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EINSTEIN-TYPE MANIFOLDS WITH COMPLETE DIVERGENCE OF WEYL AND RIEMANN TENSOR

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Abstract. In this paper, we study Einstein-type manifolds generalizing static spaces and V-static spaces. We prove that if an Einstein-type manifold has non-positive complete divergence of its Weyl tensor and non-negative complete divergence of Bach tensor, then M has harmonic Weyl curvature. Also similar results on an Einstein-type manifold with complete divergence of Riemann tensor are proved.

1. Introduction

Let (M, g) be an n-dimensional smooth Riemannian manifold of dimension $n \geq 3$. We say that (M, g, f, h) is called an Einstein-type manifold if g is a solution of

$$fr = Ddf + hg \tag{1}$$

for some smooth functions f, h on M. Here, r is Ricci curvature and Ddf is the Hessian of f.

Catino et al. considered (gradient) Einstein-type manifolds generalizing Ricci solitons [2]. They showed rigidity results of gradient Einstein-type manifolds under $i_{\nabla f} B = 0$, where B is the Bach tensor. Here, $i_{\nabla f}$ is the interior product with respect to ∇f. Recall that the Bach tensor B on an n-dimensional Riemannian manifold (M, g), $n \geq 4$, is defined by

$$B = \frac{1}{n-3}\,\delta^D \delta \mathcal{W} + \frac{1}{n-2}\,\mathring{\mathcal{W}} z,$$

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where $\mathcal{W}$ is the Weyl tensor, z is the traceless Ricci tensor, and $\mathring{\mathcal{W}}z$ is defined by

$$\mathring{\mathcal{W}}z(X,Y) = \sum_{i=1}^{n} z(\mathcal{W}(X,E_i)Y,E_i)$$

for some orthonormal basis $\{E_i\}_{i=1}^n$.

Motivated by the above work, Leandro introduced in [6] Einstein-type manifolds generalizing several interesting geometric equations, such as static vacuum Einstein equations, static perfect fluid equation, CPE equations, and V-static equations (see Section 2 for details). It should be noted that Einstein-type manifolds may have non-constant scalar curvature. For example, if (M,g,f,μ,ρ) satisfies static perfect fluid equation given by

$$fr = Ddf + \frac{(\mu-\rho)f}{n-1}g \tag{2}$$

with

$$\Delta f = \frac{(n-2)\mu + n\rho}{n-1}f, \tag{3}$$

then the scalar curvature is equal to 2μ [3]. Here, the mass-energy density μ and pressure ρ are smooth functions on M.

In [6], it was proved that an Einstein-type manifold has harmonic Weyl tensor if the complete divergence of the Weyl tensor vanishes, or

$$\operatorname{div}^4\mathcal{W} = 0$$

with zero radial Weyl curvature, or $i_{\nabla f}\mathcal{W} = 0$. Therefore, it would be interesting to find weaker curvature conditions to guarantee the rigidity of Einstein-type manifolds other than the ones mentioned above. In this direction, for example, Qing and Yuan classified Bach-flat vacuum static spaces [7]. A Riemannian manifold (M,g) is called a vacuum static space if the metric satisfies static vacuum Einstein equation

$$fr = Ddf + \frac{sf}{n-1}g, \tag{4}$$

where s is the scalar curvature. In [5] it was proved that vacuum static spaces with $\operatorname{div}^4\mathcal{W} = 0$ has harmonic curvature if $\operatorname{div}^2 B \geq 0$ for $n \geq 5$, or $\frac{1}{2}|C|^2 \geq -\langle \operatorname{div} C, z_g\rangle$ for $n = 4$. Under the same condition we also proved in [5] that a nontrivial solution (M,g) of CPE (see (9) in Section 2) is isometric to a standard sphere. Recall that the Cotton tensor $C \in \Gamma(\Lambda^2 M \otimes T^*M)$ is defined by

$$C = d^D\left(r - \frac{s}{2(n-1)}\,g\right) = d^D z + \frac{n-2}{2n(n-1)}\,ds \wedge g. \tag{5}$$

Here, $d^D z$ is defined by

$$d^D z(X,Y,Z) = D_X z(Y,Z) - D_Y z(X,Z)$$

for any vectors X, Y, Z, where D is the Levi-Civita connection of (M, g), and for a 1-form ϕ and a symmmetric 2-tensor $\eta \in C^\infty(S^2M)$, $\phi \wedge \eta$ is defined by

$$(\phi \wedge \eta)(X, Y, Z) = \phi(X)\eta(Y, Z) - \eta(Y)\eta(X, Z).$$

The purpose of this paper is to show that same result holds for generic Einstein-type manifolds. More precisely, we prove the following result.

