring with maximal ideal M, $E = (A/M)^{\infty}$, where $(A/M)^{\infty}$ is an infinite-dimensional (A/M)-vector space, and $R := A \ltimes E$ be the trivial ring extension of A by E. Then:

(1) R is an AVN-ring by Proposition 3.1.
(2) R is not a von Neumann regular ring by [24, Theorem 2.1(2)].

Now, we turn our attention to the transfer of the AVN-ring property to amalgamation of rings $A \bowtie^f J$. It is easy to see that, if $J = 0$, then $A \bowtie^f J \simeq A$, and so $A \bowtie^f J$ is an AVN-ring if and only if so is A. If $J = B$, then $A \bowtie^f J = A \times B$ is an AVN-ring if and only if so is A and B. We assume now that J is a nonzero proper ideal of B.

Theorem 3.2. *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a nonzero proper ideal of B. Then, the following statements hold*:

(1) *If $A \bowtie^f J$ is an AVN-ring, then so are A and $f(A) + J$.*
(2) *Assume that A is a local ring with maximal ideal M such that $f(M)J = 0$ and $J \subseteq Nil(B)$. Then $A \bowtie^f J$ is an AVN-ring if and only if A is an AVN-ring.*

To prove this theorem, we need the following lemma.

Lemma 3.1. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \subseteq Rad(B)$. Then $Idem(A \bowtie^f J) = \{(e, f(e)) : e \in Idem(A)\}$.*

Proof. Let $(e, f(e) + j)$ be an idempotent element of $A \bowtie^f J$. It is clear that e must be an idempotent element of A. We only need to show that $j = 0$. Indeed, $(f(e) + j)^2 = f(e) + j$. Thus, $j - j^2 = 2f(e)j$ and since $f(e)^2 = f(e)$, then $j - j^2 = 2f(e)^2 j$. Therefore, $j - j^2 = f(e)(j - j^2)$, hence $j = jf(e)$ (since $J \subseteq Rad(B)$, so $1 - j \in U(B)$), then $2j = 2jf(e) = j - j^2$. Consequently, $j = 0$ since $1 + j \in U(B)$. Accordingly, $Idem(A \bowtie^f J) \subseteq \{(e, f(e)) : e \in Idem(A)\}$. The converse is clear. $\square$

Proof of Theorem 3.2. (1) The proof of this statement is an immediate consequence of [11, Proposition 2.1(3)]) and the fact that if A is an AVN-ring and I is an ideal of A, then A/I is an AVN-ring.

(2) If $A \bowtie^f J$ is an AVN-ring, then so is A by (1). Conversely, suppose that A is an AVN-ring, and let $(a, f(a) + j)$ and $(b, f(b) + k)$ be two elements of $A \bowtie^f J$. Two cases are then possible:

Case 1: a or $b \notin M$. Then a or b is invertible in A. Assume without loss of generality that a is invertible. Then $(a, f(a) + j)$ is a unit in $A \bowtie^f J$ by

[20, Lemma 2.3], so that $(A \bowtie^f J)(a, f(a)+j)+(A \bowtie^f J)(b, f(b)+k) = (A \bowtie^f J)(a, f(a)+j) = (A \bowtie^f J)(1, f(1))$.

Case 2: a and $b \in M$. Since j, $k \in J$ there are positives integers p and m such that $j^p = 0$ and $k^m = 0$. As A is an AVN-ring, there is a positive integer n such that $Aa^{npm} + Ab^{npm} = Ae$ for some $e \in Idem(A)$. By applying binomial theorem (which is valid in any commutative ring), we get $(A \bowtie^f J)(a, f(a) + j)^{npm} + (A \bowtie^f J)(b, f(b) + k)^{npm} = (A \bowtie^f J)(a^{npm}, f(a^{npm})) + (A \bowtie^f J)(b^{npm}, f(b^{npm}))$. Hence $(A \bowtie^f J)(a, f(a)+j)^{npm}+(A \bowtie^f J)(b, f(b)+k)^{npm} = (A \bowtie^f J)(e, f(e))$ with $(e, f(e)) \in Idem(A \bowtie^f J)$ (by Lemma 3.1 since $J \subseteq Rad(B)$). Thus, $A \bowtie^f J$ is an AVN-ring. □

Next, as an illustrative example for Theorem 3.2, the next example provides an original of AVN-ring which is not a von Neumann regular ring. Recall that the maximal ideals of $R := A \ltimes E$ are of the form $m \ltimes E$ where m is a maximal ideal of A ([10, Theorem 3.2]).

