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A NOTE ON ARTINIAN LOCAL RINGS

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Abstract. In this note, we prove that an Artinian local ring is G-semisimple (resp., SG-semisimple, 2-SG-semisimple) if and only if its maximal ideal is G-projective (resp., SG-projective, 2-SG-projective). As a corollary, we obtain the global statement of the above. We also give some examples of local G-semisimple rings whose maximal ideals are n-generated for some positive integer n.

1. Introduction

Throughout this note, all rings are commutative with identity and all modules are unitary.

Recall that an R-module M is called *Gorenstein projective* (G-projective for short) if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module ([9, 11]). The Gorenstein projective dimension of an R-module M is defined in terms of Gorenstein projective resolutions, and denoted by $\mathrm{Gpd}_R(M)$. Bennis and Mahdou [7] defined the global Gorenstein dimension of a ring R as $\sup\{\mathrm{Gpd}_R(M) \mid M$ is an R-module$\}$, which is denoted by G-gldim(R). A ring R is called a *Gorenstein hereditary ring* (G-hereditary ring for short) if any submodule of a projective R-module is G-projective, equivalently G-gldim$(R) \leqslant 1$. The authors in [5] introduced *strongly Gorenstein projective* (SG-projective for short) modules. An R-module M is called SG-projective if there exists an exact sequence of projective R-modules $\cdots \to P \to P \to P \to P \to \cdots$ such that all these projective modules are the same and all these arrows in this sequence the same homomorphism with M being the image of some arrow and $\mathrm{Hom}_R(-, Q)$ leaves

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the sequence exact whenever Q is a projective module. The authors in [8] introduced G-semisimple and SG-semisimple rings. A ring R is called *G-semisimple* (resp., *SG-semisimple*) if every R-module is G-projective (resp., SG-projective). It is shown in [8] that the G-semisimple rings are just the well-known quasi-Frobenius rings (QF-rings for short), i.e., Noetherian and self-injective rings. For a ring R and a positive integer $n \geqslant 1$, an R-module M is said to be *n-strongly Gorenstein projective* (n-SG-projective for short), if there exists an exact sequence of R-modules $0 \longrightarrow M \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$, where each P_i is projective, such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module (see [6]). The authors in [4] introduced n-SG-semisimple rings. A ring R is called *n-SG-semisimple* if every R-module is n-SG-projective. In particular, 2-SG-semisimple rings are well studied in [4].

As stated in [18], Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory. So it is necessary to study them from various perspectives. It is well-known that an Artinian local ring is a principal ideal ring if and only if its maximal ideal is principal ([16, Theorem 2.1] or [1, Proposition 8.8]). Motivated by this result, in this note we try to find the above type of theorem for G-semisimple (resp., SG-semisimple, 2-SG-semisimple) rings. More precisely, we prove that an Artinian local ring is G-semisimple (resp., SG-semisimple, 2-SG-semisimple) if and only if its maximal ideal is G-projective (resp., SG-projective, 2-SG-projective). Hence an Artinian ring is G-semisimple (resp., SG-semisimple, 2-SG-semisimple) if and only if its maximal ideals are G-projective (resp., SG-projective, 2-SG-projective). We also give some examples of local G-semisimple rings whose maximal ideals are n-generated for some positive integer n.

By the language of [3], SG-semisimple, 2-SG-semisimple and n-SG-semisimple rings are also called 1-QF, 2-QF and n-QF rings respectively. For unexplained concepts and notations, one can refer to [15, 17, 19].

2. When prime ideals of an Artinian ring are G-projective

It is well known that QF-rings are Artinian rings and the radical of any Artinian ring is nilpotent. We begin this section with constructions of local QF-rings. If an ideal I of a ring R is nilpotent, its index of nilpotency is defined to be the least positive integer k for which $I^k = 0$.