Theorem 1.1. *Let (M, g, f, h) be an Einstein-type manifold having non-positive complete divergence of the Weyl tensor, or $div^4\,\mathcal{W} \le 0$. Assume that* $\operatorname{div}^2 B \ge 0$ *for $n \ge 5$, or $\frac{1}{2}|C|^2 \ge -\langle \operatorname{div} C, z_g\rangle$ for $n = 4$. If f is proper, then M has harmonic Weyl curvature.*

It should be noted that M need not to be compact and the scalar curvature of (M, g) may not be constant in Theorem 1.1. We need to explain the condition for $n = 4$; in dimension 4, the divergence of B always vanishes, implying that $\operatorname{div}^2 B \ge 0$ is not an additional condition for $n = 4$. Therefore, a proper condition for $n = 4$ is needed. Note that, for $n \ge 5$ we have $\operatorname{div}^2 B \ge 0$ if and only if

$$\frac{1}{2}|C|^2 \ge -\langle \operatorname{div} C, z\rangle. \tag{6}$$

Thus, (6) is an appropriate condition replacing $\operatorname{div}^2 B \ge 0$ for $n = 4$.

On the other hand, the complete divergence of Riemannian curvature, or $\operatorname{div}^4 R$, is also an interesting condition to consider. For example, Yang and Zhang proved the rigidity of gradient shrinking Ricci solitons under $\operatorname{div}^4 R = 0$ [8]. For Einstein-type manifolds, we have the following results.

Theorem 1.2. *Let (M, g, f, h) be an Einstein-type manifold having non-positive complete divergence of Riemannian curvature tensor, or $div^4\, R \le 0$. Assume that* $\operatorname{div}^2 B \ge 0$ *for $n \ge 5$, or $\frac{1}{2}|C|^2 \ge -\langle \operatorname{div} C, z_g\rangle$ for $n = 4$. If f is proper and*

$$\int_{\Gamma_t} \frac{1}{|\nabla f|}\, r(\nabla s, \nabla f)\, d\sigma = 0$$

on $\Gamma_t = f^{-1}(t)$ for a regular value t of f, then M has harmonic Weyl curvature.

As an immediate consequence of Theorem 1.2, we have the following result.

Corollary 1.3. *Let (M, g, f, h) be an Einstein-type manifold with $div^4\, R \le 0$. Assume that* $\operatorname{div}^2 B \ge 0$ *for $n \ge 5$, or $\frac{1}{2}|C|^2 \ge -\langle \operatorname{div} C, z_g\rangle$ for $n = 4$. If f is proper and the scalar curvature is constant, then M has harmonic Weyl curvature.*

2. Preliminaries

In this section, we shall find basic properties of the scalar curvature of Einstein-type manifolds. Let λ be a smooth function given by

$$\lambda = h - \frac{s}{n-1} f.$$

We can rewrite Einstein-type equation (1) as

$$f\,z = Ddf + \left(\frac{sf}{n(n-1)} + \lambda\right) g. \tag{7}$$

By taking the trace of (7), we have

$$\Delta f = -\frac{sf}{n-1} - n\lambda = sf - nh. \tag{8}$$

Note that $\lambda = 0$ if $f = 0$ in (7). If f is a nonzero constant, then $\lambda = -\frac{sf}{n(n-1)}$ and so $f\,z = 0$, i.e., (M, g) is Einstein and both s and λ are constants. From now on, we may assume that f is a non-constant function.