Example 5. Let $(B, M) = (\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$, $A := B \ltimes B/M$ be the trivial ring extension of B by B/M, $f : A \to B$ be a surjective ring homomorphism and $J := M$ be the maximal ideal of B. Then:

(1) $A \bowtie^f J$ is an AVN-ring.

(2) $A \bowtie^f J$ is not a von Neumann regular ring.

Proof. (1) Clearly, $J \subseteq Nil(B)$, $f(M \ltimes B/M)J = 0$ since $M^2 = (0)$. On the other hand, A is an AVN-ring by Theorem 3.1(2) and since B is a local AVN-ring. Therefore, by application to Theorem 3.2, it follows that $A \bowtie^f J$ is an AVN-ring.

(2) Since B is not a von Neumann regular ring, then A is not a von Neumann regular ring by [1, Theorem 3.7]. Thus, $A \bowtie^f J$ is not a von Neumann regular ring by [8, Proposition 2.21(2)]. □

The next theorem gives necessary and sufficient conditions about when an amalgamation is an AVN-ring in case A and B are integral domains and J is a nonzero proper ideal of B.

Theorem 3.3. *Let A and B be a pair of integral domains, $f : A \to B$ be a ring homomorphism and J be a nonzero proper ideal of B. Then $A \bowtie^f J$ is an AVN-ring if and only if f is injective, $f(A) + J$ is an AVN-ring, and $f(A) \cap J = (0)$.*

The proof will use the following lemma.

Lemma 3.2. *Let A and B be a pair of integral domains, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. If $A \bowtie^f J$ is an AVN-ring and f is not injective, then $J = 0$.*

Proof. Deny, let $0 \neq j \in J$. Since f is not injective, there exists $0 \neq a \in ker f$. Using the fact, $A \bowtie^f J$ is an AVN-ring, $(A \bowtie^f J)(a, j)^n + (A \bowtie^f J)(0, j)^n = (A \bowtie^f J)(e, f(e) + i)$ for some $(e, f(e) + i) \in Idem(A \bowtie^f J)$ and since A is an integral domain, necessarily $e = 1$. Hence, there exist $(b, f(b)+k)$, $(c, f(c)+l)$, $(\alpha, f(\alpha) + s)$ and $(\beta, f(\beta) + t)$ in $A \bowtie^f J$ such that:

$$\begin{aligned}(a^n, j^n) &= (1, 1 + i)(b, f(b) + k),\\ (0, j^n) &= (1, 1 + i)(c, f(c) + l),\\ (1, 1 + i) &= (a^n, j^n)(\alpha, f(\alpha) + s) + (0, j^n)(\beta, f(\beta) + t).\end{aligned}$$

Hence, $c = 0$ and so $f(c) = 0$. Therefore, $j^n = l(1 + i)$ and $1 + i = j^n(f(\alpha) + f(\beta) + s + t)$, we get that $1 = l(f(\alpha) + f(\beta) + s + t)$ since B is an integral domain and $j \neq 0$, which is absurd since J is a proper ideal of B. It follows that $J \neq 0$. □

Proof of Theorem 3.3. Assume that $A \bowtie^f J$ is an AVN-ring, then f is injective by Lemma 3.2 since $J \neq 0$. We claim that $f(A) \cap J = (0)$. Deny, let $0 \neq f(a) \in J$, then $(0, f(a))$ is an element of $A \bowtie^f J$. Since $A \bowtie^f J$ is an AVN-ring, there exists a positive integer n such that the ideal $((a, f(a))^n, (0, f(a))^n)(A \bowtie^f J)$ is generated by an idempotent element. Hence, there exists $(e, f(e) + j) \in A \bowtie^f J$ (since $e \in Idem(A)$ and A is an integral domain, necessarily $e = 1$), $(A \bowtie^f J)(a^n, f(a^n)) + (A \bowtie^f J)(0, f(a^n)) = (A \bowtie^f J)(1, 1+j)$. So, there exist $(b, f(b) + k)$, $(c, f(c) + l)$, $(\alpha, f(\alpha) + s)$ and $(\beta, f(\beta) + t)$ in $A \bowtie^f J$ such that:

$$\begin{aligned}(0, f(a^n)) &= (1, 1 + j)(b, f(b) + k),\\ (a^n, f(a^n)) &= (1, 1 + j)(c, f(c) + l),\\ (1, 1 + j) &= (0, f(a^n))(\alpha, f(\alpha) + s) + (a^n, f(a^n))(\beta, f(\beta) + t).\end{aligned}$$

Hence, $b = 0$ and $f(b) = 0$. From the previous equalities, we deduce that:

$$f(a^n) = (1 + j)k \quad \text{and} \quad 1 + j = f(a^n)(f(\alpha) + f(\beta) + s + t).$$

Multiplying the second equality by k, we get that $1 = k(f(\alpha)+f(\beta)+s+t)$ since B is integral domain. We conclude k is a unit, but $k \in J$ hence $J = B$ which is a contradiction. On the other hand, $f(A) + J$ is an AVN-ring by Theorem 3.2(1). Conversely, assume that $f(A) + J$ is an AVN-ring, $f(A) \cap J = (0)$ and f is injective. We claim that the naturel projection:

$$\begin{aligned}p:\ \ & A \bowtie^f J \to f(A) + J\\ & (a, f(a) + j) \longmapsto f(a) + j\end{aligned}$$

is a ring isomorphism. Indeed, it is clear that p is surjective. It remains to show that p is injective. Let $(a, f(a) + j) \in ker(p)$, it is clear that $f(a) + j = 0$ and so $f(a) = -j \in f(A) \cap J = (0)$. Consequently, $f(a) = -j = 0$ and so $a = 0$ since f is injective. It follows that $(a, f(a) + j) = (0, 0)$. Hence, p is injective. Thus, p is a ring isomorphism. The conclusion is now straightforward. This completes the proof. □

It is worth to mention that in case $A = B$, $J = I$ is a proper ideal of A, and f is the identity homomorphism on A, the AVN-ring property on $A \bowtie^f J$ forces I to be the zero-ideal as it is shown by the following corollary.

Corollary 3.1. *Let A be an integral domain and I a proper ideal of A. Then $A \bowtie I$ is an AVN-ring if and only if so is A and $I = (0)$.*

Proof. $A \bowtie I = A \bowtie^f I$ where f is the identity homomorphism of A. If I is a nonzero ideal of A, by Theorem 3.3, $A \bowtie^f I$ is an AVN-ring forces $f(A) \cap I = A \cap I = (0)$, which is a contradiction. Hence $I = (0)$ as desired. $\square$

The next example illustrates the failure of Theorem 3.3, in general, beyond the context where A and B are integral domains.

Example 6. Let $(B, M) = (\mathbb{Z}/9\mathbb{Z}, 3\mathbb{Z}/9\mathbb{Z})$, $A := B \ltimes B/M$ be the trivial ring extension of B by B/M, $f : A \to B$ be a surjective ring homomorphism and $J := M$ be the maximal ideal of B. Then:

(1) A and B are not integral domains.
(2) $A \bowtie^f J$ is an AVN-ring by Example 5.
(3) $f(A) \cap J \neq 0$.

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Haitham El Alaoui
Faculty of Sciences Dhar El Mahraz
Sidi Mohamed Ben Abdellah University
Fez, Morocco
Email address: elalaoui.haithame@gmail.com

Mourad El Maalmi
Faculty of Sciences Dhar El Mahraz
Sidi Mohamed Ben Abdellah University
Fez, Morocco
Email address: mouradmalam@gmail.com

Hakima Mouanis
Faculty of Sciences Dhar El Mahraz
Sidi Mohamed Ben Abdellah University
Fez, Morocco
Email address: hamouanis@yahoo.fr