Theorem 2.1. *Let R be an Artinian local ring with maximal ideal M and $k \geqslant 2$ be the index of nilpotency of M. If R is a QF-ring (i.e., G-semisimple ring), then M^{k-1} is principal.*

Proof. Since $M^k = 0$, we have $MM^{k-1} = 0$, and so $M \subseteq \mathrm{ann}(M^{k-1})$. Since M is the only maximal ideal of R, it follows that $M = \mathrm{ann}(M^{k-1})$, and hence $\mathrm{ann}(M) = \mathrm{ann}(\mathrm{ann}(M^{k-1}))$. But by [4, Theorem 1.1] and [4, Theorem 1.2], we have $\mathrm{ann}(\mathrm{ann}(M^{k-1})) = M^{k-1}$ and $\mathrm{ann}(M)$ is principal, respectively. Thus we get that M^{k-1} is principal. □

Recall that a domain R is called a *Gorenstein Dedekind domain* (G-Dedekind domain for short) if it is a G-hereditary ring. A nontrivial example of G-Dedekind domains is $\mathbb{Q} + x^2\mathbb{Q}[x]$ (see [13] or [14]).

Example 2.2. Let $R = \mathbb{Q} + x^2\mathbb{Q}[x]$ and $J = (x^6)$. Then $R/J \cong \{q_0 + q_1x^2 + q_2x^3 + q_3x^4 + q_4x^5 + q_5x^7 \mid q_0, q_1, q_2, q_3, q_4, q_5 \in \mathbb{Q}, x^6 = x^8 = x^9 = \cdots = x^n = \cdots = 0\}$. This is an Artinian local ring with the maximal ideal $M = (x^2, x^3)$. Since J is a principal ideal, by [13, Corollary 2.7], the ring R/J is a QF-ring. It can be seen that $M^4 = 0$ and $M^3 = (x^7)$ is principal.

The following example shows that even if some power of the maximal ideal of an Artinian local ring R is nonzero and principal, it is not necessary that R is a QF-ring.

Example 2.3. Let $J := (x^2, y^2, z^2, xz, yz)$ be an ideal of the ring $\mathbb{Q}[x, y, z]$. The ring $R := \mathbb{Q}[x, y, z]/J$ is an Arinian local ring with the maximal ideal $M := (\overline{x}, \overline{y}, \overline{z})$. It can be seen that $M^3 = 0$ and $M^2 = (\overline{xy})$ is principal. Because $\mathrm{ann}(M) = (\overline{xy}, \overline{z})$ is not principal, by [4, Theorem 1.2], R is not a QF-ring.

Since every module over any QF-ring is G-projective, the maximal ideal of a local QF-ring is G-projective. Next we prove the reverse implication holds true, that is, if the maximal ideal of an Artinian local ring R is G-projective, then R is a QF-ring.

Lemma 2.4. *Let R be a Noetherian ring. If every prime ideal of R is G-projective, then R is a G-hereditary ring, i.e.,* $\text{G-gldim}(R) \leqslant 1$.

Proof. First we show that the injective dimension of any projective R-module is at most 1. Let Q be a projective R-module. It will suffice to prove that $\mathrm{Ext}^2_R(M, Q) = 0$ for any finitely generated R-module M. Since R is Noetherian and M is a finitely generated R-module, by [19, Theorem 4.2.27], there exists an ascending chain of submodules of M:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that $M_{i+1}/M_i \cong R/P_{i+1}$, where P_i's are prime ideals of R and $i = 0, 1, \ldots, n-1$. Let P be any prime ideal of R. Since P is G-projective, we have $\mathrm{Ext}^1_R(P, Q) = 0$. But $\mathrm{Ext}^1_R(P, Q) \cong \mathrm{Ext}^2_R(R/P, Q)$, and so we have $\mathrm{Ext}^2_R(R/P, Q) = 0$ and $\mathrm{Ext}^2_R(M_{i+1}/M_i, Q) = 0$. Inductively, we get that $\mathrm{Ext}^2_R(M, Q) = 0$.

Secondly, we will show that R is 1-Gorenstein. But this comes from [10, Theorem 9.1.11].

Finally, by [19, Theorem 11.7.15], we get that R is a G-hereditary ring, i.e., $\text{G-gldim}(R) \leqslant 1$. □

Corollary 2.5. *A ring R is a QF-ring* (*i.e.,* $\text{G-gldim}(R) = 0$) *if and only if it is Artinian and its prime ideals are G-projective.*

Proof. Just notice that, by [13, Corollary 2.4], the Gorenstein global dimension of an Artinian ring is either infinite or zero. Lemma 2.4 tells us that the Gorenstein global dimension of R is finite. □

Immediately, we have the following corollary.