When $\lambda = 0$ with $s = 0$, we have a static vacuum Einstein equation given by

$$fr = Ddf$$

with $\Delta f = 0$. When $\lambda = 0$, we have a static vacuum equation $s_g'^*(f) = 0$, or satisfying (4). Here, $s_g'^*$ is the L^2-adjoint of the linearization s_g' of the scalar curvature s_g. When $\lambda = -\frac{\rho+\mu}{n-1}f$, we have static perfect fluid equation (2) with (3). When $\lambda = -\frac{s}{n(n-1)}$ with $f = 1 + \varphi$, we have a CPE given by

$$(1+\varphi)\,z = Dd\varphi + \frac{s\varphi}{n(n-1)}g. \tag{9}$$

Finally, when $\lambda = \frac{\kappa}{n-1}$, we have a V-static equation given by

$$s_g'^*(f) = \kappa\, g.$$

It becomes a Miao-Tam critical equation when $\kappa = 1$. Let $\Gamma = f^{-1}(0)$. We have learned that the following result is proved in [3] in the case of static perfect fluid spacetime.

Proposition 2.1. *The scalar curvature is constant if and only if λ is constant.*

Proof. By taking the divergence of (7), we have

$$\begin{aligned} z(\nabla f, \cdot) + \frac{n-2}{2n} f\, ds &= r(\nabla f, \cdot) + d\Delta f + \frac{1}{n(n-1)} d(sf) + d\lambda \\ &= r(\nabla f, \cdot) - \frac{1}{n} d(sf) + (1-n) d\lambda. \end{aligned}$$

Here, we used the fact that $\operatorname{div} z = \frac{n-2}{2n} ds$ and

$$\operatorname{div} Ddf = r(\nabla f, \cdot) + d\Delta f.$$

Thus,

$$\frac{1}{2} f ds + (n-1) d\lambda = 0. \tag{10}$$

Therefore, if s is constant,

$$d\lambda = 0,$$

implying that λ is constant.

Conversely, if λ is constant, then

$$\frac{1}{2} f\, ds = 0 \tag{11}$$

implying that s is constant, possibly except at $f^{-1}(0)$. If 0 is a regular value of f, then we are done. Suppose that there is an open subset Σ of critical points in $\Gamma = f^{-1}(0)$. At Γ we have

$$Ddf = -\lambda g.$$

If $\lambda \neq 0$, then a critical point of f in Γ is non-degenerate and so isolated, contradicting that $\Sigma \subset \Gamma$. If $\lambda = 0$, then (1) becomes a static vacuum equation, implying that there should be no critical points of f in Γ [4], or $\Sigma = \emptyset$. These contradiction implies that 0 is a regular value of f. $\square$

Note that, since (1) becomes a static space when $\lambda = 0$, Einstein-type equations may be considered as a perturbation of static space with perturbation factor λ. As a result of Proposition 2.1, an Einstein-type manifold reduces to a V-static manifold if the scalar curvature is constant.

Remark 2.2. The following is an example of Einstein-type manifolds having constant scalar curvature. Let $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric $g = dt^2 + g_0$, where g_0 is the cannonical metric on $\mathbb{S}^{n-1}$. Take $f(t) = \cos \sqrt{n-2}\, t$ on $\mathbb{S}^1 = [0, 2\sqrt{n-2}\pi]/\sim$ with end points identified. Then, for $h(t) = (n-2) \cos \sqrt{n-2}\, t$

$$fr\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0 = Ddf\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + (n-2)\cos\sqrt{n-2}\, t,$$

since $Ddf\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -(n-2)\cos\sqrt{n-2}\, t$. Also, since $r\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right) = (n-2)\delta_{ij}$ and $Ddf\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right) = 0$, we found that (g, f, h) satisfies

$$fr = Ddf + hg.$$

3. Divergence of Bach tensor

In this section we shall prove Theorem 1.1. First note that the divergence of the Bach tensor satisfies

$$\operatorname{div} B(X) = \frac{n-4}{(n-2)^2}\langle i_X C, z\rangle.$$

Thus, the complete divergence of B satisfies

$$\operatorname{div}^2 B = \frac{n-4}{(n-2)^2}\left(\frac{1}{2}|C|^2 + \langle \operatorname{div} C, z\rangle\right). \tag{12}$$

Therefore, $\operatorname{div}^2 B \geq 0$ if and only if

$$\frac{1}{2}|C|^2 \geq -\langle \operatorname{div} C, z\rangle. \tag{13}$$

For the Cotton tensor C, it is well known (cf. [1]) that the complete divergence of the Weyl tensor is related to C by

$$\operatorname{div}\mathcal{W} = \frac{n-3}{n-2}\,C. \tag{14}$$

Also we have

$$\operatorname{div}^2 C(X) = \frac{1}{2}\langle \tilde{i}_X \mathcal{W}, C\rangle - \frac{1}{n-2}\langle i_X C, z\rangle \tag{15}$$

for any vector field X (for example, see Proposition 2 in [5]). Here, $\tilde{i}_X$ is the interior product to the final factor by

$$\tilde{i}_\xi \omega(X, Y, Z) = \omega(X, Y, Z, \xi)$$

for a 4-tensor ω and a vector field ξ.

We introduce a 3-tensor T for Einstein-type manifolds. Namely, we define T as

$$T = \frac{1}{(n-1)(n-2)}\, i_{\nabla f} z \wedge g + \frac{1}{n-2}\, df \wedge z.$$

To prove our main results, we need the following property.

Lemma 3.1. *Let* (g, f, λ) *be a solution of* (7)*. Then*

$$f\,C = \tilde{i}_{\nabla f}\mathcal{W} - (n-1)T.$$

Proof. Let (g, f, λ) be a solution of (7). By taking d^D to the both sides of (7), we have

$$\begin{aligned} d^D(fz) &= df \wedge z + f d^D z \\ &= d^D Ddf + \frac{1}{n(n-1)}\, d(sf) \wedge g + d\lambda \wedge g. \end{aligned}$$

From the

$$\begin{aligned} d^D Ddf &= \tilde{i}_{\nabla f} R \\ &= \tilde{i}_{\nabla f}\mathcal{W} - \frac{1}{n-2} i_{\nabla f} z \wedge g - \frac{1}{n(n-1)} df \wedge g - \frac{1}{n-2} df \wedge z, \end{aligned}$$

we have

$$\begin{aligned} f\, d^D z = \tilde{i}_{\nabla f}\mathcal{W} &- \frac{1}{n-2} i_{\nabla f} z \wedge g - \frac{n-1}{n-2} df \wedge z \\ &+ \frac{1}{n(n-1)} f\, ds \wedge g + d\lambda \wedge g. \end{aligned} \tag{16}$$

Our lemma follows from the definition of C together with (5) and (10). □

Lemma 3.2. *We have*

$$\operatorname{div}^2 C(\nabla f) = \frac{1}{2} f |C|^2 + \langle i_{\nabla f} C, z\rangle.$$

Proof. Note that

$$\langle T, C\rangle = \frac{1}{n-2}\langle df \wedge z, C\rangle = \frac{2}{n-2}\langle i_{\nabla f}C, z\rangle.$$

By (15) and Lemma 3.1, we have

$$\operatorname{div}^2 C(\nabla f) = \frac{1}{2}\langle \tilde{i}_{\nabla f}\mathcal{W}, C\rangle - \frac{1}{n-2}\langle i_{\nabla f}C, z\rangle = \frac{1}{2}f|C|^2 + \langle i_{\nabla f}C, z\rangle.$$

□

One of main ingredients to prove our results is a kind of monotonicity property on the integral of the divergence of the Bach tensor. For an Einstein-type manifold (M, g) with potential functions f and λ, let $M_{t,t'} = \{x \in M \mid t \leq f(x) \leq t'\}$ and $\Gamma_t = \{x \in M \mid f(x) = t\}$ for any real numbers t and t'.

Lemma 3.3. *Assume that* $\frac{1}{2}|C|^2 \geq -\langle \operatorname{div} C, z\rangle$ *for* $n \geq 4$ *on an Einstein-type manifold with compact level sets* Γ_t. *Then*

$$\int_{\Gamma_t} \frac{1}{|\nabla f|}\langle i_{\nabla f}C, z\rangle \, d\sigma$$

is monotone increasing with respect to regular values $t's$ *of* f. *Here,* $d\sigma$ *denotes the induced* $(n-1)$*-volume form on* Γ_t.