Corollary 2.6. *Let R be an Artinian local ring with maximal ideal M. Then R is G-semisimple* (*i.e., a QF-ring*) *if and only if M is G-projective.*

As a special case of [2, Proposition 2.13], we have the following lemma.

Lemma 2.7. *A ring R is SG-semisimple* (*resp.,* 2*-SG-semisimple*) *if and only if $R = R_1 \oplus \cdots \oplus R_n$, where each R_i is a local SG-semisimple* (*resp.,* 2*-SG-semisimple*) *ring.*

The structure theorem for Artinian rings states that an Artinian ring is uniquely (up to isomorphism) a finite direct product of Artinian local rings [1, Theorem 8.7]. Thus in order to characterize when an Artinian ring is SG-semisimple (resp., 2-SG-semisimple), it is sufficient to study the local cases, i.e., Artinian local rings, by Lemma 2.7 (see Corollary 3.4).

3. Main results

It was proved in [8, Theorem 3.7] that an Artinian local ring R is SG-semisimple if and only if it has at most one nonzero proper ideal. It is routine to check that this also means that the maximal ideal M of R is principal and M^2 is zero. It was also proved in [4] that an Artinian local ring R is 2-SG-semisimple if and only if its maximal ideal is principal. We begin this section with the following lemma.

Lemma 3.1. *Let R be an Artinian local ring with maximal ideal M and $k \geqslant 2$ be the index of nilpotency of M. Then M is n-SG-projective if and only if M^{k-1} is n-SG-projective and principal.*

Proof. First we prove the necessity part. If M is n-SG-projective, then R is a QF-ring by Corollary 2.5. So, by Theorem 2.1, M^{k-1} is principal. Since $M = \operatorname{ann}(M^{k-1})$, we have the following short exact sequence: $0 \longrightarrow M \longrightarrow R \longrightarrow M^{k-1} \longrightarrow 0$. Noticing that M^{k-1} is G-projective (because R is a QF-ring), an application of [12, Proposition 2.4] to this sequence tells us that M^{k-1} is also n-SG-projective.

Secondly we prove the sufficiency part. Suppose M^{k-1} is n-SG-projective and principal. As the same reason as the first part, we also have the above short exact sequence. Because M^{k-1} is n-SG-projective, by [12, Proposition 1.1], M is also n-SG-projective. □

Now we give a characterization of local SG-semisimple rings.

Theorem 3.2. *Let R be an Artinian local ring with maximal ideal M and $k \geqslant 2$ be the index of nilpotency of M. Then the following statements are equivalent*:

(1) *R is SG-semisimple.*
(2) *M is SG-projective.*
(3) *M^{k-1} is SG-projective and principal.*

Proof. (1)⇒(2) This follows from the definition of SG-simple rings.

(2)⇔(3) This is a special case of Lemma 3.1 when $n = 1$.

(2)+(3)⇒(1) First notice that, by Corollary 2.6, R is a QF-ring. As the same argument as that of the proof of Theorem 2.1, we have $M = \text{ann}(M^{k-1})$. Because M^{k-1} is principal and SG-projective, it follows by [8, Lemma 3.4] that $\text{ann}(M^{k-1}) = \text{ann}(\text{ann}(M^{k-1}))$. But the right side of this equality is just M^{k-1} because R is a QF-ring. So we get that M is principal and $M^2 = 0$. Therefore R is SG-semisimple. □

Likewise we have the following result.

Theorem 3.3. *Let R be an Artinian local ring with maximal ideal M and $k \geqslant 2$ be the index of nilpotency of M. Then the following statements are equivalent*:

(1) *R is 2-SG-semisimple.*
(2) *M is 2-SG-projective.*
(3) *M^{k-1} is 2-SG-projective and principal.*

Proof. (1)⇒(2) This follows from the definition of 2-SG-simple rings.

(2)⇔(3) This is a special case of Lemma 3.1 when $n = 2$.

(2)+(3)⇒(1) First notice that, by Corollary 2.6, R is a QF-ring. As the same argument as that of the proof of Theorem 2.1, we have $M = \text{ann}(M^{k-1})$. Because M^{k-1} is principal and 2-SG-projective, we have $\text{ann}(M^{k-1})$ is principal by [4, Corollary 2.4]. So we get that M is principal. Therefore R is 2-SG-semisimple. □

Going back to the global case, we have the following corollary.