Proof. Note that, for an orthonormal frame $\{E_i\}_{i=1}^n$ we have

$$\begin{aligned}\langle \operatorname{div} C, z\rangle &= \operatorname{div}(C(\cdot, E_i, E_j)z_{ij}) - C_{ijk}D_{E_i}z_{jk}\\ &= \operatorname{div}(C(\cdot, E_i, E_j)z_{ij}) - \frac{1}{2}|C|^2.\end{aligned}$$

Here, we used the fact that

$$2C_{ijk}D_{E_i}z_{jk} = C_{ijk}D_{E_i}z_{jk} + C_{jik}D_{E_j}z_{ik} = C_{ijk}(D_{E_i}z_{ijk} - D_{E_j}z_{ik}) = |C|^2.$$

Thus, by (13)

$$\operatorname{div}(C(\cdot, E_i, E_j)z_{ij}) = \frac{1}{2}|C|^2 + \langle \operatorname{div} C, z\rangle \geq 0$$

on M. This implies that

$$\begin{aligned}0 &\leq \int_{M_{t,t'}} \operatorname{div}(C(\cdot, E_i, E_j)z_{ij}) = \int_{\partial M_{t,t'}} \frac{1}{|\nabla f|}\langle i_{\nabla f}C, z\rangle \, d\sigma\\ &= \int_{\Gamma_{t'}} \frac{1}{|\nabla f|}\langle i_{\nabla f}C, z\rangle \, d\sigma - \int_{\Gamma_t} \frac{1}{|\nabla f|}\langle i_{\nabla f}C, z\rangle \, d\sigma.\end{aligned}$$

□

Now, we are ready to prove Theorem 1.1, which states that the non-positive complete divergences of the Cotton tensor on an Einstein-type manifold (M, g, f, λ) with non-negativity of $\operatorname{div}^2 B$ implies that (M, g) has harmonic curvature for $n \geq 4$ if f is proper.

Suppose that $\Gamma_0 = \emptyset$. Then, either $f > 0$ or $f < 0$ on M. First assume that $f > 0$. Let $M_t = \{x \in M \mid f(x) < t\}$ for $t > 0$ so that $M_{0,t} = M_t$. In this case, by Lemma 3.3 that

$$\int_{\Gamma_t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle d\sigma \geq 0.$$

For a regular value $t > 0$, the divergence theorem together with Lemmas 3.2 and 3.3 shows

$$\begin{aligned}\int_{M_t} \operatorname{div}^3 C \, dv_g = \int_{\Gamma_t} \operatorname{div}^2 C(N) \, d\sigma &= \frac{1}{2}\int_{\Gamma_t} \frac{f|C|^2}{|\nabla f|} + \int_{\Gamma_t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma \\ &\geq \frac{t}{2}\int_{\Gamma_t} \frac{|C|^2}{|\nabla f|} \, d\sigma.\end{aligned}$$

Here, $N = \nabla f/|\nabla f|$. Since $\operatorname{div}^3 C \leq 0$ and t is arbitrary, this implies that $C = 0$ on M.

For the case $f < 0$ on M, consider $M^{-t} = \{x \in M \mid f(x) > -t\}$ with a regular value $-t$ of f, $t > 0$. Then $M_{-t,0} = M^{-t}$. Since

$$0 \leq \int_{M^{-t}} \operatorname{div}(C(\cdot, E_i, E_j) z_{ij}) = -\int_{\Gamma_{-t}} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle,$$

we have

$$\begin{aligned}\int_{M^{-t}} \operatorname{div}^3 C = -\int_{\Gamma_{-t}} \operatorname{div}^2 C(N) &= -\frac{1}{2}\int_{\Gamma_{-t}} \frac{f|C|^2}{|\nabla f|} - \int_{\Gamma_{-t}} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \\ &\geq \frac{t}{2}\int_{\Gamma_{-t}} \frac{|C|^2}{|\nabla f|}.\end{aligned}$$

Since $\operatorname{div}^3 C \leq 0$, C vanishes on all of M.