Corollary 3.4. *An Artinian ring is SG-semisimple (resp., 2-SG-semisimple) if and only if its maximal ideals are SG-projective (resp., 2-SG-projective).*

Proof. Let R be an Artinian ring and $M_1, M_2, \ldots, M_s$ be all maximal ideals of R. Then we have $R \cong \bigoplus R_{M_i}$. If M_i is an SG-projective (resp., a 2-SG-projective) R-module, then $(M_i)_{M_i}$ is an SG-projective (resp., a 2-SG-projective) R_{M_i}-module. So R_{M_i} is SG-semisimple by Theorem 3.2 (resp., 2-SG-semisimple by Theorem 3.3). Therefore R is SG-semisimple (resp., 2-SG-semisimple) by Lemma 2.7. The proof of the necessity part is obvious. □

Although there exists a nice construction to produce an example of a 2-SG-projective module, but not 3-SG-projective in [20, Example 3.2], it is also interesting to find such examples using the previous results in this section, as the reviewer suggests. To do so, we need the following result.

Proposition 3.5. *Let M be an R-module. If M is both 2-SG-projective and 3-SG-projective, then M is SG-projective.*

Proof. Since M is a 2-SG-projective R-module, there exists the following exact sequence:

$$0 \to M \to P_1 \to P_0 \to M \to 0, \tag{1}$$

where P_i's are projective. Since M is also a 3-SG-projective R-module, there exist exact sequences:

$$0 \to K \to Q_1 \to Q_0 \to M \to 0 \tag{2}$$

and

$$0 \to M \to Q_2 \to K \to 0, \tag{3}$$

where Q_i's are projective.

Combining (1) and (2), we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Thus we get the following exact sequence:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & P_1 \oplus K & \xrightarrow{\quad d \quad} & P_0 \oplus Q_1 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & & & & \searrow \quad \nearrow & & & & & \\
 & & & & & \mathrm{Im}(d) & & & & & \\
 & & & & & \nearrow \quad \searrow & & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

It can be seen that $\mathrm{Im}(d)$ is projective. It follows from (3) that we have the following exact sequence:

$$0 \to M \to Q_2 \oplus P_1 \to K \oplus P_1 \to 0.$$

Now we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & Q_2 \oplus P_1 & \longrightarrow & K \oplus P_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Im}(d) & = & \mathrm{Im}(d) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that the first vertical sequence splits, and so H is projective. Therefore the top row ensures that M is SG-projective. □

Example 3.6. Let R be a 2-SG-semisimple local ring which is not SG-semisimple. Then the maximal ideal M of R is a 2-SG-projective principal ideal, but not 3-SG-projective. For a concrete example, let $R := \mathbb{Z}/p^3\mathbb{Z}$ and $M := p\mathbb{Z}/p^3\mathbb{Z}$, where p is a prime integer ([8, Corollary 3.10]). Indeed, assume on the contrary that M is 3-SG projective. Then by Proposition 3.5 M is SG-projective. Thus R is SG-semisimple by Corollary 3.4, which is a contradiction to the hypothesis.

4. Two examples of G-semisimple rings which are not 2-SG-semisimple

Let $n \geqslant 2$ be a positive integer. In this section, we construct two local G-semisimple rings whose maximal ideals are generated by at least n elements. Since the maximal ideal of any local 2-SG-semisimple ring is principal, these G-semisimple rings are not 2-SG-semisimple.

Example 4.1. Let F be a field and $X_1, X_2, \dots, X_n$ be n indeterminates. Then the factor ring $F[X_1, X_2, \dots, X_n]/(X_1^2, X_2^2, \dots, X_n^2)$ is a local G-semisimple ring whose maximal ideal is n-generated.