Now, suppose that $f^{-1}(0) \neq \emptyset$. For regular values t and t' of f with $t < t'$, from Lemma 3.3, we have

$$\begin{aligned}\int_{M_{t,t'}} \operatorname{div}^3 C &= \int_{\Gamma_{t'}} \operatorname{div}^2 C(N) - \int_{\Gamma_t} \operatorname{div}^2 C(N) \\ &\geq \frac{t'}{2}\int_{\Gamma_{t'}} \frac{|C|^2}{|\nabla f|} - \frac{t}{2}\int_{\Gamma_t} \frac{|C|^2}{|\nabla f|}.\end{aligned}$$

Here, $N = \frac{\nabla f}{|\nabla f|}$, and we used the result of Lemma 3.2 in the last equality. Therefore, it follows from the assumption that $\operatorname{div}^3 C \leq 0$ that

$$t\int_{\Gamma_t} \frac{|C|^2}{|\nabla f|} \geq t' \int_{\Gamma_{t'}} \frac{|C|^2}{|\nabla f|}.$$

By taking $t' = 0$, we may conclude that $C = 0$ on Γ_t for all regular values t of f with $t < 0$. Similarly, one can prove that $C = 0$ on $\Gamma_{t'}$ for any positive regular value t' of f by taking $t = 0$. Hence, we may conclude that $C = 0$ on all of M by continuity. In other words, M has harmonic curvature. This proves our theorem.

4. Complete divergence of Riemannian curvature tensor

In this section we investigate the complete divergence of Riemannian curvature tensor in an Einstein-type manifolds and prove Theorem 1.2. Let R be the Riemannian curvature tensor. From the decomposition of Riemannian curvature tensor, we have

$$(\operatorname{div}\mathcal{W})_{ijk} = \frac{n-3}{n-2}(\operatorname{div}R)_{ijk} - \frac{n-3}{2(n-1)(n-2)}(ds\wedge g)_{jki}.$$

Thus we can obtain

$$(\operatorname{div}^2\mathcal{W})_{ik} = \frac{n-3}{n-2}(\operatorname{div}^2 R)_{ik} - \frac{n-3}{2(n-1)(n-2)}(\Delta s\, g_{ik} - (Dds)_{ik}),$$

$$(\operatorname{div}^3\mathcal{W})_{i} = \frac{n-3}{n-2}(\operatorname{div}^3 R)_{i} + \frac{n-3}{2(n-1)(n-2)}r(\nabla s, E_i),$$

and

$$\operatorname{div}^4\mathcal{W} = \frac{n-3}{n-2}\operatorname{div}^4 R + \frac{n-3}{2(n-1)(n-2)}\left(\frac{1}{2}|\nabla s|^2 + \langle r, Dds\rangle\right). \tag{17}$$

Therefore, if $\operatorname{div}^4 R \le 0$, by (14)

$$\operatorname{div}^3 C = \frac{n-2}{n-3}\operatorname{div}^4\mathcal{W} \le \frac{1}{2(n-1)}\left(\frac{1}{2}|\nabla s|^2 + \langle r, Dds\rangle\right). \tag{18}$$

By integrating (18) on any subset Ω of M, we have

$$\int_\Omega \operatorname{div}^3 C\, dv_g \le \frac{1}{2(n-1)}\int_{\partial\Omega}\frac{1}{|\nabla f|}r(\nabla s, \nabla f).$$

Thus, assume that

$$\int_{\Gamma_t}\frac{1}{|\nabla f|}r(\nabla s,\nabla f) = 0.$$

If $f>0$ on M, by arguing as in the proof of Theorem 1.1,

$$\frac{t}{2}\int_{\Gamma_t}\frac{|C|^2}{|\nabla f|}d\sigma \le \int_{M_t}\operatorname{div}^3 C \le \frac{1}{2(n-1)}\int_{\Gamma_t}\frac{1}{|\nabla f|}r(\nabla s,\nabla f) = 0$$

for $t>0$, implying that $C=0$ on M. Here, we used Lemma 3.3. A similar argument also holds when $f<0$ on M.

In general case, or $\Gamma_0 \ne \emptyset$, for regular values t and t' with $t<t'$,

$$\begin{aligned}\frac{t'}{2}\int_{\Gamma_{t'}}\frac{|C|^2}{|\nabla f|}d\sigma - \frac{t}{2}\int_{\Gamma_t}\frac{|C|^2}{|\nabla f|}d\sigma &\le \int_{M_{t,t'}}\operatorname{div}^3 C\\ &\le \frac{1}{2(n-1)}\int_{\partial M_{t,t'}}\frac{1}{|\nabla f|}r(\nabla s,\nabla f) = 0.\end{aligned}$$

Therefore, as argued in the proof of Theorem 1.1, we may conclude that $C=0$ on M.

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