Proof. Let $R = F[X_1, X_2, \dots, X_n]/(X_1^2, X_2^2, \dots, X_n^2)$ and denote the image of X_i in the factor ring by $\overline{X_i}$. That R is local comes from the fact that the ideal $(\overline{X_1}, \overline{X_2}, \dots, \overline{X_n})$ is nilpotent and maximal. Let $M = (\overline{X_1}, \overline{X_2}, \dots, \overline{X_n})$. It can be seen that $M^{n+1} = 0$ and $M^n = (\overline{X_1 X_2 \cdots X_n})$ is principal. So $M^n = (\overline{X_1 X_2 \cdots X_n}) \subseteq \operatorname{ann}(M)$. Next we prove that $M^n = (\overline{X_1 X_2 \cdots X_n}) = \operatorname{ann}(M)$. Let $m \in M$ and m is not inside M^n. Then

$$m = \sum_{k_i \in \{0,1\},\ 0 < k_1 + k_2 + \cdots + k_n \leqslant n} a_{k_1,\dots,k_n} \overline{X_1}^{k_1} \overline{X_2}^{k_2} \cdots \overline{X_n}^{k_n},$$

where $a_{k_1,\dots,k_n} \in F$ and the minimal sum of powers of these terms is strictly less than n. Let $a_{k_1,\dots,k_n} \overline{X_1}^{k_1} \overline{X_2}^{k_2} \cdots \overline{X_n}^{k_n}$ be the term of m such that the power sum $k_1 + k_2 + \cdots + k_n$ is the smallest among others. It can be checked that $\overline{X_1}^{1-k_1} \cdots \overline{X_n}^{1-k_n}$ is an element in M such that $m\overline{X_1}^{1-k_1} \cdots \overline{X_n}^{1-k_n} = a_{k_1,\dots,k_n} \overline{X_1 X_2 \cdots X_n} \neq 0$. Therefore m is not inside $\operatorname{ann}(M)$. Since $M^n = \operatorname{ann}(M)$ is principal and nonzero, by [4, Theorem 1.2], R is a QF-ring. Obviously, the maximal ideal M is generated by at least n elements. □

Now we view the ring in Example 4.1 from another perspective. We know that, as a factor ring of a principal ideal domain, the ring $F[X_1]/(X_1^2)$ is a QF-ring (in fact it is SG-semisimple by [8, Corollary 3.9]). Thus the ring $\frac{F[X_1]}{(X_1^2)}[X_2]$ is a G-hereditary ring (i.e., G-gldim$(\frac{F[X_1]}{(X_1^2)}[X_2]) = 1$) by [19, Theorem 11.5.11]. So the ring $\frac{F[X_1, X_2]}{(X_1^2, X_2^2)}$ is a QF-ring again by [19, Theorem 11.5.7]. Inductively, we get that the ring $F[X_1, X_2, \ldots, X_n]/(X_1^2, X_2^2, \ldots, X_n^2)$ is also a QF-ring. Next, we give another example of a local QF-ring whose maximal ideal is also generated by at least n elements.

Example 4.2. Let K be a field, X be an indeterminate, and

$$D = K[X^{n+1}, X^{n+2}, \ldots, X^{2n}].$$

The ring $R = \frac{D}{(X^{2n+2})D}$ is a local QF-ring whose maximal ideal is generated by at least n elements.

Proof. Denote the image of X^i in R by $\overline{X^i}$. Since the R-ideal

$$M = (\overline{X^{n+1}}, \overline{X^{n+2}}, \ldots, \overline{X^{2n}})$$

is maximal and nilpotent, R is local. Thus

$$R = K + K\overline{X^{n+1}} + \cdots + K\overline{X^{2n}} + K\overline{X^{2n+3}} + \cdots + K\overline{X^{3n+2}} + K\overline{X^{4n+3}}.$$

This is a K-vector space of dimension $2n+2$. It can be checked that $M^3 = (\overline{X^{4n+3}})$ and $M^4 = 0$. Thus $M^3 \subseteq \operatorname{ann}(M)$. If $m \in M$ is not inside M^3, then $m = k_i\overline{X^i} + \cdots + k_{4n+3}\overline{X^{4n+3}}$ where $k_j \in K$ and $k_i \neq 0$, $i \in \{n+1, n+2, \ldots, 2n\} \cup \{2n+3, 2n+4, \ldots, 3n+2\}$. It can be seen that

$$m\overline{X^{4n+3-i}} = k_i\overline{X^{4n+3}} \neq 0.$$

This means that m is not inside $\operatorname{ann}(M)$ too. So we have $M^3 = \operatorname{ann}(M)$ is principal and nonzero. Thus R is a QF-ring by [4, Theorem 1.2]. Obviously the maximal ideal M is n-generated. □

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